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Ground States for Singularly Perturbed Planar Choquard Equation with Critical Exponential Growth*

Limin Zhang[1,2], Fangfang Liao[3], Xianhua Tang[2] and Dongdong Qin[2,†]

Abstract In this paper, we are dedicated to studying the following singularly Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}\left[I_\alpha * F(u)\right] f(u), \qquad x \in \mathbb{R}^2,$$

where $V(x)$ is a continuous real function on $\mathbb{R}^2$, $I_\alpha : \mathbb{R}^2 \to \mathbb{R}$ is the Riesz potential, and F is the primitive function of nonlinearity f which has critical exponential growth. Using the Trudinger-Moser inequality and some delicate estimates, we show that the above problem admits at least one semiclassical ground state solution, for $\varepsilon > 0$ small provided that $V(x)$ is periodic in x or asymptotically linear as $|x| \to \infty$. In particular, a precise and fine lower bound of $\frac{f(t)}{e^{\beta_0 t^2}}$ near infinity is introduced in this paper.

Keywords Choquard equation, critical exponential growth, Trudinger-Moser inequality, ground state solution

MSC(2010) 35J20, 35J62, 35Q55.

1. Introduction

This paper is devoted to studying the following Choquard equation

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = \varepsilon^{-\alpha}\left[I_\alpha * F(u)\right] f(u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

[†]The corresponding author.
Email address: liminzhangmath@163.com (L. Zhang), liaofangfang1981@126.com (F. Liao), tangxh@mail.csu.edu.cn (X. Tang), qindd132@163.com (D. Qin)
[1]Department of Mathematics, Suzhou University of Science and Technology, Suzhou, Jiangsu 215009, China
[2]HNP-LAMA, School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China
[3]Department of Mathematics, Xiangnan University, Chenzhou, Hunan 423000, China
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where $\varepsilon > 0$ is a parameter, $\alpha \in (0,2)$ and $I_\alpha : \mathbb{R}^2 \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\pi\Gamma\left(\frac{\alpha}{2}\right)2^\alpha|x|^{2-\alpha}} := \frac{A_\alpha}{|x|^{2-\alpha}}, \quad \forall\, x \in \mathbb{R}^2\backslash\{0\},$$

$F(t) = \int_0^t f(s)\mathrm{d}s$, $V \in \mathcal{C}(\mathbb{R}^2,(0,\infty))$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:

(V0) $0 < \inf_{x\in\mathbb{R}^2} V(x) := V_0 \le V(x) \le \sup_{x\in\mathbb{R}^2} V(x) := V_\infty < \infty$;

(V1) $V(x)$ is 1-periodic in x_1, x_2;

(V2) $\inf_{x\in\mathbb{R}^2} V(x) := V_0 < V_\infty := \lim_{|x|\to\infty} V(x)$;

(F1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and there exists $\beta_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \text{for all } \beta > \beta_0$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \text{for all } \beta < \beta_0;$$

(F2) $|f(t)| = o(|t|^{\alpha/2})$ as $|t| \to 0$.

The majority of the literature focuses on the study of equation (1.1) in $\mathbb{R}^N (N \ge 3)$. Let us recall some of them as follows. The singularly perturbed elliptic equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{2-N-\alpha}\left[I_\alpha * G(x,u)\right]g(x,u)$$

appears in the theory of Bose-Einstein condensation, and is used to describe the finite-range many-body interactions between particles. Here, $G(x,u) = \int_0^u g(x,s)\mathrm{d}s$. For more related results, see, for example, [6, 7, 12, 14, 15, 17, 18, 22] and so on.

In particular, the above equation is the so-called Choquard equation, when $N = 3$. For ε =1, $\alpha = 1$, $V(x) \equiv 1$ and $g(x,u) = u$, the autonomous equation

$$-\Delta u + u = \left[I_1 * |u|^2\right]u \quad \text{ in } \mathbb{R}^3$$

arises from the quantum theory of a polaron by Pekar [27]. Choquard [20] applied it as an approximation to the Hartree-Fock theory of one-component plasma. In [24], Penrose proposed it as a model of self-gravitating matter. We also mention [38], where the fractional case is treated. Concerning other mathematical and physical background on Choquard problems, see [3,25,28,29,31,33,34] the references therein.

It is well-known that when $N \ge 3$ the Sobolev embedding yields $H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for all $s \in [2,2^*]$, where $2^* = \frac{2N}{N-2}$. Different from $N \ge 3$, the case $N = 2$ is very special. In such case, the Sobolev exponent 2^* becomes ∞, but $H^1(\mathbb{R}^2) \nsubseteq L^\infty(\mathbb{R}^2)$. Thanks to the Trudinger-Moser inequality below, it provides us a perfect replacement, which was first established by Cao in [8] (also seen in other works [4, 5] and reads as follows).

Proposition 1.1 (Cao [8])**.** i) *If* $\beta > 0$ *and* $u \in H^1(\mathbb{R}^2)$, *then*

$$\int_{\mathbb{R}^2}\left(e^{\beta u^2} - 1\right)\mathrm{d}x < \infty;$$

ii) *if* $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \le 1$, $\|u\|_2 \le M < \infty$, *and* $\beta < 4\pi$, *then there exists a constant* $\mathcal{C}(M,\beta)$, *which depends only on* M *and* β *such that*

$$\int_{\mathbb{R}^2}\left(e^{\beta u^2} - 1\right)\mathrm{d}x \le \mathcal{C}(M,\beta).$$

Involving the above Trudinger-Moser inequality, we refer the readers to previous works [9,10,13,19,26,31,37]. To state our main results, in addition to (F1) and (F2), we suppose that f verifies the global growth Ambrosetti-Rabinowitz superlinear condition, i.e.,

(F3) there exists $\bar{\mu} > 1$ such that $f(t)t \geq \bar{\mu}F(t) > 0$ for all $t \in \mathbb{R}\backslash\{0\}$.

We also introduce the following assumption.

(F4) there exist $M_0 > 0$ and $t_0 > 0$ such that

$$F(t) \leq M_0 \left|f(t)\right|, \quad \forall\ |t| \geq t_0,$$

which is satisfied for f behaving as $\exp\left(\beta_0 t^2\right)$ at infinity.

Performing the scaling $u(x) = v(\varepsilon x)$, we can easily deduce that problem (1.1) is equivalent to

$$-\Delta u + V(\varepsilon x)u = \left[I_\alpha * F(u)\right] f(u). \tag{1.2}$$

In view of Proposition 1.1 i), under assumptions (V0), (F1) and (F2), the weak solutions to equation (1.2) correspond to the critical points of the following energy functional defined in E_ε by

$$\Phi_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u|^2 + V(\varepsilon x)u^2\right] \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2}\left[I_\alpha * F(u)\right] F(u)\ \mathrm{d}x. \tag{1.3}$$

Define $c_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon}\Phi_\varepsilon(u)$, where

$$\mathcal{N}_\varepsilon := \{u \in E_\varepsilon \setminus \{0\} : \langle\Phi'_\varepsilon(u), u\rangle = 0\} \tag{1.4}$$

is the Nehari mainfold of $\Phi_\varepsilon(u)$, and E_ε is defined in Section 2.

Now, we give a review of some results related to our work. When 0 lies in a gap of the spectrum of the operator $-\Delta + V$ and (V1) holds, the authors [16,30] proved the existence of a nontrivial solution of equation (1.2) with $\varepsilon = 1$

$$\begin{cases} -\Delta u + V(x)u = \left(I_\alpha * F(u)\right) f(u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2). \end{cases} \tag{1.5}$$

Precisely, the nonlinearity $f(t)$ satisfies (F1)-(F4) and the following condition.

(F5′) $\liminf_{t\to\infty}\frac{f(t)}{e^{\beta_0 t^2}} = \kappa > \frac{\sqrt{\alpha(1+\alpha)}(2+\alpha)}{\sqrt{2\pi A_\alpha}\rho^{1+\alpha/2}}e^{4(2+\alpha)\pi(1+\rho)^2\mathcal{B}_0^2/(2+\rho)}$, where $\rho > 0$ satisfies $2(2+\alpha)\pi\rho^2\mathcal{B}_0^2 < 1$, and $\mathcal{B}_0 > 0$ is an embedding constant defined by [30, 4.17].

This together with (F4) can help overcome the difficulties caused by the fact that the embedding of the Sobolev space $H^1(\mathbb{R}^2)$ into the Orlicz space $L^\varphi(\mathbb{R}^2)$ determined by the function $\varphi(t) = \exp\left(4\pi t^2\right) - 1$ is not compact. It is worthwhile to mention that they gave a precise estimation of lower bound of $\liminf_{t\to\infty}\frac{f(t)}{e^{\beta_0 t^2}}$. In this work, we only use the special case of condition (F5′), namely,

(F5) $\liminf_{t\to\infty}\frac{f(t)}{e^{\beta_0 t^2}} = \kappa > \frac{\sqrt{\alpha(1+\alpha)}(2+\alpha)}{\sqrt{2\pi A_\alpha}\rho^{1+\alpha/2}}$, where $\rho = \frac{2}{\sqrt{(2+\alpha)V_\rho}} \geq \frac{2}{\sqrt{(2+\alpha)V_\infty}} > 0$, $V_\rho := \sup_{|x|\leq\rho} V(x)$.

Assumption (F5) plays a crucial role to estimate our threshold of the mountain pass minimax level $c_V < \frac{(2+\alpha)\pi}{2\beta_0}$, where c_V will be defined later (see Section 3).

Alves et al., [2] showed the existence and concentration of semiclassical ground state solutions of problem (1.1) under (F3) and the following assumptions on f :

- (f1) (i) $f(t) = 0$, $\forall\, t \le 0$ and $0 \le f(t) \le Ce^{4\pi t^2}$, $\forall\, t \ge 0$;
  (ii) $\exists\, t_1 > 0, M_1 > 0$ and $q \in (0,1]$ such that $0 < t^q F(t) \le M_1 f(t)$, $\forall\, |t| \ge t_1$;
- (f2) there exist $p > \alpha/2$ and $C_p > 0$ such that $f(t) \sim C_p t^p$, as $t \to 0$;
- (f5) $\lim_{t\to+\infty} \frac{tf(t)F(t)}{e^{8\pi t^2}} \ge l > \inf_{\rho>0} \frac{\alpha(1+\alpha)(2+\alpha)^2}{16\pi^2\rho^{2+\alpha}} e^{(2+\alpha)V_\rho \rho^2/4}$;
- (f6) $t \mapsto f(t)$ is strictly increasing on $(0,+\infty)$.

By using the mountain pass lemma, the authors showed the mountain pass level shall be less than $(2+\alpha)/8$, which can be derived from condition (f5). Condition (f1)-(ii) guarantees the weak limit of a Palais-Smale sequence is nonzero which lies at the heart of the proof (see [2, (2.20), (2.30)]). We also point out that there are two crucial points in their arguments. First, to show the weak limit of a Palais-Smale sequence is a solution they used Radon-Nicodym theorem by considering a sequence of measures which has uniformly bounded total variation. Second, (f2) imposes strict growth restriction on f near zero which guarantees that the corresponding energy functional associated with problem (1.5) possesses mountain pass geometry. Instead of the strict monotonicity condition (f6), we only assume a weak version of it. That is,

- (F6) $t \to f(t)$ is nondecreasing on $(-\infty, 0) \cup (0,+\infty)$.

In the present paper, we will further study the existence of nontrivial ground state solutions to problems (1.5) and (1.1). To the best of our knowledge, it seems that all the previous existence results concerning the ground state solution of Nehari type for equation (1.1) depend heavily on the monotonicity condition (f6) using the method introduced in [30], and we will weaken condition (f6). Due to the appearance of the convolution term and critical exponential growth, we need to overcome the following three difficulties:

1) giving a detailed estimate for the minimax level and showing that the mountain pass level shall be less than a threshold value under which one can restore the compactness for the critical case;

2) certifying that the Cerami sequence $\{u_n\}$ does not vanish. This requires some deep analysis in order to use the Trudinger-Moser inequality;

3) showing the weak limit $\bar{u}_\varepsilon$ of the sequence $\{u_n\}$ is a nontrivial solution for the energy functional of Φ_ε.

Now, our main results can be stated as follows.

Theorem 1.1. *Assume that V and f satisfy* (V0), (V1) *and* (F1)-(F6). *Then, problem* (1.5) *has at least one ground state solution $\tilde{u} \in E \setminus \{0\}$ such that*

$$\Phi_V(\tilde{u}) = \inf_{u\in\mathcal{N}_V} \Phi_V(u),$$

where $\mathcal{N}_V$ will be given in (2.6).

Theorem 1.2. *Assume that V and f satisfy* (V0), (V2) *and* (F1)-(F6). *Then, for any $\varepsilon > 0$ small, problem* (1.1) *has at least one ground state solution $u_\varepsilon \in \mathcal{N}_\varepsilon$ such* *that*

$$\Phi_\varepsilon(u_\varepsilon)=\inf_{u\in\mathcal{N}_\varepsilon}\Phi_\varepsilon(u),$$

where $\mathcal{N}_\varepsilon$ *is given in* (1.4).

Remark 1.1. Conditions (F1), (F2), (F4)-(F6) are much more general than (f1), (f2), (f5) and (f6). Thus, Theorems 1.1 and 1.2 improve and extend the [2, Theorems 1.3 and 1.4] and the related results in the literature.

The paper is organized as follows. In Section 2, we give the variational setting and some preliminaries. In Section 3, we establish the minima estimates of the energy functional associated with equation (1.5). Theorems 1.1 and 1.2 shall be proved in Section 4.

Throughout the paper, we make use of the following notations:

- $H^1(\mathbb{R}^2)$ denotes the Sobolev space with the norm $\|u\|=\left[\int_{\mathbb{R}^2}\left(|\nabla u|^2+|u|^2\right)\mathrm{d}x\right]^{1/2}$;
- $L^s(\mathbb{R}^2)(1\le s<\infty)$ denotes the Lebesgue space with the norm $\|u\|_s=\left(\int_{\mathbb{R}^2}|u|^s\ \mathrm{d}x\right)^{1/s}$;
- for any $x\in\mathbb{R}^2$ and $r>0$, $B_r(x):=\{y\in\mathbb{R}^2:|y-x|<r\}$ and $B_r=B_r(0)$;
- $C_1,C_2,\cdots$ denote positive constants possibly different in different places.

2. Variational framework and preliminaries

In this section, we first give some preliminary propositions.

Proposition 2.1 (Hardy-Littlewood-Sobolev inequality, [21])**.** *Let* $s,r>1$ *and* $0<\mu<2$ *with* $\frac{1}{s}+\frac{1}{r}=\frac{4-\mu}{2}$, $g\in L^s\left(\mathbb{R}^2\right)$ *and* $h\in L^r\left(\mathbb{R}^2\right)$. *Then there exists a sharp constant* $\mathcal{C}(\mu,s,r)$, *independent of* g,h *such that*

$$\int_{\mathbb{R}^2}\left(I_{2-\mu}*g\right)h\ \mathrm{d}x\le\mathcal{C}(\mu,s,r)\|g\|_s\|h\|_r.$$

In particular,

$$\int_{\mathbb{R}^2}\left(I_\alpha*g\right)h\ \mathrm{d}x\le\mathcal{C}_0\|g\|_{4/(2+\alpha)}\|h\|_{4/(2+\alpha)},$$

where $\mathcal{C}_0:=\mathcal{C}\left(2-\alpha,4/(2+\alpha),4/(2+\alpha)\right)$.

Proposition 2.2 (Cauchy-Schwarz type inequality, [1] and [23])**.** *For* $g,h\in L^1_{loc}\left(\mathbb{R}^2\right)$, *there holds*

$$\int_{\mathbb{R}^2}\left(I_\alpha*|g|\right)|h|\ \mathrm{d}x\le\left[\int_{\mathbb{R}^2}\left(I_\alpha*|g|\right)|g|\ \mathrm{d}x\int_{\mathbb{R}^2}\left(I_\alpha*|h|\right)|h|\ \mathrm{d}x\right]^{\frac{1}{2}}.$$

Let Ω be an open subset of $\mathbb{R}^2$. Just for the convenience of description, we denote

$$\Psi_\Omega(u)=\frac{1}{2}\int_\Omega\left[I_\alpha*F(u)\right]F(u)\ \mathrm{d}x,\quad\left\langle\Psi'_\Omega(u),v\right\rangle=\int_\Omega\left[I_\alpha*F(u)\right]f(u)v\ \mathrm{d}x$$

for any $u,v\in H^1(\mathbb{R}^2)$. In particular, when $\Omega=\mathbb{R}^2$, $\Psi_{\mathbb{R}^2}(u)$, and $\left\langle\Psi'_{\mathbb{R}^2}(u),v\right\rangle$ are simply written by $\Psi(u)$ and $\langle\Psi'(u),v\rangle$ respectively.

Fixing $\beta > \beta_0$, for any $\epsilon > 0$ and $q > 0$, it follows from (F1) and (F2) that there exists $C = C(\epsilon, \beta, q) > 0$ such that

$$|f(t)| \leq \epsilon |t|^{\alpha/2} + C|t|^{q-1}\left(e^{\beta t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{2.1}$$

Consequently,

$$|F(t)| \leq \epsilon |t|^{(2+\alpha)/2} + C|t|^{q}\left(e^{\beta t^2} - 1\right), \quad \forall\, t \in \mathbb{R}. \tag{2.2}$$

By virtue of (2.2), Propositions 1.1 and 2.1, we can demonstrate that the energy functional

$$\begin{aligned}\Phi_V(u) &= \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2 + V(x)|u|^2\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2}\left[I_\alpha * F(u)\right]F(u)\,\mathrm{d}x \\ &:= \frac{1}{2}\|u\|_V^2 - \Psi(u)\end{aligned} \tag{2.3}$$

associated with equation (1.5) is of class $\mathcal{C}^1(H^1\left(\mathbb{R}^2\right), \mathbb{R})$, and

$$\begin{aligned}\langle \Phi_V'(u), v\rangle &= \int_{\mathbb{R}^2}\left(\nabla u \nabla v + V(x)uv\right)\mathrm{d}x - \int_{\mathbb{R}^2}\left[I_\alpha * F(u)\right]f(u)v\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2}\left(\nabla u \nabla v + V(x)uv\right)\mathrm{d}x - \langle \Psi'(u), v\rangle\end{aligned} \tag{2.4}$$

for all $u, v \in H^1(\mathbb{R}^2)$.

Let E denote the space $H^1\left(\mathbb{R}^2\right)$ equipped with the norm $\|\cdot\|_V$, which is equivalent to the standard Sobolev norm. In addition, for $s \in [2, \infty)$, there exists $\tau_s > 0$ such that

$$\|u\|_s \leq \tau_s \|u\|_V, \quad \forall\, u \in E. \tag{2.5}$$

Define

$$\mathcal{N}_V := \left\{u \in E \setminus \{0\} : \langle \Phi_V'(u), u\rangle = 0\right\}, \tag{2.6}$$

which is the Nehari mainfold of $\Phi_V(u)$.

Under assumption (V0), for $\varepsilon > 0$, the set

$$E_\varepsilon := \left\{u \in E : \int_{\mathbb{R}^2} V(\varepsilon x)|u|^2\,\mathrm{d}x < \infty\right\} \tag{2.7}$$

is a Hilbert space endowed with the norm

$$\|u\|_\varepsilon := \left[\int_{\mathbb{R}^2}\left(|\nabla u|^2 + V(\varepsilon x)|u|^2\right)\mathrm{d}x\right]^{1/2},$$

which is equivalent to the standard Sobolev norm. Note that conditions (V0), and (V2) were introduced by Rabinowitz in [32]. Hereafter, we will denote by

$$M := \left\{x \in \mathbb{R}^2 : V(x) \equiv V_0\right\} \tag{2.8}$$

the minimum points set of $V(x)$.

By Sobolev embedding theorem, for $s \in [2, \infty)$, there exists $\gamma_s > 0$ such that

$$\|u\|_s \leq \gamma_s \|u\|_\varepsilon, \quad \forall\, u \in E_\varepsilon. \tag{2.9}$$

Particularly, $\gamma_2 \leq 1/\sqrt{V_0}$. By (V0), (2.2) and Proposition 1.1, we know that the functional Φ_ε defined by (1.3) is well defined on E_ε. Moreover, by standard arguments, $\Phi_\varepsilon \in \mathcal{C}^1(E_\varepsilon, \mathbb{R})$ with

$$\langle \Phi'_\varepsilon(u), v\rangle = \int_{\mathbb{R}^2} [\nabla u \nabla v + V(\varepsilon x)uv]\,\mathrm{d}x - \langle \Psi'(u), v\rangle, \qquad \forall\, u, v \in E_\varepsilon. \tag{2.10}$$

Hence, the solutions of equation (1.2) are the critical points of function (2.10).

In order to prove our results, we give the following preliminary lemmas.

Lemma 2.1. *Assume that* (V0), (F1) *and* (F2) *hold. Then, there exists* $\varrho > 0$, *independent of* ε *such that*

$$\|u\|_\varepsilon \geq \varrho, \quad \forall\, u \in \mathcal{N}_\varepsilon.$$

Proof. By (V0), there exists $\vartheta_0 = 1 > 0$ such that

$$\|\nabla u\|_2 \leq \|u\|_\varepsilon, \qquad \forall\, u \in E_\varepsilon. \tag{2.11}$$

Let $u \in \mathcal{N}_\varepsilon$. It follows from (2.1) and (2.2) that there exist constants $\beta > \beta_0$ and $C_1 > 0$ such that

$$|f(t)| \leq \left(\frac{|t|}{\sqrt{2}\gamma_2}\right)^{\alpha/2} + C_1|t|\left(e^{\beta t^2} - 1\right), \qquad \forall\, t \in \mathbb{R} \tag{2.12}$$

and

$$|F(t)| \leq \left(\frac{|t|}{\sqrt{2}\gamma_2}\right)^{(2+\alpha)/2} + C_1|t|^2\left(e^{\beta t^2} - 1\right), \qquad \forall\, t \in \mathbb{R}. \tag{2.13}$$

In view of Proposition 1.1 ii), one has

$$\begin{aligned}\int_{\mathbb{R}^2}\left(e^{4\beta u^2/\alpha} - 1\right)\mathrm{d}x &= \int_{\mathbb{R}^2}\left(e^{4\beta^2\|u\|_\varepsilon^2(u/\|u\|_\varepsilon)^2/\alpha} - 1\right)\mathrm{d}x \\ &\leq \mathcal{C}\left(\gamma_2, 2\pi\right), \qquad \forall\, \|u\|_\varepsilon \leq \sqrt{\alpha\pi/2\beta}.\end{aligned} \tag{2.14}$$

Similarly,

$$\int_{\mathbb{R}^2}\left(e^{8\beta u^2/\alpha} - 1\right)\mathrm{d}x \leq \mathcal{C}\left(\gamma_2, 2\pi\right), \qquad \forall\, \|u\|_\varepsilon \leq \sqrt{\alpha\pi/4\beta}. \tag{2.15}$$

Then, from (2.12)-(2.15) and the Hölder inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}|F(u)|^{4/(2+\alpha)}\,\mathrm{d}x \\ \leq\ &\int_{\mathbb{R}^2}\left[\left(\frac{|u|}{\sqrt{2}\gamma_2}\right)^{(2+\alpha)/2} + C_1\left(e^{\beta u^2} - 1\right)|u|^2\right]^{4/(2+\alpha)}\mathrm{d}x \\ \leq\ &2\int_{\mathbb{R}^2}\left[\frac{1}{2\gamma_2^2}|u|^2 + C_1^{4/(2+\alpha)}\left(e^{\beta u^2} - 1\right)^{4/(2+\alpha)}|u|^{8/(2+\alpha)}\right]\mathrm{d}x \\ \leq\ &2\left\{\frac{1}{2\gamma_2^2}\|u\|_2^2 + C_1^{4/(2+\alpha)}\left[\int_{\mathbb{R}^2}\left(e^{\beta u^2} - 1\right)^{4/\alpha}\mathrm{d}x\right]^{\alpha/(2+\alpha)}\|u\|_4^{8/(2+\alpha)}\right\}\end{aligned} \tag{2.16}$$

$$
\begin{aligned}
&\leq \frac{1}{\gamma_2^2}\|u\|_2^2+2C_1^{4/(2+\alpha)}\left[\int_{\mathbb{R}^2}\left(e^{4\beta u^2/\alpha}-1\right)\mathrm{d}x\right]^{\alpha/2+\alpha}\|u\|_4^{8/(2+\alpha)}\\
&\leq \|u\|_\varepsilon^2+2\left(\gamma_4^8C_1^4\mathcal{C}^\alpha\left(\gamma_2,2\pi\right)\right)^{1/(2+\alpha)}\|u\|_\varepsilon^{8/(2+\alpha)}\\
&:=\|u\|_\varepsilon^2+C_2\|u\|_\varepsilon^{8/(2+\alpha)},\qquad\forall\,\|u\|_\varepsilon\leq\sqrt{\alpha\pi/2\beta},
\end{aligned}
$$

and for all $\|u\|_\varepsilon\leq\sqrt{\alpha\pi/4\beta}$, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^2}|f(u)|^{4/\alpha}\mathrm{d}x\\
&\leq\int_{\mathbb{R}^2}\left[\left(\frac{|u|}{\sqrt{2}\gamma_2}\right)^{\alpha/2}+C_1\left(e^{\beta u^2}-1\right)|u|\right]^{4/\alpha}\mathrm{d}x\\
&\leq 2^{4/\alpha}\int_{\mathbb{R}^2}\left[\frac{1}{2\gamma_2^2}|u|^2+C_1^{4/\alpha}\left(e^{\beta u^2}-1\right)^{4/\alpha}|u|^{4/\alpha}\right]\mathrm{d}x\\
&\leq 2^{4/\alpha}\left\{\frac{1}{2\gamma_2^2}\|u\|_2^2+C_1^{4/\alpha}\left[\int_{\mathbb{R}^2}\left(e^{\beta u^2}-1\right)^{8/\alpha}\mathrm{d}x\right]^{1/2}\|u\|_{8/\alpha}^{4/\alpha}\right\}\\
&\leq\frac{2^{4/\alpha-1}}{\gamma_2^2}\|u\|_2^2+(2C_1)^{4/\alpha}\left[\int_{\mathbb{R}^2}\left(e^{8\beta u^2/\alpha}-1\right)\mathrm{d}x\right]^{1/2}\|u\|_{8/\alpha}^{4/\alpha}\\
&\leq 2^{4/\alpha-1}\|u\|_\varepsilon^2+\left(2\gamma_{8/\alpha}C_1\right)^{4/\alpha}\mathcal{C}^{1/2}\left(\gamma_2,2\pi\right)\|u\|_\varepsilon^{4/\alpha}.
\end{aligned}
\tag{2.17}
$$

Hence, combining with (1.4), (2.16), (2.17), $u\in\mathcal{N}_\varepsilon$, Proposition 2.1 and the Hölder inequality, we have

$$
\begin{aligned}
\|u\|_\varepsilon^2&=\int_{\mathbb{R}^2}\left[I_\alpha*F(u)\right]f(u)u\;\mathrm{d}x\\
&\leq\mathcal{C}_0\|F(u)\|_{4/(2+\alpha)}\|f(u)u\|_{4/(2+\alpha)}\\
&\leq\mathcal{C}_0\|F(u)\|_{4/(2+\alpha)}\|f(u)\|_{4/\alpha}\|u\|_2\\
&\leq\gamma_2\mathcal{C}_0\left[\|u\|_\varepsilon^2+C_2\|u\|_\varepsilon^{8/(2+\alpha)}\right]^{(2+\alpha)/4}\|u\|_\varepsilon\\
&\quad\times\left[2^{4/\alpha-1}\|u\|_\varepsilon^2+\left(2\gamma_{8/\alpha}C_1\right)^{4/\alpha}\mathcal{C}^{1/2}\left(\gamma_2,2\pi\right)\|u\|_\varepsilon^{4/\alpha}\right]^{\alpha/4}\\
&\leq 4\gamma_2\mathcal{C}_0\|u\|_\varepsilon^{2+\alpha}+C_3\|u\|_\varepsilon^4.
\end{aligned}
$$

Therefore, there exists $0<\varrho<\sqrt{\alpha\pi/4\beta}$ such that the conclusion holds. □

Lemma 2.2. *Assume that* (V0), (F1), (F2) *and* (F3) *hold. Then, there exist a sequence* $\{u_n\}\subset E_\varepsilon$ *and* $\kappa_0>0$ *independent of* ε *such that*

$$
\Phi_\varepsilon(u_n)\to c_\varepsilon\geq\kappa_0,\qquad\|\Phi_\varepsilon'(u_n)\|\left(1+\|u_n\|_\varepsilon\right)\to 0,
\tag{2.18}
$$

as $n\to+\infty$, *where* c_ε *is given by*

$$
c_\varepsilon=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\varepsilon\left(\gamma(t)\right)
$$

with

$$
\Gamma=\left\{\gamma\in\mathcal{C}\left([0,1],E_\varepsilon\right):\gamma(0)=0,\Phi_\varepsilon(\gamma(1))<0\right\}.
$$

Proof. By using the argument which is used in the proof of Lemma 2.1, one can easily obtain that there exists $\kappa_0, \rho > 0$ independent of ε such that

$$\Phi_\varepsilon(u) \geq \kappa_0, \quad \forall\, u \in S := \{u \in E_\varepsilon : \|u\|_\varepsilon = \rho\}. \tag{2.19}$$

On the other hand, for any fixed $u_0 \in E_\varepsilon \backslash \{0\}$, we set

$$\zeta(t) = \Psi\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right), \quad \forall\, t \in \mathbb{R}\backslash\{0\}.$$

Obviously, $\zeta(t) > 0$, by (F3) and

$$\frac{\zeta'(t)}{\zeta(t)} \geq \frac{2\bar{\mu}}{t},$$

which implies that $\frac{\zeta(t)}{|t|^{2\bar{\mu}}}$ is increasing on $(-\infty, 0) \cup (0, +\infty)$. Consequently, choosing $t > 1/\|u_0\|_\varepsilon$, we have

$$\Psi(tu_0) \geq (t\|u_0\|_\varepsilon)^{2\bar{\mu}}\, \Psi\left(\frac{u_0}{\|u_0\|_\varepsilon}\right). \tag{2.20}$$

Hence, by (1.3) and (2.20), for any $t > 1/\|u_0\|_\varepsilon$, one has

$$\begin{aligned}\Phi_\varepsilon(tu_0) &\leq \frac{1}{2}\|tu_0\|_\varepsilon^2 - (t\|u_0\|_\varepsilon)^{2\bar{\mu}}\, \Psi\left(\frac{u_0}{\|u_0\|_\varepsilon}\right) \\ &= \frac{t^2}{2}\left[\|u_0\|_\varepsilon^2 - 2t^{2\bar{\mu}-2}\|u_0\|_\varepsilon^{2\bar{\mu}}\Psi\left(\frac{u_0}{\|u_0\|_\varepsilon}\right)\right].\end{aligned}$$

Then, $\bar{\mu} > 1$, and we assert $\lim_{t\to\infty} \Phi_\varepsilon(tu_0) = -\infty$. Thus, combining with (2.19), we can choose $T > 0$ independent of ε such that $e = Tu_0 \in \{u \in E_\varepsilon : \|u\|_\varepsilon > \rho\}$ and $\Phi_\varepsilon(e) < 0$. Then, in view of the mountain pass lemma [35], we deduce that there exist $c_\varepsilon \in \left[\kappa_0,\ \sup_{t\geq 0} \Phi_\varepsilon(tu_0)\right]$ and a sequence $\{u_n\} \subset E_\varepsilon$ satisfying (2.18). □

Lemma 2.3. *Assume that* (V0), (F1), (F2), (F3) *and* (F6) *hold. Then,*

$$\Phi_\varepsilon(u) \geq \Phi_\varepsilon(tu) + \frac{1-t^2}{2}\left\langle \Phi_\varepsilon'(u), u\right\rangle, \quad \forall\, u \in E_\varepsilon,\ t \geq 0, \tag{2.21}$$

$$\Phi_\varepsilon(u) > \Phi_\varepsilon(tu) + \frac{1-t^2}{2}\left\langle \Phi_\varepsilon'(u), u\right\rangle, \quad \forall\, u \in E_\varepsilon \setminus \{0\},\ t \geq 0 \text{ and } t \not\equiv 1. \tag{2.22}$$

Proof. It is easy to see that (F3) implies

$$\frac{F(t)}{t} \text{ strictly increasing on } (-\infty, 0) \cup (0, +\infty). \tag{2.23}$$

For every $t \geq 0$, let

$$\xi(t) := \Psi(tu) - \Psi(u) + \frac{1-t^2}{2}\left\langle \Psi'(u), u\right\rangle.$$

Then, it follows from (F6) and (2.23) that

$$\xi'(t) = \langle \Psi'(tu), u\rangle - \langle \Psi'(u), tu\rangle$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} [I_\alpha * F(tu)] f(tu)u \, \mathrm{d}x - \int_{\mathbb{R}^2} [I_\alpha * F(u)] f(u)tu \, \mathrm{d}x \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} I_\alpha(x-y) \left[\frac{F(tu(y))}{tu(y)} f(tu(x)) - \frac{F(u(y))}{u(y)} f(u(x)) \right] tu(x)u(y) \, \mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} I_\alpha(x-y) \left\{ \frac{F(tu(y))}{tu(y)} [f(tu(x)) - f(u(x))] \right. \\
&\quad \left. + \left[\frac{F(tu(y))}{tu(y)} - \frac{F(u(y))}{u(y)} \right] f(u(x)) \right\} tu(x)u(y) \, \mathrm{d}x\mathrm{d}y \\
&\quad \begin{cases} \geq 0, \, t \geq 1; \\ \leq 0, \, 0 < t < 1, \end{cases}
\end{aligned} \tag{2.24}
$$

which derives $\xi(t) \geq \xi(1) = 0$ immediately. Moreover, the above inequality is strict, if $u \in E_\varepsilon \setminus \{0\}$ and $t \equiv 1$. Hence, from (1.3) and (2.10), one has

$$
\begin{aligned}
\Phi_\varepsilon(u) - \Phi_\varepsilon(tu) &= \frac{1-t^2}{2} \|u\|_\varepsilon^2 - \Psi(u) + \Psi(tu) \\
&= \frac{1-t^2}{2} \langle \Phi_\varepsilon'(u), u \rangle - \Psi(u) + \Psi(tu) + \frac{1-t^2}{2} \langle \Psi'(u), u \rangle \\
&\geq \frac{1-t^2}{2} \langle \Phi_\varepsilon'(u), u \rangle, \qquad \forall \, u \in E_\varepsilon, \; t \geq 0.
\end{aligned}
$$

This shows that (2.21) holds. □

From Lemma 2.3, we have the following corollary at once.

Corollary 2.1. *Assume that* (V0), (F1), (F2), (F3) *and* (F6) *hold. Then,*

$$
\Phi_\varepsilon(u) \geq \max_{t \geq 0} \Phi_\varepsilon(tu), \qquad \forall \, u \in \mathcal{N}_\varepsilon,
$$

and the above inequality is strict, if $t \not\equiv 1$.

Lemma 2.4. *Assume that* (V0), (F1), (F2) *and* (F6) *hold. Then, for any $u \in E_\varepsilon \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$.*

Proof. Let $u \in E_\varepsilon \setminus \{0\}$ be fixed and define a function $\varsigma(t) := \Phi_\varepsilon(tu)$ on $[0, \infty)$. Clearly, by (1.3), we have for $t \neq 0$

$$
\begin{aligned}
\varsigma'(t) = 0 \;&\Leftrightarrow\; t\|u\|_\varepsilon^2 - \langle \Psi_\varepsilon'(tu), u \rangle = 0 \\
&\Leftrightarrow\; t^2\|u\|_\varepsilon^2 - \langle \Psi_\varepsilon'(tu), tu \rangle = 0 \\
&\Leftrightarrow\; \langle \Phi_\varepsilon'(tu), tu \rangle = 0 \\
&\Leftrightarrow\; tu \in \mathcal{N}_\varepsilon.
\end{aligned}
$$

By (F1), (F2) and (2.19), one has $\varsigma(0) = 0$ and $\varsigma(t) > 0$ for $t > 0$ small and $\varsigma(t) < 0$ for t large. Therefore, $\max_{t \in (0,\infty)} \varsigma(t)$ is achieved at some $t_u > 0$, so that $\varsigma'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. By Corollary 2.1, t_u is unique. □

From (2.19), Corollary 2.1 and Lemma 2.4, we can similarly prove the following lemma as in [36, Lemma 2.6].

Lemma 2.5. *Assume that* (V0), (F1), (F2), (F3) *and* (F6) *hold. Then,*

$$
c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \Phi_\varepsilon(u) = \inf_{u \in E_\varepsilon \setminus \{0\}} \max_{t \geq 0} \Phi_\varepsilon(tu) \geq \kappa_0.
$$

Lemma 2.6. *Assume that* (V0), (F1), (F2), (F3) *and* (F6) *hold. Then, any sequence* $\{u_n\}$ *satisfying* (2.18) *is bounded in* E_ε.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|_\varepsilon \to \infty$ as $n \to \infty$. From (F3), (1.3), (2.10) and (2.18), one has

$$\begin{aligned} c_\varepsilon + o(1) &= \Phi_\varepsilon(u_n) - \frac{1}{2}\left\langle \Phi'_\varepsilon(u_n), u_n \right\rangle \\ &= \frac{1}{2}\left\langle \Psi'(u_n), u_n \right\rangle - \Psi(u_n) \\ &\geq \frac{\bar{\mu}-1}{2\bar{\mu}}\left\langle \Psi'(u_n), u_n \right\rangle . \end{aligned} \tag{2.25}$$

Then, it follows from (2.18) and (2.25) that

$$1 = \frac{1}{\|u_n\|_\varepsilon^2}\left\langle \Psi'(u_n), u_n \right\rangle + o(1) = o(1).$$

This contradiction shows that $\{u_n\}$ is bounded in E_ε. □

3. Minimax estimates

In this section, we will estimate the minimax level of the energy functional defined by (2.3). Let

$$\rho := \frac{2}{\sqrt{(2+\alpha)V_\rho}}. \tag{3.1}$$

Then, (F5) implies that

$$\kappa^2 > \frac{\alpha(1+\alpha)(2+\alpha)^2}{2\pi e A_\alpha \rho^{2+\alpha}} e^{(2+\alpha)V_\rho \rho^2/4}. \tag{3.2}$$

As in [11], we define Moser type functions $w_n(x)$ supported in B_ρ as follows:

$$w_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & 0 \leq |x| \leq \rho/n; \\ \frac{\log(\rho/|x|)}{\sqrt{\log n}}, & \rho/n \leq |x| \leq \rho; \\ 0, & |x| \geq \rho. \end{cases} \tag{3.3}$$

By an elemental computation, we have

$$\|\nabla w_n\|_2^2 = \int_{\mathbb{R}^2} |\nabla w_n|^2 \,\mathrm{d}x = 1 \tag{3.4}$$

and

$$\int_{\mathbb{R}^2} V(x) w_n^2 \,\mathrm{d}x \leq V_\rho \int_{B_\rho} w_n^2 \,\mathrm{d}x = V_\rho \rho^2 \delta_n, \tag{3.5}$$

where

$$V_\rho := \sup_{|x|\leq\rho} V(x), \quad \delta_n := \frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2} > 0. \tag{3.6}$$

In view of the proof of [2, (2.11)], one has

$$\int_{B_{\rho/n}} \mathrm{d}x \int_{B_{\rho/n}} \frac{\mathrm{d}y}{|x-y|^{2-\alpha}} \ \geq\ \frac{4\pi^2}{\alpha(1+\alpha)(2+\alpha)} \left(\frac{\rho}{n}\right)^{2+\alpha}. \tag{3.7}$$

Applying Lemma 2.2 to the functional Φ_V, there exists a sequence $\{u_n\} \subset E_\varepsilon$ such that

$$\|\Phi'_V(u_n)\|(1+\|u_n\|_V) \to 0, \quad \Phi_V(u_n) \to c_V, \tag{3.8}$$

where c_V can be characterized by

$$0 < c_V := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi_V(\gamma(t)) \tag{3.9}$$

with

$$\Gamma := \left\{\gamma \in \mathcal{C}^1([0,1],E) : \gamma(0)=0, \Phi_V(\gamma(1))<0\right\}.$$

We shall control the minimax-levels c_V by a fine threshold, which can help restore the compactness.

Lemma 3.1. *Assume that* (V0), (F1), (F2), (F3) *and* (F5) *hold. Then, there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$\max_{t\geq 0} \Phi_V(tw_{\bar{n}}) < \frac{(2+\alpha)\pi}{2\beta_0}. \tag{3.10}$$

Proof. By (3.2), we can choose $\epsilon > 0$ such that

$$\frac{(2+\alpha)V_\rho\rho^2}{4} + \log\frac{(1+\epsilon)\alpha(1+\alpha)(2+\alpha)^2}{2\pi A_\alpha(\kappa-\epsilon)^2\rho^{2+\alpha}} < 1. \tag{3.11}$$

Note that

$$\liminf_{t\to\infty}\frac{tF(t)}{e^{\beta_0 t^2}} \geq \liminf_{t\to\infty}\frac{\int_0^t sf(s)\,\mathrm{d}s}{e^{\beta_0 t^2}} = \liminf_{t\to\infty}\frac{f(t)}{2\beta_0 e^{\beta_0 t^2}}. \tag{3.12}$$

It follows from (F5) and (3.12) that there exists $t_\epsilon > 0$ such that

$$f(t) \geq (\kappa-\epsilon)e^{\beta_0 t^2}, \quad tF(t) \geq \frac{\kappa-\epsilon}{2\beta_0}e^{\beta_0 t^2}, \quad \forall\, t \geq t_\epsilon. \tag{3.13}$$

From (2.3), (3.4) and (3.5), one has

$$\begin{aligned}\Phi_V(tw_n) &= \frac{t^2}{2}\|w_n\|_V^2 - \Psi(tw_n) \\ &\leq \frac{t^2}{2}\left(1+V_\rho\rho^2\delta_n\right) - \Psi(tw_n), \quad \forall\, t\geq 0.\end{aligned} \tag{3.14}$$

There are four possible cases as follows. From now on, in the sequel, all inequalities hold for large $n \in \mathbb{N}$.

Case (i). $t \in \left[0, \sqrt{\frac{(2+\alpha)\pi}{2\beta_0}}\right]$. Then, it follows from (F3) and (3.14) that

$$\begin{aligned}\Phi_V(tw_n) &\leq \frac{t^2}{2}\left(1+V_\rho\rho^2\delta_n\right) - \Psi(tw_n) \\ &\leq \frac{(2+\alpha)\pi}{4\beta_0}\left(1+V_\rho\rho^2\delta_n\right).\end{aligned}$$

Clearly, there exists $\bar{n}\in\mathbb{N}$ such that (3.10) holds.

Case (ii). $t \in \left[\sqrt{\frac{(2+\alpha)\pi}{2\beta_0}}, \sqrt{\frac{(2+\alpha)\pi}{\beta_0}}\right]$. Let $M_n = \frac{1}{\sqrt{2\pi}}\sqrt{\log n}$. Then $tw_n(x) \geq t_\epsilon$ for $x \in B_{\rho/n}$ and for large $n\in\mathbb{N}$, it follows from (F3), (3.3) and (3.13), and we have

$$\Psi(tw_n)$$

$$
\begin{aligned}
&\geq \Psi_{B_{\rho/n}}(tw_n) \\
&= \frac{A_\alpha}{2}\int_{B_{\rho/n}} F(tw_n(x)) \left[\int_{B_{\rho/n}} \frac{1}{|x-y|^{2-\alpha}} F(tw_n(y))\, \mathrm{d}y\right] \mathrm{d}x \\
&\geq \frac{A_\alpha}{2}[F(tM_n)]^2 \int_{B_{\rho/n}} \mathrm{d}x \int_{B_{\rho/n}} \frac{\mathrm{d}y}{|x-y|^{2-\alpha}} \\
&\geq \frac{2\pi^2 A_\alpha \rho^{2+\alpha}}{\alpha(1+\alpha)(2+\alpha)n^{2+\alpha}}[F(tM_n)]^2 \\
&\geq \frac{\pi^2 A_\alpha (\kappa-\varepsilon)^2 \rho^{2+\alpha}}{2\alpha(1+\alpha)(2+\alpha)\beta_0^2 n^{2+\alpha}(tM_n)^2} e^{2\beta_0 (tM_n)^2} \\
&\geq \frac{\pi^2 A_\alpha (\kappa-\varepsilon)^2 \rho^{2+\alpha}}{\alpha(1+\alpha)(2+\alpha)^2 \beta_0 n^{2+\alpha}\log n} e^{\pi^{-1}\beta_0 t^2 \log n},
\end{aligned} \tag{3.15}
$$

which together with (3.14) that

$$
\begin{aligned}
&\Phi_V(tw_n) \\
\leq\ & \frac{t^2}{2}\left(1+V_\rho \rho^2 \delta_n\right) - \frac{\pi^2 A_\alpha (\kappa-\varepsilon)^2 \rho^{2+\alpha}}{\alpha(1+\alpha)(2+\alpha)^2 \beta_0 n^{2+\alpha}\log n} e^{\pi^{-1}\beta_0 t^2 \log n} \\
:=\ & \widehat{\varphi}_n(t).
\end{aligned} \tag{3.16}
$$

Let $\hat{t}_n > 0$ such that $\widehat{\varphi}_n'(\hat{t}_n) = 0$. Then,

$$
1 + V_\rho \rho^2 \delta_n = \frac{2\hat{A}_{\alpha,\rho}\beta_0}{\pi n^{2+\alpha}} e^{\pi^{-1}\beta_0 \hat{t}_n^2 \log n}, \tag{3.17}
$$

where $\hat{A}_{\alpha,\rho} = \pi^2 A_\alpha (\kappa-\epsilon)^2 \rho^{2+\alpha}/[\alpha(1+\alpha)(2+\alpha)^2\beta_0]$. It follows from (3.17) that

$$
\lim_{n\to\infty} \hat{t}_n^2 = \frac{(2+\alpha)\pi}{\beta_0}. \tag{3.18}
$$

Let

$$
D_1 := \log \frac{(1+\epsilon)\pi}{2\hat{A}_{\alpha,\rho}\beta_0}. \tag{3.19}
$$

Then, (3.11) and (3.19) imply that

$$
\frac{(2+\alpha)V_\rho \rho^2}{4} + D_1 - 1 < 0. \tag{3.20}
$$

From (3.17), (3.18) and (3.19), one has

$$
\begin{aligned}
\hat{t}_n^2 &= \frac{(2+\alpha)\pi}{\beta_0}\left[1 + \frac{\log\left(1+V_\rho\rho^2\delta_n\right)\pi - \log(2\hat{A}_{\alpha,\rho}\beta_0)}{(2+\alpha)\log n}\right] \\
&\leq \frac{(2+\alpha)\pi}{\beta_0}\left[1 + \frac{D_1}{(2+\alpha)\log n}\right] + O\left(\frac{1}{\log^2 n}\right)
\end{aligned} \tag{3.21}
$$

and

$$
\widehat{\varphi}_n(t) \leq \widehat{\varphi}_n(\hat{t}_n) = \frac{1}{2}\left(1+V_\rho\rho^2\delta_n\right)\left(\hat{t}_n^2 - \frac{\pi}{\beta_0 \log n}\right), \qquad \forall\ t \geq 0. \tag{3.22}
$$

In view of (3.18), (3.21) and (3.22), we get

$$\begin{aligned}
\widehat{\varphi}_n(t) &\leq \frac{1}{2}\left(1+V_\rho\rho^2\delta_n\right)\left(\hat{t}_n^2-\frac{\pi}{\beta_0\log n}\right)\\
&\leq \frac{(2+\alpha)\pi}{2\beta_0}\left(1+V_\rho\rho^2\delta_n\right)\left[1+\frac{D_1-1}{(2+\alpha)\log n}+O\left(\frac{1}{\log^2 n}\right)\right] \\
&\leq \frac{(2+\alpha)\pi}{2\beta_0}+\frac{\pi}{2\beta_0\log n}\left[\frac{(2+\alpha)V_\rho\rho^2}{4}+D_1-1\right]+O\left(\frac{1}{\log^2 n}\right).
\end{aligned} \tag{3.23}$$

Hence, combining (3.16) with (3.23), one has

$$\begin{aligned}
\Phi_V(tw_n) \leq{}& \frac{(2+\alpha)\pi}{2\beta_0}+\frac{\pi}{2\beta_0\log n}\left[\frac{(2+\alpha)V_\rho\rho^2}{4}+D_1-1\right]\\
&+O\left(\frac{1}{\log^2 n}\right).
\end{aligned} \tag{3.24}$$

Clearly, in this case, (3.20) and (3.24) show that there exists $\bar{n}\in\mathbb{N}$ such that (3.10) holds.

Case (iii). $t\in\left[\sqrt{\frac{(2+\alpha)\pi}{\beta_0}},\sqrt{\frac{(2+\alpha)\pi}{\beta_0}(1+\epsilon)}\right]$. Then, $tw_n(x)\geq t_\epsilon$ for $x\in B_{\rho/n}$ and for large $n\in\mathbb{N}$, it follows (3.15), and then one has

$$\begin{aligned}
&\Psi(tw_n)\\
\geq{}& \frac{\pi^2A_\alpha(\kappa-\varepsilon)^2\rho^{2+\alpha}}{2\alpha(1+\alpha)(2+\alpha)\beta_0^2n^{2+\alpha}(tM_n)^2}e^{2\beta_0(tM_n)^2}\\
\geq{}& \frac{\pi^2A_\alpha(\kappa-\varepsilon)^2\rho^{2+\alpha}}{(1+\epsilon)\alpha(1+\alpha)(2+\alpha)^2\beta_0n^{2+\alpha}\log n}e^{\pi^{-1}\beta_0t^2\log n}.
\end{aligned} \tag{3.25}$$

Using (F3), (3.14) and (3.25), we have

$$\begin{aligned}
&\Phi_V(tw_n)\\
\leq{}& \frac{t^2}{2}\left(1+V_\rho\rho^2\delta_n\right)-\frac{\tilde{A}_{\alpha,\rho}}{n^{2+\alpha}\log n}e^{\pi^{-1}\beta_0t^2\log n}\\
:={}& \widetilde{\varphi}_n(t),
\end{aligned} \tag{3.26}$$

where

$$\tilde{A}_{\alpha,\rho}=\frac{\pi^2A_\alpha(\kappa-\epsilon)^2\rho^{2+\alpha}}{(1+\epsilon)\alpha(1+\alpha)(2+\alpha)^2\beta_0}.$$

Let $\tilde{t}_n>0$ such that $\widetilde{\varphi}_n'(\tilde{t}_n)=0$. Then,

$$1+V_\rho\rho^2\delta_n=\frac{2\tilde{A}_{\alpha,\rho}\beta_0}{\pi n^{2+\alpha}}e^{\pi^{-1}\beta_0\tilde{t}_n^2\log n}, \tag{3.27}$$

which yields

$$\lim_{n\to\infty}\tilde{t}_n^2=\frac{(2+\alpha)\pi}{\beta_0}. \tag{3.28}$$

Let

$$D_2:=\log\frac{\pi}{2\tilde{A}_{\alpha,\rho}\beta_0}. \tag{3.29}$$

Then, (3.11) and (3.29) imply that

$$\frac{(2+\alpha)V_\rho\rho^2}{4}+D_2-1<0. \tag{3.30}$$

It follows from (3.27), (3.28) and (3.29) that

$$\begin{aligned}\tilde{t}_n^2&=\frac{(2+\alpha)\pi}{\beta_0}\left[1+\frac{\log\left(1+V_\rho\rho^2\delta_n\right)\pi-\log(2\tilde{A}_{\alpha,\rho}\beta_0)}{(2+\alpha)\log n}\right]\\&\le\frac{(2+\alpha)\pi}{\beta_0}\left[1+\frac{D_2}{(2+\alpha)\log n}\right]+O\left(\frac{1}{\log^2 n}\right)\end{aligned} \tag{3.31}$$

and

$$\widetilde{\varphi}_n(t)\le\widetilde{\varphi}_n(\tilde{t}_n)=\frac{1}{2}\left(1+V_\rho\rho^2\delta_n\right)\left(\tilde{t}_n^2-\frac{\pi}{\beta_0\log n}\right),\quad\forall\, t\ge 0. \tag{3.32}$$

From (3.28), (3.31) and (3.32), we have

$$\begin{aligned}&\widetilde{\varphi}_n(t)\\&\le\frac{1}{2}\left(1+V_\rho\rho^2\delta_n\right)\left(\tilde{t}_n^2-\frac{\pi}{\beta_0\log n}\right)\\&\le\frac{(2+\alpha)\pi}{2\beta_0}\left(1+V_\rho\rho^2\delta_n\right)\left[1+\frac{D_2-1}{(2+\alpha)\log n}+O\left(\frac{1}{\log^2 n}\right)\right]\\&\le\frac{(2+\alpha)\pi}{2\beta_0}+\frac{\pi}{2\beta_0\log n}\left[\frac{(2+\alpha)V_\rho\rho^2}{4}+D_2-1\right]+O\left(\frac{1}{\log^2 n}\right),\end{aligned}$$

which together with (3.26) that

$$\begin{aligned}\Phi_V(tw_n)\le&\frac{(2+\alpha)\pi}{2\beta_0}+\frac{\pi}{2\beta_0\log n}\left[\frac{(2+\alpha)V_\rho\rho^2}{4}+D_2-1\right]\\&+O\left(\frac{1}{\log^2 n}\right).\end{aligned} \tag{3.33}$$

Clearly, in this case, (3.30) and (3.33) imply that there exists $\bar{n}\in\mathbb{N}$ such that (3.10) holds.

Case (iv). $t\in\left(\sqrt{\frac{(2+\alpha)\pi}{\beta_0}(1+\epsilon)},+\infty\right)$. Then, $tw_n(x)\ge t_\epsilon$ for $x\in B_{\rho/n}$ and for large $n\in\mathbb{N}$, it follows from (F3), (3.3) and (3.15) that

$$\begin{aligned}\Phi_V(tw_n)\le&\frac{t^2}{2}\left(1+V_\rho\rho^2\delta_n\right)-\Psi(tw_n)\\\le&\frac{t^2}{2}\left(1+\frac{V_\rho\rho^2}{4\log n}\right)-\frac{\pi^3A_\alpha(\kappa-\epsilon)^2\rho^{2+\alpha}}{\alpha(1+\alpha)(2+\alpha)\beta_0^2t^2n^{2+\alpha}\log n}e^{\pi^{-1}\beta_0t^2\log n}\\\le&\frac{(1+\epsilon)(2+\alpha)\pi}{2\beta_0}\left(1+\frac{V_\rho\rho^2}{4\log n}\right)\\&-\frac{\pi^2A_\alpha(\kappa-\epsilon)^2\rho^{2+\alpha}}{(1+\epsilon)\alpha(1+\alpha)(2+\alpha)^2\beta_0\log n}e^{(1+\epsilon)(2+\alpha)\log n},\end{aligned}$$

which implies that there exists $\bar{n} \in \mathbb{N}$ such that (3.10) holds. In the above derivation process, we use the fact that the function

$$\frac{t^2}{2}\left(1+\frac{V_\rho \rho^2}{4\log n}\right)-\frac{\pi^3 A_\alpha(\kappa-\epsilon)^2\rho^{2+\alpha}}{\alpha(1+\alpha)(2+\alpha)\beta_0^2 t^2 n^{2+\alpha}\log n}e^{\pi^{-1}\beta_0 t^2\log n}$$

is decreasing on $t \in \left(\sqrt{\frac{(2+\alpha)\pi}{\beta_0}(1+\epsilon)}, +\infty\right)$, since its stagnation points tend to $\sqrt{\frac{(2+\alpha)\pi}{\beta_0}}$, as $n \to \infty$. □

Now, we deduce the following corollary from (3.9) and Lemma 3.1.

Corollary 3.1. *Assume that* (V0), (F1), (F2), (F3) *and* (F5) *hold. Then,*

$$c_V < \frac{(2+\alpha)\pi}{2\beta_0}.$$

Lemma 3.2. *Assume that* (V0), (F1), (F2), (F5) *and* (F6) *hold. Then,*

$$\lim_{\varepsilon\to 0} c_\varepsilon = c_{V_0},$$

where c_{V_0} *is the minimax value defined in* (3.9) *with* $V(x) \equiv V_0$.

Proof. By virtue of Lemma 3.1 and [30, Theorem 1.3], it is not complicated to verify that there exists $w \in E$ being the nontrivial solution of equation (1.5) with $V(x) \equiv V_0$. In fact, by a standard argument, we can easily check that w is a ground state solution of equation (1.5), and then $w \in \mathcal{N}_{V_0}$. In what follows, for any given $\delta > 0$, $w_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ being fixed and verifying

$$w_\delta \in \mathcal{N}_{V_0}, \quad w_\delta \to w \quad \text{in } E \quad \text{and} \quad \Phi_{V_0}(w_\delta) < c_{V_0} + \delta. \tag{3.34}$$

Without loss of generality, let us suppose $0 \in M$, where M is given in (2.8). Consequently, we choose a smooth cut-off function $\iota \in \mathcal{C}_0^\infty\left(\mathbb{R}^2, [0,1]\right)$ to be such that $\iota = 1$ on B_1 and $\iota = 0$ on $\mathbb{R}^2 \backslash B_2$. Further, we choose $v_n(x) = \iota(\varepsilon_n x)\, w_\delta(x)$, where $\varepsilon_n \to 0$ as $n \to +\infty$. Then, we obtain

$$v_n \to w_\delta \qquad \text{in } E,$$

which implies $\|v_n\|_{V_0} \to \|w_\delta\|_{V_0}$. From Lemma 2.4, there exists $t_n > 0$ such that $t_n v_n \in \mathcal{N}_{\varepsilon_n}$. We claim that $\{t_n\}$ is bounded. Otherwise, we assume $|t_n| \to +\infty$. Combining with (F5), (3.7), (3.13) and Fatou's Lemma, one has

$$\begin{aligned}
\|v_n\|_{\varepsilon_n}^2 &= t_n^{-2}\left\langle \Psi'\left(t_n v_n\right), t_n v_n\right\rangle + o(1) \\
&= \liminf_{n\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} I_\alpha(x-y)F(t_n v_n(y)f(t_n v_n(x))t_n^{-1}v_n(x)\, \mathrm{d}x\mathrm{d}y \quad (3.35)\\
&\geq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \liminf_{n\to\infty} I_\alpha(x-y)F(t_n v_n(y)f(t_n v_n(x))t_n^{-1}v_n(x)\, \mathrm{d}x\mathrm{d}y \\
&= +\infty,
\end{aligned}$$

which shows that $\{t_n\}$ is bounded. Thus, up to a subsequence, we may assume $\{t_n\} \to \hat{t}_0 \geq 0$. Note that there exists a constant $\kappa_0 > 0$ independent of ε such that $c_{\varepsilon_n} > \kappa_0 > 0$, which yields $\hat{t}_0 > 0$. Using this and (3.34), we have

$$\langle \Psi'(w), w\rangle = \lim_{\delta\to 0}\langle \Psi'(w_\delta), w_\delta\rangle$$

$$
\begin{aligned}
&= \lim_{\delta\to 0}\int_{\mathbb{R}^2}\left(|\nabla w_\delta|^2+V_0\,|w_\delta|^2\right)\mathrm{d}x\\
&= \lim_{\delta\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(|\nabla v_n|^2+V(\varepsilon_n x)\,|v_n|^2\right)\mathrm{d}x\\
&= \lim_{\delta\to 0}\lim_{n\to\infty}t_n^{-2}\int_{\mathbb{R}^2}\left(|\nabla (t_n v_n)|^2+V(\varepsilon_n x)\,|t_n v_n|^2\right)\mathrm{d}x\\
&= \lim_{\delta\to 0}\lim_{n\to\infty}t_n^{-2}\left\langle \Psi'(t_n v_n), t_n v_n\right\rangle\\
&= \lim_{\delta\to 0}t_0^{-2}\left\langle \Psi'(t_0 w_\delta), t_0 w_\delta\right\rangle\\
&= t_0^{-2}\left\langle \Psi'(t_0 w), t_0 w\right\rangle.
\end{aligned}
$$

Together with the monotonicity assumption (F6), we derive $\hat{t}_0 = 1$. Hence, by (2.19), (3.34) and Lemma 2.5, we can deduce

$$
\begin{aligned}
c_{\varepsilon_n} &\le \Phi_{\varepsilon_n}(t_n v_n)\\
&= \frac{t_n^2}{2}\int_{\mathbb{R}^2}\left(|\nabla v_n|^2+V\left(\varepsilon_n x\right)|v_n|^2\right)\mathrm{d}x-\Psi(t_n v_n)\\
&= \frac{t_n^2}{2}\int_{\mathbb{R}^2}\left(|\nabla v_n|^2+V_0\,|v_n|^2\right)\mathrm{d}x-\Psi(t_n v_n)+\frac{t_n^2}{2}\int_{\mathbb{R}^2}\left[V\left(\varepsilon_n x\right)-V_0\right]|v_n|^2\ \mathrm{d}x\\
&= \Phi_{V_0}(t_n v_n)+\frac{t_n^2}{2}\int_{\mathbb{R}^2}\left[V\left(\varepsilon_n x\right)-V_0\right]|v_n|^2\ \mathrm{d}x\\
&= \Phi_{V_0}(w_\delta)+o_n(1)\\
&\le c_{V_0}+\delta.
\end{aligned}
$$

As δ is arbitrary, we have

$$\limsup_{n\to\infty} c_{\varepsilon_n}\le c_{V_0}.$$

This shows that

$$\limsup_{\varepsilon\to 0} c_{\varepsilon}\le c_{V_0}. \tag{3.36}$$

On the other hand, it follows from (V0) that

$$c_\varepsilon \ge c_{V_0},\quad \forall\ \varepsilon>0,$$

which implies

$$\liminf_{\varepsilon\to 0} c_{\varepsilon}\ge c_{V_0}. \tag{3.37}$$

From (3.36) and (3.37), we get the conclusion

$$\lim_{\varepsilon\to 0} c_{\varepsilon}= c_{V_0}.$$

□

Using Corollary 3.1 and Lemma 3.2, we can get the following lemma immediately.

Lemma 3.3. *Assume that* (V0), (F1), (F2), (F3), (F5) *and* (F6) *hold. Then, there exists* $\varepsilon_0>0$ *such that*

$$c_\varepsilon<\frac{(2+\alpha)\pi}{2\beta_0},\quad \forall\ \varepsilon\in[0,\varepsilon_0). \tag{3.38}$$

Moreover, since $c_{V_0}\le c_{V_\infty}$*, we also have*

$$\lim_{\varepsilon\to 0} c_\varepsilon\le c_{V_\infty}.$$

4. The proofs of main results

In this section, we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Applying Lemmas 2.2 and 2.6 with $\varepsilon = 1$, we deduce that there exists a sequence $\{u_n\} \subset E$ satisfying (3.8) and $\|u_n\|_V^2 \le C_4/2$ for the constant $C_4 > 0$, it follows from (F3), (1.3) and (2.18) that for n large

$$2\bar{\mu}\Psi(u_n) \le \langle \Psi'(u_n), u_n\rangle \le C_4. \tag{4.1}$$

Set

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} |u_n|^2 \, \mathrm{d}x.$$

We claim $\delta > 0$. The proof is essentially contained in [30, proof of (4.92)]. We give the details here for the convenience of readers. If $\delta = 0$, then by Lions' concentration compactness principle [35, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$.

For any given $\epsilon > 0$, we choose $M_\epsilon > \max\{M_0 C_4/\epsilon, t_0\}$, and then it follows from (F3), (F4) and (4.1) that

$$\begin{aligned} \Psi_{\{|u_n|\ge M_\epsilon\}}(u_n) &\le M_0 \int_{\{|u_n|\ge M_\epsilon\}} [I_\alpha * F(u_n)]\,|f(u_n)| \, \mathrm{d}x \\ &\le \frac{M_0}{M_\epsilon}\left\langle \Psi'_{\{|u_n|\ge M_\epsilon\}}(u_n), u_n\right\rangle \\ &\le \frac{M_0}{M_\epsilon}\langle \Psi'(u_n), u_n\rangle \\ &< \epsilon. \end{aligned} \tag{4.2}$$

Define $\bar{\epsilon} = \epsilon\bar{\mu}^{1/2}/\left[\mathcal{C}_0^{1/2}\gamma_2^{(2+\alpha)/2}C_4^{(3+\alpha)/2}\right]$. Using (F2), (F3), (2.5), (4.1), Propositions 2.1 and 2.2, we can choose $N_\epsilon \in (0,1)$ such that

$$\begin{aligned} \Psi_{\{|u_n|\le N_\epsilon\}}(u_n) &\le \frac{1}{\bar{\mu}}\left\langle \Psi'_{\{|u_n|\le N_\epsilon\}}(u_n), u_n\right\rangle \\ &\le \bar{\epsilon}\int_{\{|u_n|\le N_\epsilon\}} [I_\alpha * F(u_n)]\,|u_n|^{(2+\alpha)/2} \, \mathrm{d}x \\ &\le \bar{\epsilon}(2\Psi(u_n))^{1/2}\left\{\int_{\{|u_n|\le N_\epsilon\}} \left[I_\alpha * |u_n|^{(2+\alpha)/2}\right]|u_n|^{(2+\alpha)/2} \, \mathrm{d}x\right\}^{1/2} \\ &\le \bar{\epsilon}\left(\frac{\mathcal{C}_0 C_4}{\bar{\mu}}\right)^{1/2}\|u_n\|_2^{(2+\alpha)/2} \\ &\le \bar{\epsilon}\left(\frac{\mathcal{C}_0 C_4}{\bar{\mu}}\right)^{1/2}\tau_2^{(2+\alpha)/2}\|u_n\|_V^{(2+\alpha)/2} \\ &\le \epsilon, \end{aligned} \tag{4.3}$$

By (F1), (F3), (4.1), Propositions 2.1 and 2.2, we have

$$\begin{aligned} \Psi_{\{N_\epsilon\le|u_n|\le M_\epsilon\}}(u_n) &\le \frac{1}{\bar{\mu}}\left\langle \Psi'_{\{N_\epsilon\le|u_n|\le M_\epsilon\}}(u_n), u_n\right\rangle \\ &\le C_5\int_{\{N_\epsilon\le|u_n|\le M_\epsilon\}} [I_\alpha * F(u_n)]\,|u_n|^3 \, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\le C_5\left(2\Psi(u_n)\right)^{1/2}\left\{\int_{\{N_\epsilon\le|u_n|\le M_\epsilon\}}\left[I_\alpha*|u_n|^3\right]|u_n|^3\,\mathrm{d}x\right\}^{1/2} \qquad (4.4)\\
&\le C_5\left(\frac{\mathcal{C}_0C_4}{\bar{\mu}}\right)^{\frac{1}{2}}\|u_n\|^3_{12/(2+\alpha)}\\
&\le o_n(1).
\end{aligned}
$$

Following from (4.2), (4.3) and (4.4), one has

$$
\Psi(u_n)=o_n(1). \tag{4.5}
$$

Then, we can deduce from (2.3), (3.8) and (4.5) that there exists $\nu\in(0,1/2)$ small enough such that for n large

$$
0<2\kappa_0<\|u_n\|_V^2=2c_V+o_n(1)\le\frac{(2+\alpha)\pi}{\beta_0}(1-\nu). \tag{4.6}
$$

Let us choose $q\in(1,2)$ such that

$$
(1-\nu^2)q<1. \tag{4.7}
$$

By (F1), there exists $C_6>0$ such that

$$
|f(t)|\le C_6\left[e^{\beta_0(1+\nu)t^2}-1\right],\qquad\forall\ |t|\ge1. \tag{4.8}
$$

In view of (4.6), (4.7), (4.8) and Proposition 1.1 ii), we have

$$
\begin{aligned}
\int_{\{|u_n|\ge1\}}|f(u_n)|^{\frac{4q}{2+\alpha}}\mathrm{d}x&\le C_6\int_{\{|u_n|\ge1\}}\left[e^{\beta_0(1+\nu)u_n^2}-1\right]^{\frac{4q}{2+\alpha}}\mathrm{d}x \qquad (4.9)\\
&\le C_6\int_{\mathbb{R}^2}\left[\exp\left(\frac{4\beta_0(1+\nu)q\|u_n\|_V^2}{2+\alpha}\left(\frac{u_n}{\|u_n\|_V}\right)^2\right)-1\right]\mathrm{d}x\\
&\le C_7.
\end{aligned}
$$

Let $q'=q/(q-1)$. Then, from (4.9), Propositions 2.1 and 2.2, we get

$$
\begin{aligned}
&\left\langle\Psi'_{\{|u_n|\ge1\}}(u_n),u_n\right\rangle\\
&\le(2\Psi(u_n))^{\frac{1}{2}}\left[\int_{\{|u_n|\ge1\}}\left[I_\alpha*(f(u_n)u_n\chi_{\{|u_n|\ge1\}})\right]f(u_n)u_n\chi_{\{|u_n|\ge1\}}\mathrm{d}x\right]^{\frac{1}{2}}\\
&\le\sqrt{C_4\mathcal{C}_0}\left[\int_{\{|u_n|\ge1\}}|f(u_n)|^{4/(2+\alpha)}|u_n|^{4/(2+\alpha)}\right]^{\frac{2+\alpha}{4}} \qquad (4.10)\\
&\le\sqrt{C_4\mathcal{C}_0}\left[\int_{\{|u_n|\ge1\}}|f(u_n)|^{4q/(2+\alpha)}\right]^{\frac{2+\alpha}{4q}}\left[\int_{\{|u_n|\ge1\}}|u_n|^{4q'/(2+\alpha)}\right]^{\frac{2+\alpha}{4q'}}\\
&\le C_8\|u_n\|_{4q'/(2+\alpha)}\\
&=o_n(1).
\end{aligned}
$$

By (4.3), (4.4) and (4.10) that

$$
\langle\Psi'(u_n),u_n\rangle=o_n(1),
$$

which together with (3.8) implies

$$\|u_n\|_V^2 = o_n(1),$$

which contradicts with (4.6). Thus, $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists $\{y_n\} \subset \mathbb{Z}^2$ such that $\int_{B_{1+\sqrt{2}}(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}$. Let us define $\tilde{u}_n(x) = u_n(x+y_n)$. Then, there exists $\tilde{u} \in E$ such that $u_n(\cdot + y_n) \rightharpoonup \tilde{u} \neq 0$ in E. By (V1), we have $\|\tilde{u}_n\|_V = \|u_n\|_V$,

$$\|\Phi_V'(\tilde{u}_n)\|(1 + \|\tilde{u}_n\|_V) \to 0 \quad \text{and} \quad \Phi_V(\tilde{u}_n) \to c_V. \tag{4.11}$$

By [30, Lemma 4.8], for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\lim_{n\to\infty} \langle \Psi_V'(\tilde{u}_n), \varphi\rangle = \langle \Psi_V'(\tilde{u}), \varphi\rangle. \tag{4.12}$$

Thus,

$$o_n(1) = \langle \Phi_V'(\tilde{u}_n), \varphi\rangle = \langle \Phi_V'(\tilde{u}), \varphi\rangle,$$

which shows that $\tilde{u}$ is a solution of equation (1.5). Using the information and (F3), (2.3) and Fatou's Lemma, one has

$$\begin{aligned} c_V &= \lim_{n\to\infty}\left[\Phi_V(\tilde{u}_n) - \frac{1}{2}\langle \Phi_V'(\tilde{u}_n), \tilde{u}_n\rangle\right] \\ &= \liminf_{n\to\infty}\left[\frac{1}{2}\langle \Psi_V'(\tilde{u}_n), \tilde{u}_n\rangle - \Psi_V(\tilde{u}_n)\right] \\ &\geq \frac{1}{2}\langle \Psi_V'(\tilde{u}), \tilde{u}\rangle - \Psi_V(\tilde{u}) \\ &= \Phi_V(\tilde{u}) - \frac{1}{2}\langle \Phi_V'(\tilde{u}), \tilde{u}\rangle \\ &\geq c_V, \end{aligned}$$

which yields $\Phi_V(\tilde{u}) = c_V$. Thus, $\tilde{u}$ is a ground state solution of equation (1.5).

Proof of Theorem 1.2. Applying Theorem 1.1 to Φ_{V_∞}, we can deduce that Φ_{V_∞} has a critical point $u_\infty \in \mathcal{N}_{V_\infty}$. That is,

$$u_\infty \in \mathcal{N}_{V_\infty}, \qquad \Phi_{V_\infty}'(u_\infty) = 0, \qquad \Phi_{V_\infty}(u_\infty) = c_{V_\infty}. \tag{4.13}$$

Since Φ_{V_∞} is autonomous and $V_0 < V_\infty$, there exist $\tilde{x} \in \mathbb{R}^2$ and $\tilde{r} > 0$ such that

$$V_\infty - V(x) > 0 \ \text{ and } \ |u_\infty(x)| > 0 \ \text{ for a.e. } |x - \tilde{x}| \leq \tilde{r}. \tag{4.14}$$

From Lemma 2.4, there exists $t_\infty > 0$ such that $t_\infty u_\infty \in \mathcal{N}_\varepsilon$. Then, following from (V2), (1.3), (2.3), (4.13) and Corollary 2.1, one has

$$\begin{aligned} c_{V_\infty} &= \Phi_{V_\infty}(u_\infty) \geq \Phi_{V_\infty}(t_\infty u_\infty) \\ &= \Phi_\varepsilon(t_\infty u_\infty) + \frac{t_\infty^2}{2}\int_{\mathbb{R}^2}[V_\infty - V(\varepsilon x)]|u_\infty|^2 \ \mathrm{d}x \\ &\geq c_\varepsilon + \frac{t_\infty^2}{2}\int_{\mathbb{R}^2}[V_\infty - V(\varepsilon x)]|u_\infty|^2 \ \mathrm{d}x \\ &> c_\varepsilon. \end{aligned} \tag{4.15}$$

In virtue of Lemmas 2.2, 2.5, 2.6 and 3.3, we get that there exists a bounded sequence $\{u_n\} \subset E_\varepsilon$ satisfying (2.18) and $\|u_n\|_\varepsilon^2 \le C_9/2$ for some constant $C_9 > 0$ and

$$\kappa_0 \le c_\varepsilon < \frac{(2+\alpha)\,\pi}{2\beta_0}, \quad \forall\, \varepsilon \in (0, \varepsilon_0)\,. \tag{4.16}$$

It follows from (F3), (1.3) and (2.18) that

$$2\bar{\mu}\Psi(u_n) \le \langle \Psi'(u_n), u_n \rangle \le C_9. \tag{4.17}$$

Passing to a subsequence if necessary, we may assume $u_n \rightharpoonup \bar{u}_\varepsilon$ in E_ε, $u_n \to \bar{u}_\varepsilon$ in $L^s_{\rm loc}(\mathbb{R}^2)$ for $s \in [2,\infty)$ and $u_n \to \bar{u}_\varepsilon$ a.e. on $\mathbb{R}^2$. For any $n \in \mathbb{N}$, from Lemma 2.4, there exists $t_n > 0$ such that $t_n u_n \in \mathcal{N}_{V_\infty}$. Hence,

$$\Phi_{V_\infty}(t_n u_n) \ge c_{V_\infty} \text{ and } \left\langle \Phi'_{V_\infty}(t_n u_n), t_n u_n \right\rangle = 0. \tag{4.18}$$

Letting us define

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} |u_n|^2 \,\mathrm{d}x$$

by using an argument which is in the proof of Theorem 1.1, we deduce $\delta > 0$. Going if necessary to a subsequence, there exists $\{y_n\} \subset \mathbb{R}^2$ such that $\int_{B_{1+\sqrt{2}}(y_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}$. Defining $\tilde{u}_n(x) = u_n(x+y_n)$, and then we have

$$\int_{B_{1+\sqrt{2}}(0)} |\tilde{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{4.19}$$

By the equivalence of the norm $\|\cdot\|_\varepsilon$ and the standard Sobolev norm, then we have $\{\|\tilde{u}_n\|_\varepsilon\}$ is bounded and

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u}_\varepsilon, & \text{in } E_\varepsilon, \\ \tilde{u}_n \to \tilde{u}_\varepsilon, & \text{in } L^s_{\rm loc}(\mathbb{R}^2) \text{ for } s \in [2,\infty), \\ \tilde{u}_n \to \tilde{u}_\varepsilon, & \text{a.e. on } \mathbb{R}^2. \end{cases}$$

Clearly, following from (4.19) that $\tilde{u}_\varepsilon \ne 0$. By (F3) and (4.18), one has

$$\begin{aligned} 0 &= t_n^{-2} \left\langle \Phi'_{V_\infty}(t_n u_n), t_n u_n \right\rangle \\ &= t_n^{-2} \left\langle \Phi'_{V_\infty}(t_n \tilde{u}_n), t_n \tilde{u}_n \right\rangle \\ &= \int_{\mathbb{R}^2} \left(|\nabla \tilde{u}_n|^2 + V_\infty |\tilde{u}_n|^2 \right) \mathrm{d}x - t_n^{-2} \langle \Psi'(t_n\tilde{u}_n), t_n\tilde{u}_n \rangle \\ &\le \int_{\mathbb{R}^2} \left(|\nabla \tilde{u}_n|^2 + V_\infty |\tilde{u}_n|^2 \right) \mathrm{d}x - 2\bar{\mu} t_n^{-2} \Psi(t_n \tilde{u}_n). \end{aligned} \tag{4.20}$$

Thereby, combining with (F1), (4.20) and the boundedness of $\{\|\tilde{u}_n\|\}$, we derive that $\{t_n\}$ is bounded. Thus, up to a subsequence, we may assume $0 \le t_n \le \tilde{t}_0$.

Now, we prove $\bar{u}_\varepsilon \ne 0$. Arguing by the contradiction, suppose $\bar{u}_\varepsilon \equiv 0$, and then $u_n \rightharpoonup 0$ in E_ε, $u_n \to 0$ in $L^s_{\rm loc}\left(\mathbb{R}^2\right)$ for $s \in [1,\infty)$ and $u_n \to 0$, a.e. on $\mathbb{R}^2$. From (2.18), (4.14), (4.15), (4.16) (4.18) and Lemma 2.3, we have

$$\begin{aligned} & c_\varepsilon + o_n(1) \\ = {} & \Phi_\varepsilon(u_n) \end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&\geq \Phi_\varepsilon\left(t_n u_n\right)+\frac{1-t_n^2}{2}\left\langle\Phi_\varepsilon^{\prime}\left(u_n\right), u_n\right\rangle \\
&=\Phi_\varepsilon\left(t_n u_n\right)+o_n(1) \\
&=\Phi_{V_\infty}\left(t_n u_n\right)+\frac{t_n^2}{2} \int_{\mathbb{R}^2}\left[V(\varepsilon x)-V_\infty\right]\left|u_n\right|^2 \mathrm{~d} x+o_n(1) \\
&\geq c_{V_\infty}+\frac{t_n^2}{2} \int_{|x| \leq R / \varepsilon}\left[V(\varepsilon x)-V_\infty\right]\left|u_n\right|^2 \mathrm{~d} x \\
&\quad+\frac{t_n^2}{2} \int_{|x|>R / \varepsilon}\left[V(\varepsilon x)-V_\infty\right]\left|u_n\right|^2 \mathrm{~d} x+o_n(1) \\
&\geq c_{V_\infty}-\frac{\left(V_\infty-V_0\right) \tilde{t}_0^2}{2} \int_{|x| \leq R / \varepsilon}\left|u_n\right|^2 \mathrm{~d} x-\frac{\tilde{t}_0^2}{2} \sup _{|x|>R / \varepsilon}\left[V_\infty-V(\varepsilon x)\right]\left\|u_n\right\|_2^2+o_n(1) \\
&\geq c_\varepsilon+\frac{t_\infty^2}{2} \int_{\mathbb{R}^2}\left[V_\infty-V(\varepsilon x)\right]\left|u_\infty\right|^2 \mathrm{~d} x-\frac{\tilde{t}_0^2}{2} \sup _{|x|>R / \varepsilon}\left[V_\infty-V(\varepsilon x)\right]\left\|u_n\right\|_2^2+o_n(1) \\
&\geq c_\varepsilon+\frac{t_\infty^2}{4} \int_{\mathrm{R}^2}\left[V_\infty-V(\varepsilon x)\right]\left|u_\infty\right|^2 \mathrm{~d} x+o_R(1)+o_n(1) \\
&>c_\varepsilon .
\end{aligned}
$$

This contradiction shows $\bar{u}_\varepsilon \neq 0$. From (4.17) and [30, Lemma 4.8], one has

$$
\lim_{n\to\infty}\langle\Psi_\varepsilon'(u_n),\varphi\rangle=\langle\Psi_\varepsilon'(\bar{u}_\varepsilon),\varphi\rangle, \qquad \forall\ \varphi\in\mathcal{C}_0^\infty(\mathbb{R}^2),
$$

which yields

$$
o_n(1)=\langle\Phi_\varepsilon'(u_n),\varphi\rangle=\langle\Phi_\varepsilon'(\bar{u}_\varepsilon),\varphi\rangle.
$$

Hence, $\bar{u}_\varepsilon$ is a solution of equation (1.2). Using this and (F3), (1.3), (2.18) and Fatou's Lemma, we have

$$
\begin{aligned}
c_\varepsilon &= \lim_{n\to\infty}\left[\Phi_\varepsilon(u_n)-\frac{1}{2}\langle\Phi_\varepsilon'(u_n),u_n\rangle\right] \\
&=\liminf_{n\to\infty}\left[\frac{1}{2}\langle\Psi_\varepsilon'(u_n),u_n\rangle-\Psi_\varepsilon(u_n)\right] \\
&\geq \frac{1}{2}\langle\Psi_\varepsilon'(\bar{u}_\varepsilon),\bar{u}_\varepsilon\rangle-\Psi_\varepsilon(\bar{u}_\varepsilon) \\
&=\Phi_\varepsilon(\bar{u}_\varepsilon)-\frac{1}{2}\langle\Phi_\varepsilon'(\bar{u}_\varepsilon),\bar{u}_\varepsilon\rangle \\
&\geq c_\varepsilon,
\end{aligned}
$$

which implies $\Phi_\varepsilon(\bar{u}_\varepsilon)=c_\varepsilon$. Therefore, for $\varepsilon\in(0,\varepsilon_0)$, $\bar{u}_\varepsilon$ is a ground state solution of equation (1.2).

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