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Dynamics of Stochastic Ginzburg-Landau Equations Driven by Colored Noise on Thin Domains*

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Abstract This work is concerned with the asymptotic behaviors of solutions to a class of non-autonomous stochastic Ginzburg-Landau equations driven by colored noise and deterministic non-autonomous terms defined on thin domains. The existence and uniqueness of tempered pullback random attractors are proved for the stochastic Ginzburg-Landau systems defined on $(n+1)$-dimensional narrow domain. Furthermore, the upper semicontinuity of these attractors is established, when a family of $(n+1)$-dimensional thin domains collapse onto an n-dimensional domain.

Keywords Stochastic Ginzburg-Landau equation, colored noise, thin domain, random attractor, upper semicontinuity

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1. Introduction

In this paper, we investigate the asymptotic behavior of solutions of the following non-autonomous stochastic Ginzburg-Landau equations driven by colored noise on $\mathcal{O}_\varepsilon$ with Neumann boundary conditions: for $t > \tau$ with $\tau \in \mathbb{R}$ and $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$,

$$\begin{cases} \dfrac{\partial \hat{u}^\varepsilon}{\partial t} - (1 + \mathrm{i}\mu)\Delta \hat{u}^\varepsilon + \rho \hat{u}^\varepsilon = f(t, x, \hat{u}^\varepsilon) + G(t, x) + R(t, x, \hat{u}^\varepsilon)\zeta_\delta(\theta_t \omega), \quad x \in \mathcal{O}_\varepsilon, \\ \dfrac{\partial \hat{u}^\varepsilon}{\partial \nu_\varepsilon} = 0, \quad x \in \partial \mathcal{O}_\varepsilon, \end{cases} \tag{1.1}$$

with the initial condition

$$\hat{u}^\varepsilon(\tau, x) = \hat{u}^\varepsilon_\tau(x), \quad x \in \mathcal{O}_\varepsilon, \tag{1.2}$$

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where $\hat{u}^\varepsilon(t,x)$ is a complex-valued function on $\mathbb{R}\times\mathcal{O}_\varepsilon$. In (1.1), i is the imaginary unit, and μ,ρ are real constants and $\rho>0$. ν_ε is the unit outward normal vector to $\partial\mathcal{O}_\varepsilon$. The so-called thin domain $\mathcal{O}_\varepsilon$ (ε small) is given by

$$\mathcal{O}_\varepsilon=\Big\{x=(x^*,x_{n+1})|\ x^*=(x_1,x_2,\cdots,x_n)\in\mathcal{Q},\ 0<x_{n+1}<\varepsilon g(x^*)\Big\} \tag{1.3}$$

with $0<\varepsilon\le 1$ and $g\in C^2(\overline{\mathcal{Q}},(0,+\infty))$, where $\mathcal{Q}$ is a smooth bounded domain in $\mathbb{R}^n$. Since $g\in C^2\big(\overline{\mathcal{Q}},(0,+\infty)\big)$, there exist two positive constants β_1 and β_2 such that

$$\beta_1\le g(x^*)\le\beta_2,\quad \forall\ x^*\in\overline{\mathcal{Q}}. \tag{1.4}$$

Denote $\mathcal{O}=\mathcal{Q}\times(0,1)$ and $\widetilde{\mathcal{O}}=\mathcal{Q}\times(0,\beta_2)$ which contain $\mathcal{O}_\varepsilon$ for $0<\varepsilon\le 1$. The nonlinearity f and the body force G satisfy some conditions, which are to be specified later. $\zeta_\delta(\theta_t\omega)$ with $0<\delta\le 1$ is an Ornstein-Uhlenbeck (O-U) process (also known as a colored noise).

The O-U process is a stationary Gaussian process with zero mathematical expectation, and the O-U process is the only existing Markovian Gaussian colored noise (see, e.g. [6] and [23]). Furthermore, the O-U process is also called a colored noise, because its power spectrum is not flat compared with the white noise (see [2,7,9,23–25,28,30]).

As we know, the Wiener process W can be chosen as a stochastic process to represent the position of the Brownian particle. But the velocity of the particle cannot be obtained from the Wiener process because of the nowhere differentiability of the sample paths of W. However, the O-U process was originally constructed to approximately describe the stochastic behavior of the velocity [25,30]. Hence, it can be further used to determine the position of the particle. Furthermore, as demonstrated in [23], in many complex systems, stochastic fluctuations are actually correlated. Therefore, they should be modeled by colored noise rather than white noise.

During the study of stochastic dynamics, one of the most crucial issues arises from the modeling of random forcing. To study such a random forcing, we need to consider the time scale τ_d of the deterministic system and the time scale τ_r of the random forcing. The stochastic forcing is modeled in different ways based on the ratio of τ_r/τ_d. If $\tau_r/\tau_d\gg 1$, and the dynamical system is very slow with respect to the temporal variability of its random drivers. Hence, the random forcing could be modeled as white noise. If $\tau_r/\tau_d\simeq 1$, then the dynamics of the system is sensitive to the autocorrelation of the random forcing, and therefore the random forcing should be modeled by colored noise. Based on these considerations, the colored noise has been used in many works to study the dynamics of physical and biological system (see, e.g. [2,7,12–14,23,25,30] and the reference therein).

As $\varepsilon\to 0$, the thin domain $\mathcal{O}_\varepsilon$ collapses to an n-dimensional domain. In this paper, we will see that the limiting behavior of the equation is determined by the following system on the lower dimensional spatial domain $\mathcal{Q}$: for $t>\tau$ with $\tau\in\mathbb{R}$

and $y^*=(y_1,\cdots,y_n)\in\mathcal{Q}$,

$$\begin{cases}\dfrac{\partial u^0}{\partial t}-(1+\mathrm{i}\mu)\dfrac{1}{g}\displaystyle\sum_{i=1}^{n}(gu^0_{y_i})_{y_i}+\rho u^0=f(t,y^*,0,u^0)+G(t,y^*,0)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad +R(t,x,u^0)\zeta_\delta(\theta_t\omega),\quad y^*\in\mathcal{Q},\\ \dfrac{\partial u^0}{\partial\nu_0}=0,\quad y^*\in\partial\mathcal{Q},\end{cases}\tag{1.5}$$

with the initial condition

$$u^0(\tau,y^*)=u^0_\tau(y^*),\quad y^*\in\mathcal{Q},\tag{1.6}$$

where ν_0 is the unit outward normal vector to $\partial\mathcal{Q}$. Note that $u^0_{y_i}$ means $\frac{\partial u^0}{\partial y_i}$ in (1.5) and similar notations will be used throughout this paper.

The study of the asymptotic behavior of deterministic PDEs defined on thin domains was first initiated by Hale and Raugel [10, 11]. Then, their results were extended to various problems including stochastic problems (see, for instance, [1, 4, 5, 15, 19–22]). However, almost all the studied stochastic equations are driven by white noise. There are few equations driven by colored noise.

It is well-known that the Ginzburg-Landau equation is an important nonlinear evolution equation, which is used to simplify mathematical models for pattern formation in mechanics, physics and chemistry. For the deterministic Ginzburg-Landau equation, the long-time behavior of solutions was investigated in [16, 17, 32]. For the stochastic Ginzburg-Landau equation, the study of the random attractor can be found in [18, 29, 31]. In this work, we mainly focus on the dynamics of the stochastic system (1.1) driven by colored noise and defined on the thin domain $\mathcal{O}_\varepsilon$ for small ε, and explore the limiting behavior of the system as $\varepsilon\to 0$.

The rest of the paper is organized as follows. In Section 2, we establish the existence of a continuous cocycle in $L^2(\mathcal{O})$ for the stochastic equation defined on the fixed domain $\mathcal{O}$, which is converted from (1.1) and (1.2). We also describe the existence of a continuous cocycle in $L^2(\mathcal{Q})$ for the stochastic equation (1.5) and (1.6). In Section 3, we deduce all necessary uniform estimates of the solutions. In Section 4, we prove the existence and uniqueness of tempered attractors for the stochastic equation. In Section 5, we establish the upper semicontinuity of these attractors.

2. Continuous Cocycles for random Ginzburg-Landau systems

In this section, we will define a continuous cocycle for the following non-autonomous Ginzburg-Landau systems driven by colored noise for $x\in(x^*,x_{n+1})\in\mathcal{O}_\varepsilon$

$$\begin{cases}\dfrac{\partial u^\varepsilon}{\partial t}-(1+\mathrm{i}\mu)\Delta\hat{u}^\varepsilon+\rho\hat{u}^\varepsilon=f(t,x,\hat{u}^\varepsilon)+G(t,x)+R(t,x,\hat{u}^\varepsilon)\zeta_\delta(\theta_t\omega),\quad t>\tau,\\ \dfrac{\partial\hat{u}^\varepsilon}{\partial\nu_\varepsilon}=0,\quad x\in\partial\mathcal{O}_\varepsilon,\\ \hat{u}^\varepsilon(\tau,x)=\hat{u}^\varepsilon_\tau(x),\end{cases}\tag{2.1}$$

where $\tau \in \mathbb{R}$, $\mu, \rho > 0$ are constants, and $G \in L^2_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$. $\zeta_\delta(\theta_t\omega)(0 < \delta \le 1)$ is an Ornstein-Uhlenbeck (O-U) process defined on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ is equipped with the compact-open topology, $\mathcal{F} = \mathfrak{B}(\Omega)$ is the Borel sigma-algebra of Ω, $\mathbb{P}$ is the Wiener measure, and $\{\theta_t\}_{t\in\mathbb{R}}$ is the measure-preserving transformation group on Ω given by $\theta_t\omega(\cdot) = \omega(\cdot+t)-\omega(t)$ for all $(\omega, t) \in \Omega\times\mathbb{R}$. In this paper, f, $R : \mathbb{R}\times\widetilde{\mathcal{O}}\times\mathbb{C} \to \mathbb{R}$ are continuous functions such that for all $x \in \widetilde{\mathcal{O}}$ and $t, s \in \mathbb{R}$,

$$\mathrm{Re} f(t,x,u)\bar{u} \le -\gamma|u|^p + \psi_1(t,x), \tag{2.2}$$

$$\left|\frac{\partial f(t,x,u)}{\partial u}\right| \le \beta, \tag{2.3}$$

$$\left|\frac{\partial f}{\partial x}(t,x,u)\right| \le \psi_2(t,x), \tag{2.4}$$

$$\mathrm{Re} R(t,x,u)\bar{u} \le -\lambda|u|^q + \psi_3(t,x), \tag{2.5}$$

$$\left|\frac{\partial R}{\partial u}(t,x,u)\right| \le \kappa, \tag{2.6}$$

$$\left|\frac{\partial R}{\partial x}(t,x,u)\right| \le \psi_4(t,x), \tag{2.7}$$

for $u \in \mathbb{C}$, where $p > q \ge 2$, γ, β and κ are positive constants, $\psi_1, \psi_3 \in L^1_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$, $\psi_2, \psi_4 \in L^2_{loc}(\mathbb{R}, L^\infty(\widetilde{\mathcal{O}}))$.

Remark 2.1. One may take $f(t,x,u) = (1+iv)|u|^{2\sigma_1}u$ and $R(t,x,u) = |u|^{2\sigma_2}u$ with $0 < \sigma_1 < \sigma_2$, which satisfy the above conditions.

Next, we transfer problem (2.1) into the boundary value problem on the fixed domain $\mathcal{O}$. For $0 < \varepsilon \le 1$, we define a transformation $T_\varepsilon : \mathcal{O}_\varepsilon \to \mathcal{O}$ by $T_\varepsilon(x^*, x_{n+1}) = (x^*, \frac{x_{n+1}}{\varepsilon g(x^*)})$ for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$. Let $y = (y^*, y_{n+1}) = T_\varepsilon(x^*, x_{n+1})$. Then, we have $x^* = y^*$, $x_{n+1} = \varepsilon g(y^*)y_{n+1}$. By some calculations, we find that the Jacobian matrix of T_ε is given by

$$J = \frac{\partial(y_1, \cdots, y_{n+1})}{\partial(x_1, \cdots, x_{n+1})} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ -\frac{y_{n+1}}{g}g_{y_1} & -\frac{y_{n+1}}{g}g_{y_2} & \cdots & -\frac{y_{n+1}}{g}g_{y_n} & \frac{1}{\varepsilon g(y^*)} \end{pmatrix}.$$

The determinant of J is $|J| = \frac{1}{\varepsilon g(y^*)}$. Let J^* be the transpose of J. Then, we have

$$JJ^* = \begin{pmatrix} 1 & 0 & \cdots & 0 & -\frac{y_{n+1}}{g}g_{y_1} \\ 0 & 1 & \cdots & 0 & -\frac{y_{n+1}}{g}g_{y_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\frac{y_{n+1}}{g}g_{y_n} \\ -\frac{y_{n+1}}{g}g_{y_1} & -\frac{y_{n+1}}{g}g_{y_2} & \cdots & -\frac{y_{n+1}}{g}g_{y_n} & \sum_{i=1}^{n}\left(\frac{y_{n+1}}{g}g_{y_i}\right)^2 + \left(\frac{1}{\varepsilon g(y^*)}\right)^2 \end{pmatrix}.$$

It follows from [10] that the gradient operator, the Laplace operator in the original variable $x \in \mathcal{O}_\varepsilon$ and the new variable $y \in \mathcal{O}$ are related by

$$\nabla_x \hat{u}(x) = J^* \nabla_y u(y) \text{ and } \triangle_x \hat{u}(x) = |J|\text{div}_y(|J|^{-1}JJ^*\nabla_y u(y)) = \frac{1}{g}\text{div}_y(P_\varepsilon u(y)),$$

where $\hat{u}(x) = u(y)$, ∇_x is the gradient operator in $x \in \mathcal{O}_\varepsilon$, $\triangle_x$ is the Laplace operator in $x \in \mathcal{O}_\varepsilon$, div_y is the divergence operator, ∇_y is the gradient operator in $y \in \mathcal{O}$, and P_ε is the operator given by

$$P_\varepsilon u(y) = \begin{pmatrix} g u_{y_1} - g_{y_1} y_{n+1} u_{y_{n+1}} \\ \vdots \\ g u_{y_n} - g_{y_n} y_{n+1} u_{y_{n+1}} \\ -\sum_{i=1}^{n} y_{n+1} g_{y_i} u_{y_i} + \frac{1}{\varepsilon^2 g}\left(1 + \sum_{i=1}^{n}(\varepsilon y_{n+1} g_{y_i})^2\right) u_{y_{n+1}} \end{pmatrix}.$$

In the sequel, for $x = (x^*, x_{n+1}) \in \mathcal{O}_\varepsilon$, $y = (y^*, y_{n+1}) \in \mathcal{O}$ and $t, s \in \mathbb{R}$, we denote

$$\begin{aligned} &u^\varepsilon(y) = \hat{u}^\varepsilon(x), \quad f(t,x,s) = f(t,x^*,x_{n+1},s), \quad f_0(t,y^*,s) = f(t,y^*,0,s), \\ &f_\varepsilon(t,y^*,y_{n+1},s) = f(t,y^*,\varepsilon, g(y^*)y_{n+1},s), \quad G_\varepsilon(t,y^*,y_{n+1}) = G(t,y^*,\varepsilon g(y^*)y_{n+1}), \\ &G_0(t,y^*) = f(t,y^*,0), \quad R_\varepsilon(t,y^*,y_{n+1},s) = R(t,y^*,\varepsilon g(y^*)y_{n+1},s), \\ &R_0(t,y^*,s) = R(t,y^*,0,s). \end{aligned}$$

Then, problem (2.1) is equivalent to the following system for $y = (y^*, y_{n+1}) \in \mathcal{O}$ and $t > \tau$,

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - (1+\mathrm{i}\mu)\dfrac{1}{g}\text{div}_y(P_\varepsilon u^\varepsilon) + \rho u^\varepsilon = f_\varepsilon(t,y,u^\varepsilon) + G_\varepsilon(t,y) + R_\varepsilon(t,y,u^\varepsilon)\zeta_\delta(\theta_t\omega), \\ P_\varepsilon u^\varepsilon \cdot \nu = 0, \quad y \in \partial\mathcal{O}, \\ u^\varepsilon(\tau,y) = u^\varepsilon_\tau(y) = \hat{u}^\varepsilon_\tau(T_\varepsilon^{-1}(y)), \end{cases} \tag{2.8}$$

where ν is the unit outward normal vector to $\partial\mathcal{O}$.

To write problem (2.8) as an abstract system, we introduce an inner product $(\cdot,\cdot)_{H_g(\mathcal{O})}$ on $L^2(\mathcal{O})$ by

$$(u,v)_{H_g(\mathcal{O})} = \int_{\mathcal{O}} g u \bar{v} dy, \quad \text{for all } \ u, v \in L^2(\mathcal{O}),$$

and denote by $H_g(\mathcal{O})$ the space equipped with this inner product. Since g is a continuous function on $\overline{\mathcal{Q}}$, and satisfies (1.4), one can easily show that $H_g(\mathcal{O})$ is a Hilbert space with the norm equivalent to the natural norm of $L^2(\mathcal{O})$. For $0<\varepsilon\le 1$, we introduce a bilinear form $a_\varepsilon(\cdot,\cdot): H^1(\mathcal{O})\times H^1(\mathcal{O})\to\mathbb{C}$ given by

$$a_\varepsilon(u,v)=\big(J^*\nabla_y u, J^*\nabla_y v\big)_{H_g(\mathcal{O})}\quad \text{for } u,v\in H^1(\mathcal{O}), \tag{2.9}$$

where $J^*\nabla_y u=\big(u_{y_1}-\frac{g_{y_1}}{g}y_{n+1}u_{y_{n+1}},\cdots,u_{y_n}-\frac{g_{y_n}}{g}y_{n+1}u_{y_{n+1}},\frac{1}{\varepsilon g}u_{y_{n+1}}\big)$. Let $H^1_\varepsilon(\mathcal{O})$ be the space $H^1(\mathcal{O})$ endowed with the norm

$$\|u\|_{H^1_\varepsilon(\mathcal{O})}=\left(\|u\|^2_{H^1(\mathcal{O})}+\frac{1}{\varepsilon^2}\|u_{y_{n+1}}\|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}. \tag{2.10}$$

It yields from [10] that there exist positive constants ε_0, η_1 and η_2 such that for all $0<\varepsilon<\varepsilon_0$ and $u\in H^1(\mathcal{O})$,

$$\eta_1\|u\|^2_{H^1_\varepsilon(\mathcal{O})}\le a_\varepsilon(u,u)+\|u\|^2_{L^2(\mathcal{O})}\le\eta_2\|u\|^2_{H^1_\varepsilon(\mathcal{O})}. \tag{2.11}$$

Denoted by A_ε, the linear self-adjoint operator is

$$A_\varepsilon u=-\frac{1}{g}\mathrm{div}_y(P_\varepsilon u),\quad u\in D(A_\varepsilon)=\Big\{u\in H^2(\mathcal{O}): P_\varepsilon u\cdot\nu=0 \text{ on } \partial\mathcal{O}\Big\}.$$

Then, we have

$$a_\varepsilon(u,v)=(A_\varepsilon u,v)_{H_g(\mathcal{O})},\quad \forall u\in D(A_\varepsilon),\quad \forall v\in H^1(\mathcal{O}). \tag{2.12}$$

Note that system (2.8) can be rewritten as

$$\begin{cases}\dfrac{\partial u^\varepsilon}{\partial t}+(1+\mathrm{i}\mu)A_\varepsilon u^\varepsilon+\rho u^\varepsilon=f_\varepsilon(t,y,u^\varepsilon)+G_\varepsilon(t,y)\\ \qquad\qquad\qquad\qquad +R_\varepsilon(t,y,u^\varepsilon)\zeta_\delta(\theta_t\omega),\ y\in\mathcal{O},\ t>\tau,\\ u^\varepsilon(\tau)=u^\varepsilon_\tau.\end{cases} \tag{2.13}$$

For systems (1.5)-(1.6), we introduce an inner product $(\cdot,\cdot)_{H_g(\mathcal{Q})}$ on $L^2(\mathcal{Q})$ by

$$(u,v)_{H_g(\mathcal{Q})}=\int_{\mathcal{Q}}gu\bar{v}dy^*,\quad \text{for all}\ \ u,v\in L^2(\mathcal{Q}),$$

and denote by $H_g(\mathcal{Q})$ the space $L^2(\mathcal{Q})$ equipped with this product. Let $a_0(\cdot,\cdot)$: $H^1(\mathcal{Q})\times H^1(\mathcal{Q})\to\mathbb{C}$ be a bilinear form given by

$$a_0(u,v)=\int_{\mathcal{Q}}g\nabla u\cdot\nabla\bar{v}dy^*.$$

Denoted by A_0, the unbounded operator on $H_g(\mathcal{Q})$ with domain $D(A_0)=\{u\in H^2(\mathcal{Q}), \frac{\partial u}{\partial\nu_0}=0 \text{ on } \partial\mathcal{Q}\}$ is defined by

$$A_0u=-\frac{1}{g}\sum_{i=1}^n(gu_{y_i})_{y_i},\quad u\in D(A_0).$$

Then, one has

$$a_0(u,v)=(A_0u,v)_{H_g(\mathcal{Q})},\quad \forall u\in D(A_0),\ \forall v\in H^1(\mathcal{Q}).$$

Therefore, systems (1.5)-(1.6) can be rewritten as

$$\begin{cases} \dfrac{\partial u^0}{\partial t} + (1+\mathrm{i}\mu)A_0 u^0 + \rho u^0 = f_0(t,y^*,u^0) + G_0(t,y^*) \\ \qquad\qquad\qquad\qquad + R_0(t,y^*,u^0)\zeta_\delta(\theta_t\omega),\ y^* \in \mathcal{Q},\ t > \tau, \\ u^0(\tau) = u^0_\tau. \end{cases} \tag{2.14}$$

For the rest of this paper, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}. \tag{2.15}$$

Then, $\big(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}}\big)$ is a metric dynamical system. It follows from [3] that there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset of full measure (still denoted by Ω) such that

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0 \ \text{ for every } \omega \in \Omega. \tag{2.16}$$

Throughout this paper, for every $\omega \in \Omega$ and $\delta \in (0,1]$, we write

$$\zeta_\delta(\omega) = \frac{1}{\delta}\int_{-\infty}^0 e^{\frac{s}{\delta}} dW = -\frac{1}{\delta^2}\int_{-\infty}^0 e^{\frac{s}{\delta}}\omega(s)ds. \tag{2.17}$$

In addition, this process has the following properties from [8].

Lemma 2.1. *For every $\omega \in \Omega$, the mapping $t \to \zeta_\delta(\theta_t\omega)$ is continuous, and for every $0 < \delta \le 1$,*

$$\lim_{t\to\pm\infty} \frac{|\zeta_\delta(\theta_t\omega)|}{|t|} = 0 \tag{2.18}$$

and

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \zeta_\delta(\theta_s\omega)ds = 0 \ \text{uniformly for } 0 < \delta \le 1. \tag{2.19}$$

Lemma 2.2. *Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then, for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\tau,\omega,T,\varepsilon)$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau+T]$,*

$$\left|\int_0^t \zeta_\delta(\theta_s\omega)ds - \omega(t)\right| < \varepsilon. \tag{2.20}$$

By Lemma 2.2 and the continuity of ω, one has the following estimates immediately.

Corollary 2.1. *Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then, there exist $\delta_0 = \delta_0(\tau,\omega,T)$ and $M = M(\tau,\omega,T) > 0$ such that for all $0 < \delta < \delta_0$ and $t \in [\tau, \tau+T]$,*

$$\left|\int_0^t \zeta_\delta(\theta_s\omega)ds\right| \le M. \tag{2.21}$$

Note that (2.13) is a deterministic equation which is parametrized by $\omega \in \Omega$. By the Galerkin method, one can show that if f satisfies (2.2)–(2.4), then, for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $u^\varepsilon_\tau \in L^2(\mathcal{O})$, system (2.13) has a unique solution $u^\varepsilon(\cdot,\tau,\omega,u^\varepsilon_\tau) \in$

$C\big([\tau,\infty),L^2(\mathcal{O})\big)\cap L^2\big((\tau,\tau+T),H^1(\mathcal{O})\big)$ for every $T>0$. Furthermore, one may show that $u^\varepsilon(t,\tau,\omega,u_\tau^\varepsilon)$ is $\big(\mathcal{F},\mathcal{B}(L^2(\mathcal{O}))\big)$-measure in $\omega\in\Omega$ and continuous in u_τ^ε with respect to the norm of $L^2(\mathcal{O})$. Now, we define a mapping $\Psi_\varepsilon:\mathbb{R}^+\times\mathbb{R}\times\Omega\times L^2(\mathcal{O})\to L^2(\mathcal{O})$ for the problem (2.13). Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $u_\tau^\varepsilon\in L^2(\mathcal{O})$. Let

$$\Psi_\varepsilon(t,\tau,\omega,u_\tau^\varepsilon)=u^\varepsilon(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau^\varepsilon). \tag{2.22}$$

As stated in [26], the mapping Ψ_ε is a continuous cocycle on $L^2(\mathcal{O})$ over the space $\big(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}}\big)$.

Let $H_\varepsilon:L^2(\mathcal{O}_\varepsilon)\to L^2(\mathcal{O})$ be an affine mapping of the form

$$(H_\varepsilon\hat{u}(y))=\hat{u}(T_\varepsilon^{-1}y),\quad \forall\hat{u}\in L^2(\mathcal{O}_\varepsilon).$$

Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $\hat{u}_\tau^\varepsilon\in L^2(\mathcal{O}_\varepsilon)$, we can define a continuous cocycle $\hat{\Psi}_\varepsilon$ for problem (2.1) by the formula

$$\hat{\Psi}_\varepsilon(t,\tau,\omega,u_\tau^\varepsilon)=H_\varepsilon^{-1}\Psi_\varepsilon(t,\tau,\omega,H_\varepsilon u_\tau^\varepsilon),$$

where Ψ_ε is the continuous cocycle for problem (2.13) on $L^2(\mathcal{O})$.

According to the arguments, it is easy to see that system (2.14) generates a continuous cocycle $\Psi_0(t,\tau,\omega,u_\tau^0)$ in the space $L^2(\mathcal{Q})$. Denote $X_\varepsilon=L^2(\mathcal{O}_\varepsilon)$, $X_0=L^2(\mathcal{Q})$ and $X_1=L^2(\mathcal{O})$. For each $i=\varepsilon,0$ or 1, let $D_i=\{D_i(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ be a family of nonempty subsets of X_i. Then, D_i is called tempered (or subexponentially growing) if for every $c>0$, the following holds:

$$\lim_{t\to-\infty}e^{ct}\|D_i(\tau+t,\theta_t\omega)\|_{X_i}=0,$$

where $\|D_i\|_{X_i}=\sup_{x\in D_i}\|x\|_{X_i}$. This definition is a straightforward extension of the concept of tempered random subsets for autonomous random dynamical systems. We also denote by $\mathcal{D}_i$, the collection of all families of tempered nonempty subsets of X_i, i.e.,

$$\mathcal{D}_i=\{D_i=\{D_i(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}:D_i\text{ is tempered in }X_i\}.$$

The following condition will be needed when deriving uniform estimates of solutions

$$\int_{-\infty}^{\tau}e^{\rho s}\left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)ds<\infty,\quad\forall\tau\in\mathbb{R}. \tag{2.23}$$

When constructing tempered pullback attractors for the cocycle Ψ_ε, we will assume for any $\sigma>0$ and $\tau\in\mathbb{R}$,

$$\lim_{r\to-\infty}e^{\sigma r}\int_{-\infty}^{\tau}e^{\rho s}\left(\|G(s+r,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s+r,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_3(s+r,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)ds=0. \tag{2.24}$$

3. Uniform estimates of solutions

In this section, we derive uniform estimates of solutions for system (2.13). To get started, we derive the estimates of solutions for problem (2.13) in $H_g(\mathcal{O})$.

Lemma 3.1. *Assume that* (2.2), (2.5) *and* (2.23) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D_1 = \{D_1(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D_1, \delta) > 0$, *independent of* ε, *such that for all* $t \geq T$, *the solution* u^ε *of system* (2.13) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned} u^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, u^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} \leq 1 + M_1 \int_{-\infty}^0 e^{\rho s} \Big(\|G(s+\tau, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \\ + \psi_1(s+\tau, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s+\tau, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega) \Big) ds \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\int_{\tau-t}^{\tau} e^{\rho(s-\tau)} \Big(\|u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, v^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} + \|u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})} \Big) ds \\ &\leq M_2 + M_2 \int_{-\infty}^0 e^{\rho s} \Big(\|G(s+\tau, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s+\tau, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} \\ &\quad + \|\psi_3(s+\tau, \cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega) \Big) ds, \end{aligned} \tag{3.2}$$

where $u^\varepsilon_{\tau-t} \in D_1(\tau - t, \theta_{-t}\omega)$, *and* M_1 *and* M_2 *are positive constants independent of* $\tau, \omega, \varepsilon$ *and* D_1.

Proof. Taking the inner product of (2.13) with u^ε in $H_g(\mathcal{O})$ and taking the real part, we obtain

$$\begin{aligned} &\frac{d}{dt}\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + 2\mathrm{Re}(1 + \mathrm{i}\mu)(A_\varepsilon u^\varepsilon, u^\varepsilon)_{H_g(\mathcal{O})} + 2\rho\|u^\varepsilon\|^2_{H_g(\mathcal{O})} \\ =&2\mathrm{Re}(f_\varepsilon(t, y, u^\varepsilon), u^\varepsilon)_{H_g(\mathcal{O})} + 2\mathrm{Re}(G_\varepsilon(t, y), u^\varepsilon)_{H_g(\mathcal{O})} \\ &+ 2\zeta_\delta(\theta_t\omega)\mathrm{Re}(R_\varepsilon(t, y, u^\varepsilon), u^\varepsilon)_{H_g(\mathcal{O})}. \end{aligned} \tag{3.3}$$

For the second term on the left-hand side of (3.3), applying (2.12), one has

$$2\mathrm{Re}(1 + \mathrm{i}\mu)(A_\varepsilon u^\varepsilon, u^\varepsilon)_{H_g(\mathcal{O})} = 2a_\varepsilon(u^\varepsilon, u^\varepsilon). \tag{3.4}$$

For the first term on the right-hand side of (3.3), using (2.2) and (1.4), we have

$$\begin{aligned} 2\mathrm{Re}(f_\varepsilon(t, y, u^\varepsilon), u^\varepsilon)_{H_g(\mathcal{O})} =&2\mathrm{Re} \int_{\mathcal{O}} g f(t, y^*, \varepsilon g(y^*)y_{n+1}, u^\varepsilon)\overline{u^\varepsilon} dy \\ \leq& -2\gamma \int_{\mathcal{O}} g|u^\varepsilon|^p dy + 2 \int_{\mathcal{O}} g\psi_1(t, y^*, \varepsilon g(y^*)y_{n+1}) dy \\ \leq& -2\gamma\beta_1 \int_{\mathcal{O}} |u^\varepsilon|^p dy + c\|\psi_1(t, \cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}. \end{aligned} \tag{3.5}$$

Applying Hölder's inequality and Young's inequality, the second term on the right-hand side of (3.3) is bounded by

$$\begin{aligned} 2\mathrm{Re}(G_\varepsilon(t, y), u^\varepsilon)_{H_g(\mathcal{O})} \leq& 2\|G_\varepsilon(t, y)\|_{H_g(\mathcal{O})}\|u^\varepsilon\|_{H_g(\mathcal{O})} \\ \leq& \frac{1}{2}\rho\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \frac{2}{\rho}\|G_\varepsilon(t, y)\|^2_{H_g(\mathcal{O})} \end{aligned}$$

$$\leq \frac{1}{2}\rho\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + c\|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}. \tag{3.6}$$

Applying (2.5), (1.4) and Young's inequality, the last term on the right-hand side of (3.3) is bounded by

$$\begin{aligned}
&2\zeta_\delta(\theta_t\omega)\mathrm{Re}(R_\varepsilon(t,y,u^\varepsilon),u^\varepsilon)_{H_g(\mathcal{O})} \\
&\leq -2\lambda\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} g|u^\varepsilon|^q dy + 2\zeta_\delta(\theta_t\omega)\int_{\mathcal{O}} g\psi_3(t,y^*,\varepsilon g(y^*)y_{n+1})dy \\
&\leq \gamma\beta_1\|u^\varepsilon\|^p_p + c|\zeta_\delta(\theta_t\omega)|^{\frac{p}{p-q}} + c|\zeta_\delta(\theta_t\omega)|^2 + c\|\psi_3(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}.
\end{aligned} \tag{3.7}$$

By (3.3)–(3.7), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \frac{3\rho}{2}\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + 2a_\varepsilon(u^\varepsilon,u^\varepsilon) + \gamma\beta_1\|u^\varepsilon\|^p_p \\
&\leq c\left(\|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(t,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_t\omega)\right),
\end{aligned} \tag{3.8}$$

where $\eta_\delta(\theta_t\omega) = |\zeta_\delta(\theta_t\omega)|^{\frac{p}{p-q}} + |\zeta_\delta(\theta_t\omega)|^2$. Multiplying (3.8) by $e^{\rho t}$ and then integrating the resulting inequality on $(\tau-t,\tau)$ with $\tau\geq 0$, one has, for every $\omega\in\Omega$,

$$\begin{aligned}
&\|u^\varepsilon(\tau,\tau-t,\omega,u^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} \\
&+ 2\int_{\tau-t}^{\tau} e^{\rho(s-\tau)}a_\varepsilon(u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t}),u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t}))ds \\
&+ \frac{1}{2}\rho\int_{\tau-t}^{\tau} e^{\rho(s-\tau)}\|u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})}ds \\
&+ \gamma\beta_1\int_{\tau-t}^{\tau} e^{\rho(s-\tau)}\|u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}ds \\
&\leq e^{-\rho t}\|u^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} + ce^{-\rho\tau}\int_{-\infty}^{\tau} e^{\rho s}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} \\
&\quad + \psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega)\Big)ds.
\end{aligned} \tag{3.9}$$

Now, replacing ω by $\theta_{-\tau}\omega$ in (3.9), we get

$$\begin{aligned}
&\|u^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})} \\
&+ 2\int_{\tau-t}^{\tau} e^{\rho(s-\tau)}a_\varepsilon(u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t}),u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t}))ds \\
&+ \frac{1}{2}\rho\int_{\tau-t}^{\tau} e^{\rho(s-\tau)}\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^2_{H_g(\mathcal{O})}ds \\
&+ \gamma\beta_1\int_{\tau-t}^{\tau} e^{\rho(s-\tau)}\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}ds \\
&\leq e^{-\rho t}\|u^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} + ce^{-\rho\tau}\int_{-\infty}^{\tau} e^{\rho s}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} \\
&\quad + \psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_{s-\tau}\omega)\Big)ds \\
&\leq e^{-\rho t}\|u^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})} + c\int_{-\infty}^{0} e^{\rho s}\eta_\delta(\theta_s\omega)ds + c\int_{-\infty}^{0} e^{\rho s}\Big(\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} \\
&\quad + \|\psi_1(s+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\Big)ds.
\end{aligned} \tag{3.10}$$

Note that $u^\varepsilon_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ and D_1 is tempered. We have $e^{-\rho t}\|u^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})}\to 0$, as $t\to\infty$. Thus, there exists $T=T(\tau,\omega,D_1,\delta)>0$ such that for all $t\geq T$, $e^{-\rho t}\|u^\varepsilon_{\tau-t}\|^2_{H_g(\mathcal{O})}\leq 1$. Due to (2.18), the second term on the right-hand side of (3.10) is well-defined. Then, the lemma follows immediately from (3.10) and (2.23). □

As a consequence of Lemma 3.1, we obtain the following inequality which is useful for deriving the uniform estimates of solutions in $H^1_\varepsilon(\mathcal{O})$.

Lemma 3.2. *Assume that* (2.2), (2.5) *and* (2.23) *hold. Then, there exists* $\varepsilon_0>0$ *such that for every* $0<\varepsilon<\varepsilon_0$, $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $D_1=\{D_1(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, *there exists* $T_1=T_1(\tau,\omega,D_1,\delta)\geq 1$, *independent of* ε, *such that for all* $t\geq T_1$, *the solution* u^ε *of system* (2.13) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned}&\int_{\tau-1}^{\tau}\left(\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}+\|u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^p_{L^p(\mathcal{O})}\right)ds\\&\leq M_3+M_3\int_{-\infty}^{0}e^{\rho s}\left(\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}\right.\\&\quad\left.+\psi_3(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\eta_\delta(\theta_s\omega)\right)ds,\end{aligned}\tag{3.11}$$

where $u^\varepsilon_{\tau-t}\in D_1(\tau-t,\theta_{-t}\omega)$ *and* M_3 *is a positive constant independent of* τ,ω,ε *and* D_1.

The following inequality is needed to deduce the uniform estimates of solutions u^ε in $H^1_\varepsilon(\mathcal{O})$.

Lemma 3.3. *Assume that* (2.2)–(2.4) *hold. One has, for* $u\in D(A_\varepsilon)$,

$$Re(f_\varepsilon(t,y,u),A_\varepsilon u)_{H_g(\mathcal{O})}\leq M\left(a_\varepsilon(u,u)+\|\psi_2\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right),$$

where M *is a positive constant independent of* ε.

Proof. By (2.9) and (2.12), we infer that

$$\begin{aligned}&\mathrm{Re}\left(f_\varepsilon(t,y,u),A_\varepsilon u\right)_{H_g(\mathcal{O})}=\mathrm{Re}\ a_\varepsilon\left(f_\varepsilon(t,y,u),u\right)\\&=\mathrm{Re}\sum_{i=1}^{n}\int_{\mathcal{O}}\left(f_{\varepsilon y_i}+f_{\varepsilon u}u_{y_i}-\frac{g_{y_i}}{g}y_{n+1}(f_{\varepsilon y_{n+1}}+f_{\varepsilon u}u_{y_{n+1}})\right)\left(\bar u_{y_i}-\frac{g_{y_i}}{g}y_{n+1}\bar u_{y_{n+1}}\right)gdy\\&\quad+\mathrm{Re}\int_{\mathcal{O}}\frac{1}{\varepsilon^2 g}(f_{\varepsilon y_{n+1}}(t,y,u)+f_{\varepsilon u}(t,y,u)u_{y_{n+1}})\bar u_{y_{n+1}}dy\\&=\mathrm{Re}\sum_{i=1}^{n}\int_{\mathcal{O}}f_{\varepsilon u}(t,y,u)\left|u_{y_i}-\frac{g_{y_i}}{g}y_{n+1}u_{y_{n+1}}\right|^2gdy\\&\quad+\mathrm{Re}\sum_{i=1}^{n}\int_{\mathcal{O}}\left(f_{\varepsilon y_i}(t,y,u)-\frac{g_{y_i}}{g}y_{n+1}f_{\varepsilon y_{n+1}}(t,y,u)\right)\left(\bar u_{y_i}-\frac{g_{y_i}}{g}y_{n+1}\bar u_{y_{n+1}}\right)gdy\\&\quad+\mathrm{Re}\int_{\mathcal{O}}\frac{1}{\varepsilon^2 g}f_{\varepsilon y_{n+1}}(t,y,u)\bar u_{y_{n+1}}dy+\int_{\mathcal{O}}\frac{1}{\varepsilon^2 g^2}f_{\varepsilon u}(t,y,u)|u_{y_{n+1}}|^2gdy.\end{aligned}$$

Together with (2.3) and (2.4), one has

$$\mathrm{Re}\left(f_\varepsilon(t,y,u),A_\varepsilon u\right)_{H_g(\mathcal{O})}=\mathrm{Re}a_\varepsilon\left(f_\varepsilon(t,y,u),u\right)$$

$$\leq \beta\ a_\varepsilon(u,u) + \int_{\mathcal{O}} \frac{1}{\varepsilon^2 g} \left|f_{\varepsilon y_{n+1}}(t,y,u)\right| \left|u_{y_{n+1}}\right| dy$$
$$+\sum_{i=1}^{n} \int_{\mathcal{O}} \left|f_{\varepsilon y_i}(t,y,u) - \frac{g_{y_i}}{g} y_{n+1} f_{\varepsilon y_{n+1}}(t,y,u)\right| \left|u_{y_i} - \frac{g_{y_i}}{g} y_{n+1} u_{y_{n+1}}\right| g dy$$
$$\leq \beta\ a_\varepsilon(u,u) + \frac{1}{2}\ a_\varepsilon(u,u) + \frac{1}{2}\int_{\mathcal{O}} \frac{1}{\varepsilon^2 g^2} |f_{\varepsilon y_{n+1}}(t,y,u)|^2 g dy$$
$$+\frac{1}{2}\sum_{i=1}^{n} \int_{\mathcal{O}} \left|f_{\varepsilon y_i}(t,y,u) - \frac{g_{y_i}}{g} y_{n+1} f_{\varepsilon y_{n+1}}(t,y,u)\right|^2 g dy$$
$$\leq \left(\beta + \frac{1}{2}\right)\ a_\varepsilon(u,u) + c\|\psi_2\|^2_{L^\infty(\widetilde{\mathcal{O}})}.$$

This completes the proof.

□

Similar to Lemma 3.3, we obtain the following lemma for the function $R_\varepsilon(t,y,u)$.

Lemma 3.4. *Assume that* (2.5)–(2.7) *hold. One has, for* $u \in D(A_\varepsilon)$,

$$Re(R_\varepsilon(t,y,u), A_\varepsilon u)_{H_g(\mathcal{O})} \leq M\left(a_\varepsilon(u,u) + \|\psi_4\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right),$$

where M *is a positive constant independent of* ε.

Lemma 3.5. *Assume that* (2.2)–(2.7) *and* (2.23) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *there exists* $T_1 = T_1(\tau,\omega,D_1,\delta) \geq 1$, *independent of* ε, *such that for all* $t \geq T_1$, *the solution* u^ε *of system* (2.13) *with* ω *replaced by* $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned}
&\|u^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})}\\
\leq& M_4 + M_4\int_{-\infty}^{0} e^{\rho s}\left(\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}\right.\\
&\left.+\|\psi_3(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega)\ \right) ds,
\end{aligned} \tag{3.12}$$

where $u^\varepsilon_{\tau-t} \in D_1(\tau-t,\theta_{-t}\omega)$, *and* M_4 *is a positive constant independent of* ε.

Proof. Taking the inner product of (2.13) with $A_\varepsilon u^\varepsilon$ in $H_g(\mathcal{O})$ and taking the real part, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\ a_\varepsilon(u^\varepsilon,u^\varepsilon) + \|A_\varepsilon u^\varepsilon\|^2_{H_g(\mathcal{O})} + \rho\ a_\varepsilon(u^\varepsilon,u^\varepsilon)\\
&= \mathrm{Re}(f_\varepsilon(t,y,u^\varepsilon), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})} + \mathrm{Re}(G_\varepsilon(t,y), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})}\\
&\quad + \zeta_\delta(\theta_t\omega)\mathrm{Re}(R_\varepsilon(t,y,u^\varepsilon), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})}.
\end{aligned} \tag{3.13}$$

For the first term of the right-hand side of (3.13), by Lemma 3.3, we have

$$\mathrm{Re}(f_\varepsilon(t,y,u^\varepsilon), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})} \leq c a_\varepsilon(u^\varepsilon,u^\varepsilon) + c\|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}. \tag{3.14}$$

For the second term of the right-hand side of (3.13), applying Young's inequality, we get

$$\mathrm{Re}(G_\varepsilon(t,y), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})} \leq \frac{1}{2}\|A_\varepsilon u^\varepsilon\|^2_{H_g(\mathcal{O})} + \frac{1}{2}\|G_\varepsilon(t,y)\|^2_{H_g(\mathcal{O})}$$

$$\leq \frac{1}{2}\|A_\varepsilon u^\varepsilon\|^2_{H_g(\mathcal{O})} + c\|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}. \tag{3.15}$$

For the last term of the right-hand side of (3.13), by Lemma 3.3, we deduce

$$\begin{aligned}&\zeta_\delta(\theta_t\omega)\mathrm{Re}(R_\varepsilon(t,y,u^\varepsilon), A_\varepsilon u^\varepsilon)_{H_g(\mathcal{O})}\\ &\leq c\,|\zeta_\delta(\theta_t\omega)|\,a_\varepsilon(u^\varepsilon,u^\varepsilon) + c\,|\zeta_\delta(\theta_t\omega)|\,\|\psi_4(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}.\end{aligned} \tag{3.16}$$

By (3.13)–(3.16), one has

$$\begin{aligned}&\frac{d}{dt}\,a_\varepsilon(u^\varepsilon,u^\varepsilon) + \|A_\varepsilon u^\varepsilon\|^2_{H_g(\mathcal{O})} + 2\rho a_\varepsilon(u^\varepsilon,u^\varepsilon) \leq c\,(1+|\zeta_\delta(\theta_t\omega)|)\,a_\varepsilon(u^\varepsilon,u^\varepsilon)\\ &\qquad + c\left(\|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + |\zeta_\delta(\theta_t\omega)|\,\|\psi_4(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right),\end{aligned} \tag{3.17}$$

which implies

$$\begin{aligned}&\frac{d}{dt}\,a_\varepsilon(u^\varepsilon,u^\varepsilon) \leq c\,(1+|\zeta_\delta(\theta_t\omega)|)\,a_\varepsilon(u^\varepsilon,u^\varepsilon)\\ &\qquad + c\left(\|\psi_2(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + |\zeta_\delta(\theta_t\omega)|\,\|\psi_4(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(t,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right).\end{aligned} \tag{3.18}$$

Given $t\in\mathbb{R}^+$, $\tau\in\mathbb{R}$ and $s\in(\tau-1,\tau)$, by integrating (3.18) on (s,τ), we have

$$\begin{aligned}&a_\varepsilon\left(u^\varepsilon(\tau,\tau-t,\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(\tau,\tau-t,\omega,u^\varepsilon_{\tau-t})\right)\\ &\leq a_\varepsilon\left(u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t})\right)\\ &\quad + c\int_s^\tau (1+|\zeta_\delta(\theta_\xi\omega)|)\,a_\varepsilon\left(u^\varepsilon(\xi,\tau-t,\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(\xi,\tau-t,\omega,u^\varepsilon_{\tau-t})\right)d\xi\\ &\quad + c\int_s^\tau \left(\|\psi_2(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + |\zeta_\delta(\theta_\xi\omega)|\,\|\psi_4(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)d\xi.\end{aligned}$$

Now, we integrate the above with respect to s on $(\tau-1,\tau)$ to obtain

$$\begin{aligned}&a_\varepsilon\left(u^\varepsilon(\tau,\tau-t,\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(\tau,\tau-t,\omega,u^\varepsilon_{\tau-t})\right)\\ &\leq \int_{\tau-1}^\tau a_\varepsilon\left(u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t})\right)ds\\ &\quad + c\int_{\tau-1}^\tau (1+|\zeta_\delta(\theta_s\omega)|)\,a_\varepsilon\left(u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(s,\tau-t,\omega,u^\varepsilon_{\tau-t})\right)ds\\ &\quad + c\int_{\tau-1}^\tau \left(\|\psi_2(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + |\zeta_\delta(\theta_s\omega)|\,\|\psi_4(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)ds.\end{aligned}$$

Replacing ω by $\theta_{-\tau}\omega$ gives

$$\begin{aligned}&a_\varepsilon\left(u^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(\tau,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\right)\\ &\leq (c_1+1)\int_{\tau-1}^\tau a_\varepsilon\left(u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t}), u^\varepsilon(s,\tau-t,\theta_{-\tau}\omega,u^\varepsilon_{\tau-t})\right)ds\\ &\quad + c_2\int_{\tau-1}^\tau \left(\|\psi_2(\xi,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_4(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}\right)ds,\end{aligned} \tag{3.19}$$

where $c_1 = c_1(\tau,\omega) > 0$ and $c_2 = c_2(\tau,\omega) > 0$. Together with Lemma 3.2, we obtain the result.

□

4. Existence of pullback random attractors

We establish the existence of $\mathcal{D}_1$-pullback attractor for the cocycle Ψ_ε associated with the stochastic problem (2.13) and $\mathcal{D}_0$-pullback attractor for the cocycle Ψ_0 associated with the stochastic problem (2.14) respectively. First, we show that the problem (2.13) has a tempered pullback absorbing set as stated below.

Lemma 4.1. *Suppose that* (2.2)–(2.7), (2.23) *and* (2.24) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, *the continuous cocycle* Ψ_ε *associated with problem* (2.13) *has a closed measurable* $\mathcal{D}_1$*-pullback absorbing set* $K \in \mathcal{D}_1$ *which is given by, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$K(\tau,\omega) = \left\{ u^\varepsilon \in L^2(\mathcal{O}) : \|u^\varepsilon\|^2_{L^2(\mathcal{O})} \le L(\tau,\omega) \right\},$$

where

$$L(\tau,\omega) = M' + M' \times \int_{-\infty}^0 e^{\rho s} \left(\|G(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega) \right) ds,$$

and M' *is a positive constant independent of* ε.

Proof. For $u^\varepsilon_{\tau-t} \in D_1(\tau - t, \theta_{-t}\omega)$, by Lemma 3.5, we obtain

$$\|u^\varepsilon(\tau, \tau - t, \theta_{-\tau}\omega, u^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} \le L(\tau,\omega). \tag{4.1}$$

Therefore, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D_1 \in \mathcal{D}_1$, there exists $T = T(\tau,\omega,\delta,D_1) \ge 1$, independent of ε, such that for all $t \ge T$,

$$\Psi_\varepsilon(t, \tau - t, \theta_{-t}\omega, D_1(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau,\omega).$$

Next, we prove that $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered. Let σ be an arbitrary positive constant and consider

$$\begin{aligned}
&\lim_{r\to-\infty} e^{\sigma r}\|K(\tau + r, \theta_r\omega)\|^2_{L^2(\mathcal{O})} \le \lim_{r\to-\infty} e^{\sigma r} L(\tau + r, \theta_r\omega) \\
&= \lim_{r\to-\infty} M' e^{\sigma r} + \lim_{r\to-\infty} M' e^{\sigma r} \int_{-\infty}^0 e^{\rho s}\eta_\delta(\theta_{s+r}\omega) ds + \lim_{r\to-\infty} M' e^{\sigma r} \int_{-\infty}^0 e^{\rho s}\mathcal{K}_1 ds \\
&= M' \lim_{r\to-\infty} e^{\sigma r} + M' \lim_{r\to-\infty} e^{(\sigma-\rho) r} \int_{-\infty}^t e^{\rho s}\eta_\delta(\theta_s\omega) ds + M' e^{-\rho\tau} \lim_{r\to-\infty} e^{\sigma r} \int_{-\infty}^\tau e^{\rho s}\mathcal{K}_2 ds,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_1 &= \|G(s+\tau+r,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s+\tau+r,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s+\tau+r,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}, \\
\mathcal{K}_2 &= \|G(s+r,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s+\tau,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s+\tau,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}.
\end{aligned}$$

With (2.24) and Lemma 2.1, we deduce

$$\lim_{r\to-\infty} e^{\sigma r}\|K(\tau + r, \theta_r\omega)\|^2_{L^2(\mathcal{O})} = 0.$$

Hence, $K(\tau,\omega)$ is tempered in $L^2(\mathcal{O})$. On the other hand, it is evident that, for every $\tau \in \mathbb{R}$, $L(\tau,\cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. Consequently, K is a closed measurable $\mathcal{D}_1$-pullback absorbing set for Ψ_ε in $\mathcal{D}_1$.

□

Lemma 4.2. *Suppose that* (2.2)–(2.7) *and* (2.23) *hold. Then, for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D_1 = \{D_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$, *the sequence* $\Psi_\varepsilon(t_n, \tau - t_n, \theta_{-t_n}\omega, u^\varepsilon_{\tau-t,n})$ *has a convergent subsequence in* $L^2(\mathcal{O})$, *provided* $t_n \to \infty$ *and* $u^\varepsilon_{\tau-t,n} \in D_1(\tau - t_n, \theta_{-t_n}\omega)$.

Proof. First, for $u^\varepsilon_{\tau-t} \in D_1(\tau - t, \theta_{-t}\omega)$, by Lemmas 3.1, 3.2 and 3.5, there exist $T_1 = T_1(\tau,\omega,D,\delta) \geq 1$ and $c_1(\tau,\omega,\delta) > 0$ such that for all $t \geq T_1$,

$$\|\Psi_\varepsilon(t, \tau - t, \theta_{-t}\omega, u^\varepsilon_{\tau-t})\|^2_{H^1_\varepsilon(\mathcal{O})} \leq c_1. \tag{4.2}$$

Let $N_1 = N_1(\tau,\omega,D,\delta) \geq 1$ be large enough such that $t_n \geq T_1$ for $n \geq N_1$. Then, by (4.2), for all $n \geq N_1$,

$$\|\Psi_\varepsilon(t_n, \tau - t_n, \theta_{-t_n}\omega, u^\varepsilon_{\tau-t_n})\|^2_{H^1_\varepsilon(\mathcal{O})} \leq c_1. \tag{4.3}$$

By the compactness of embedding $H^1_\varepsilon(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, it follows from (4.3) that there is $\phi \in L^2(\mathcal{O})$ such that, up to some subsequence,

$$\Psi_\varepsilon(t_n, \tau - t_n, \theta_{-t_n}\omega, u^\varepsilon_{\tau-t,n}) \to \phi \ \text{ strongly in } L^2(\mathcal{O}),$$

as desired.

□

Theorem 4.1. *Suppose that* (2.2)–(2.7), (2.23) *and* (2.24) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, *the continuous cocycle* Ψ_ε *has a unique* $\mathcal{D}_1$*-pullback attractor* $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_1$ *in* $L^2(\mathcal{O})$. *In addition, if* G, f, ψ_1, ψ_2 *are* T*-periodic with respect to* t *with* $T > 0$, *then the attractor* $\mathcal{A}_\varepsilon$ *is also* T*-periodic.*

Proof. From Lemma 4.1, we know that Ψ_ε has a closed measurable $\mathcal{D}_1$-pullback absorbing set K. Applying Lemma 4.2, we get that Ψ_ε is $\mathcal{D}_1$-pullback asymptotically compact in $L^2(\mathcal{O})$. Hence, we obtain the existence of a unique $\mathcal{D}_1$-pullback attractor for the cocycle Ψ_ε following from [27] immediately. If G, f, ψ_1, ψ_2 are T-periodic with respect to t, then the continuous cocycle Ψ_ε and the absorbing set K are also T-periodic, which implies the T-periodicity of the attractor.

□

Similar results also hold for the solutions of problem (2.14), and more precisely, we have the following theorem.

Theorem 4.2. *Suppose that* (2.2)–(2.7), (2.23) *and* (2.24) *hold. Then, the continuous cocycle* Ψ_0 *has a unique* $\mathcal{D}_0$*-pullback attractor* $\mathcal{A}_0 = \{\mathcal{A}_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_0$ *in* $L^2(\mathcal{O})$. *In addition, if* G, f, ψ_1, ψ_2 *are* T*-periodic with respect to* t *with* $T > 0$, *then the attractor* $\mathcal{A}_0$ *is also* T*-periodic.*

5. Upper-semicontinuity of random attractors

Now, we establish the upper semicontinuity of the random attractor $\mathcal{A}_\varepsilon$. To that end, we first derive the uniform estimates of solutions.

Lemma 5.1. *Suppose that* (2.2)–(2.7) *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ *and* $u^\varepsilon_\tau \in H_g(\mathcal{O})$, *the solution* u^ε *of* (2.13) *satisfies, for all* $t \in [\tau, \tau + T]$,

$$\int_\tau^t \|u^\varepsilon(s,\tau,\omega,u^\varepsilon_\tau)\|^2_{H^1_\varepsilon(\mathcal{O})} ds \leq \hat{M}\|u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})}$$

$$+\hat{M}\int_{\tau}^{\tau+T}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\eta_\delta(\theta_s\omega)\Big)\,ds,$$

where $\hat{M}$ is a positive constant independent of ε.

Proof. Multiplying (3.8) by $e^{\rho t}$ and then integrating the resulting inequality on (τ,t), we deduce that for every $\omega\in\Omega$ and $t\in[\tau,\tau+T]$,

$$\begin{aligned}
&\|u^\varepsilon(t,\tau,\omega,u^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})}+2\int_\tau^t e^{\rho(s-t)}a_\varepsilon(u^\varepsilon(s,\tau,\omega,u^\varepsilon_\tau),u^\varepsilon(s,\tau,\omega,u^\varepsilon_\tau))ds\\
&+\frac{1}{2}\rho\int_\tau^t e^{\rho(s-t)}\|u^\varepsilon(s,\tau,\omega,v^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})}ds+\gamma\beta_1\int_\tau^t e^{\rho(s-t)}\|u^\varepsilon(s,\tau,\omega,u^\varepsilon_\tau)\|^p_{L^p(\mathcal{O})}ds\\
&\le e^{-\rho(t-\tau)}\|u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})}\\
&+c\int_\tau^t e^{\rho(s-t)}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\eta_\delta(\theta_s\omega)\Big)\,ds\\
&\le \|u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})}\\
&+c\int_\tau^{\tau+T}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\eta_\delta(\theta_s\omega)\Big)\,ds,
\end{aligned}\tag{5.1}$$

which along with the same argument as that of Lemma 3.2 completes the proof. □

Similarly, we can obtain the following estimates.

Lemma 5.2. *Suppose that* (2.2)–(2.7) *hold. Then, for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, $T>0$ and $u^0_\tau\in H_g(\mathcal{O})$, the solution u^0 of* (2.14) *satisfies, for all $t\in[\tau,\tau+T]$,*

$$\begin{aligned}
&\int_\tau^t\|u^0(s,\tau,\omega,u^0_\tau)\|^2_{H^1(\mathcal{O})}ds\le\hat{M}\|u^0_\tau\|^2_{H_g(\mathcal{O})}\\
&\qquad+\hat{M}\int_{\tau}^{\tau+T}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})}+\|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})}+\eta_\delta(\theta_s\omega)\Big)\,ds,
\end{aligned}$$

where $\hat{M}$ is a positive constant independent of ε.

Given $u\in L^2(\mathcal{O})$, and let $\mathcal{M}u$ be the average function of u in y_{n+1} defined by

$$\mathcal{M}u=\int_0^1 u(y^*,y_{n+1})dy_{n+1}.$$

The following result on the average function can be found in [10].

Lemma 5.3. *If $u\in H^1(\mathcal{O})$, then $\mathcal{M}u\in H^1(\mathcal{Q})$ and $\|u-\mathcal{M}u\|_{H_g(\mathcal{O})}\le c\varepsilon\|u\|_{H^1_\varepsilon(\mathcal{O})}$, where c is a constant, independent of ε.*

In the sequel, we further assume that the functions f and G satisfy

$$\|f_\varepsilon(t,\cdot,s)-f_0(t,\cdot,s)\|_{L^2(\mathcal{O})}\le\varphi_1(t)\varepsilon,\quad\text{for all } t,s\in\mathbb{R},\tag{5.2}$$

$$\|G_\varepsilon(t,\cdot)-G_0(t,\cdot)\|_{L^2(\mathcal{O})}\le\varphi_2(t)\varepsilon,\quad\text{for all } t\in\mathbb{R}\tag{5.3}$$

and

$$\|R_\varepsilon(t,\cdot,s) - R_0(t,\cdot,s)\|_{L^2(\mathcal{O})} \leq \varphi_3(t)\varepsilon, \quad \text{for all } t, s \in \mathbb{R}, \tag{5.4}$$

where $\varphi_i(t) \in L^2_{loc}(\mathbb{R})$ for $i = 1, 2, 3$. Since $L^2(\mathcal{Q})$ can be embedded naturally into $L^2(\mathcal{O})$ as the subspace of functions independent of y_{n+1}, we can consider the cocycle Ψ_0 as a mapping from $L^2(\mathcal{Q})$ into $L^2(\mathcal{O})$. In this sense, we can compare Ψ_0 and Ψ_ε.

Theorem 5.1. *Suppose that* (2.2)–(2.7) *and* (5.2)–(5.4) *hold. Given* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and a positive number* $\hat{\eta}(\tau,\omega)$, *if* $u^\varepsilon_\tau \in H^1_\varepsilon(\mathcal{O})$ *such that* $\|u^\varepsilon_\tau\|_{H^1_\varepsilon(\mathcal{O})} \leq \hat{\eta}(\tau,\omega)$, *then one has, for any* $t \geq \tau$,

$$\lim_{\varepsilon\to 0} \|\Psi_\varepsilon(t,\tau,\omega,u^\varepsilon_\tau) - \Psi_0(t,\tau,\omega,\mathcal{M}u^0_\tau)\|_{L^2(\mathcal{O})} = 0.$$

Proof. Taking the inner product of (2.14) with $g\phi$, where $\phi \in H^1(\mathcal{Q})$, we infer

$$\begin{aligned}&\int_{\mathcal{Q}} g\frac{du^0}{dt}\bar{\phi}dy^* + (1+\mathrm{i}\mu)\sum_{i=1}^n \int_{\mathcal{Q}} gu^0_{y_i}\bar{\phi}_{y_i}dy^* + \rho\int_{\mathcal{Q}} gu^0\bar{\phi}dy^* \\ =& \int_{\mathcal{Q}} gf(t,y^*,0,u^0)\bar{\phi}dy^* + \int_{\mathcal{Q}} gG(t,y^*,0)\bar{\phi}dy^* + \zeta_\delta(\theta_t\omega)\int_{\mathcal{Q}} gR(t,y^*,0,u^0)\bar{\phi}dy^*.\end{aligned}$$

If $\xi \in H^1(\mathcal{O})$, then $\int_0^1 \xi(y^*,y_{n+1})dy_{n+1} \in H^1(\mathcal{Q})$. Therefore, for any $\xi \in H^1(\mathcal{O})$, we have

$$\begin{aligned}&\left(\frac{du^0}{dt},\xi\right)_{H_g(\mathcal{O})} + (1+\mathrm{i}\mu)\sum_{i=1}^n \left(u^0_{y_i},\xi_{y_i}\right)_{H_g(\mathcal{O})} + \rho\left(u^0,\xi\right)_{H_g(\mathcal{O})} \\ =& \left(f(t,y^*,0,u^0),\xi\right)_{H_g(\mathcal{O})} + \left(G(t,y^*,0),\xi\right)_{H_g(\mathcal{O})} + \zeta_\delta(\theta_t\omega)\left(R(t,y^*,0,u^0),\xi\right)_{H_g(\mathcal{O})}.\end{aligned}$$

Since u^0 is independent of y_{n+1}, the above equality gives, for any $\xi \in H^1(\mathcal{O})$ and $0 < \varepsilon \leq 1$,

$$\begin{aligned}&\left(\frac{du^0}{dt},\xi\right)_{H_g(\mathcal{O})} + (1+\mathrm{i}\mu)a_\varepsilon\left(u^0,\xi\right) + \rho\left(u^0,\xi\right)_{H_g(\mathcal{O})} \\ =& \left(f(t,y^*,0,u^0),\xi\right)_{H_g(\mathcal{O})} + \left(G(t,y^*,0),\xi\right)_{H_g(\mathcal{O})} \\ &+ \zeta_\delta(\theta_t\omega)\left(R(t,y^*,0,u^0),\xi\right)_{H_g(\mathcal{O})} - (1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}u^0_{y_i}, y_{n+1}\xi_{y_{n+1}}\right)_{H_g(\mathcal{O})}.\end{aligned} \tag{5.5}$$

Due to (5.5) and (2.13), one has, for any $\xi \in, H^1(\mathcal{O})$

$$\begin{aligned}&\left(\frac{du^\varepsilon}{dt} - \frac{du^0}{dt},\xi\right)_{H_g(\mathcal{O})} + (1+\mathrm{i}\mu)a_\varepsilon\left(u^\varepsilon - u^0,\xi\right) + \rho\left(u^\varepsilon - u^0,\xi\right)_{H_g(\mathcal{O})} \\ =& \left(f_\varepsilon(t,y^*,y_{n+1},u^\varepsilon) - f(t,y^*,0,u^0),\xi\right)_{H_g(\mathcal{O})} + \left(G_\varepsilon(t,y^*,y_{n+1}) - G(t,y^*,0),\xi\right)_{H_g(\mathcal{O})} \\ &+ \zeta_\delta(\theta_t\omega)\left(R_\varepsilon(t,y^*,y_{n+1},u^\varepsilon) - R(t,y^*,0,u^0),\xi\right)_{H_g(\mathcal{O})} \\ &+ (1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}u^0_{y_i}, y_{n+1}\xi_{y_{n+1}}\right)_{H_g(\mathcal{O})}.\end{aligned} \tag{5.6}$$

Setting $\xi = u^\varepsilon - u^0$ and then taking the real part, (5.6) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})} + a_\varepsilon\left(u^\varepsilon - u^0, u^\varepsilon - u^0\right) + \rho\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})}\\
= &\ \mathrm{Re}\left(f_\varepsilon(t, y^*, y_{n+1}, u^\varepsilon) - f(t, y^*, 0, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&+\mathrm{Re}\left(G_\varepsilon(t, y^*, y_{n+1}) - G(t, y^*, 0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&+\zeta_\delta(\theta_t\omega)\mathrm{Re}\left(R_\varepsilon(t, y^*, y_{n+1}, u^\varepsilon) - R(t, y^*, 0, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&+\mathrm{Re}(1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}u^0_{y_i}, y_{n+1}(u^\varepsilon_{y_{n+1}} - u^0_{y_{n+1}})\right)_{H_g(\mathcal{O})}.
\end{aligned} \tag{5.7}$$

By (2.3) and (5.2), we have

$$\begin{aligned}
&\mathrm{Re}\left(f_\varepsilon(t, y^*, y_{n+1}, u^\varepsilon) - f(t, y^*, 0, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
= &\ \mathrm{Re}\left(f(t, y^*, \varepsilon g(y^*)y_{n+1}, u^\varepsilon) - f(t, y^*, \varepsilon g(y^*)y_{n+1}, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&+\mathrm{Re}\left(f(t, y^*, \varepsilon g(y^*)y_{n+1}, u^0) - f(t, y^*, 0, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
\le &\ \beta\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})} + c\varepsilon\varphi_1^2(t) + c\varepsilon\left(\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \|u^0\|^2_{H_g(\mathcal{O})}\right).
\end{aligned} \tag{5.8}$$

By (5.3), we obtain

$$\begin{aligned}
&\mathrm{Re}\left(G_\varepsilon(t, y^*, y_{n+1}) - G(t, y^*, 0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
\le &\ \|G_\varepsilon(t, y^*, y_{n+1}) - G(t, y^*, 0)\|_{H_g(\mathcal{O})}\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})}\\
\le &\ c\varphi_2(t)\varepsilon\|v^\varepsilon - v^0\|^2_{H_g(\mathcal{O})}\\
\le &\ c\varepsilon\varphi_2^2(t) + c\varepsilon\left(\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \|u^0\|^2_{H_g(\mathcal{O})}\right).
\end{aligned} \tag{5.9}$$

By (2.6) and (5.4), we deduce

$$\begin{aligned}
&\zeta_\delta(\theta_t\omega)\mathrm{Re}\left(R_\varepsilon(t, y^*, y_{n+1}, u^\varepsilon) - R(t, y^*, 0, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&= \zeta_\delta(\theta_t\omega)\mathrm{Re}\left(R(t, y^*, \varepsilon g(y^*)y_{n+1}, u^\varepsilon) - R(t, y^*, \varepsilon g(y^*)y_{n+1}, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&\quad + \zeta_\delta(\theta_t\omega)\mathrm{Re}\left(R(t, y^*, \varepsilon g(y^*)y_{n+1}, u^0) - R(t, y^*, 0, u^0), u^\varepsilon - u^0\right)_{H_g(\mathcal{O})}\\
&\le \kappa\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})} + c\varepsilon|\zeta_\delta(\theta_t\omega)|^2\varphi_3^2(t) + c\varepsilon|\zeta_\delta(\theta_t\omega)|^2\left(\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \|u^0\|^2_{H_g(\mathcal{O})}\right).
\end{aligned} \tag{5.10}$$

Finally, by (2.10), we get

$$\begin{aligned}
&\mathrm{Re}(1+\mathrm{i}\mu)\sum_{i=1}^n\left(\frac{g_{y_i}}{g}u^0_{y_i}, y_{n+1}(u^\varepsilon_{y_{n+1}} - u^0_{y_{n+1}})\right)_{H_g(\mathcal{O})}\\
= &\ \mathrm{Re}(1+\mathrm{i}\mu)\sum_{i=1}^n\left(g_{y_i}u^0_{y_i}, y_{n+1}(u^\varepsilon_{y_{n+1}} - u^0_{y_{n+1}})\right)_{L^2(\mathcal{O})}\\
\le &\ c\varepsilon\|u^0\|_{H^1(\mathcal{Q})}\|u^\varepsilon - u^0\|^2_{H^1_\varepsilon(\mathcal{O})}\\
\le &\ c\varepsilon\left(\|u^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|u^0\|^2_{H^1(\mathcal{Q})}\right).
\end{aligned} \tag{5.11}$$

From (5.7)–(5.11), we obtain, for $t \geq \tau$,

$$\begin{aligned}\frac{d}{dt}\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})} &\leq \lambda\|u^\varepsilon - u^0\|^2_{H_g(\mathcal{O})} + c\varepsilon\left(\|u^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|u^0\|^2_{H^1(\mathcal{Q})}\right)\\ &\quad + c\varepsilon\sum_{i=1}^{2}\varphi_i^2(t) + c\varepsilon|\zeta_\delta(\theta_t\omega)|^2\varphi_3^2(t) + c\varepsilon(1+|\zeta_\delta(\theta_t\omega)|^2)\left(\|u^\varepsilon\|^2_{H_g(\mathcal{O})}\right.\\ &\quad \left.+\|u^0\|^2_{H_g(\mathcal{O})}\right),\end{aligned} \tag{5.12}$$

where $\lambda = 2(\beta+\kappa)$. Multiplying (5.12) by $e^{-\lambda t}$ and then integrating the resulting inequality on (τ, t), we deduce

$$\begin{aligned}&\|u^\varepsilon(t) - u^0(t)\|^2_{H_g(\mathcal{O})}\\ &\leq e^{\lambda(t-\tau)}\|u^\varepsilon(\tau) - u^0(\tau)\|^2_{H_g(\mathcal{O})} + c\varepsilon\int_\tau^t e^{\lambda(t-s)}\left(\|u^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|u^0\|^2_{H^1(\mathcal{Q})}\right)ds\\ &\quad + c\varepsilon\sum_{i=1}^{2}\int_\tau^t e^{\lambda(t-s)}\varphi_i^2(s)ds + c\varepsilon\int_\tau^t e^{\lambda(t-s)}|\zeta_\delta(\theta_s\omega)|^2\varphi_3^2(s)ds\\ &\quad + c\varepsilon\int_\tau^t e^{\lambda(t-s)}(1+|\zeta_\delta(\theta_s\omega)|^2)\left(\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \|u^0\|^2_{H_g(\mathcal{O})}\right)ds\\ &\leq e^{\lambda(t-\tau)}\|u^\varepsilon(\tau) - u^0(\tau)\|^2_{H_g(\mathcal{O})} + c\varepsilon e^{\lambda(t-\tau)}\int_\tau^t\left(\|u^\varepsilon\|^2_{H^1_\varepsilon(\mathcal{O})} + \|u^0\|^2_{H^1(\mathcal{Q})}\right)ds\\ &\quad + c\varepsilon e^{\lambda(t-\tau)}\sum_{i=1}^{2}\int_\tau^t\varphi_i^2(s)ds + c\varepsilon e^{\lambda(t-\tau)}\max_{\tau\leq s\leq t}|\zeta_\delta(\theta_s\omega)|^2\int_\tau^t\varphi_3^2(s)ds\\ &\quad + c\varepsilon e^{\lambda(t-\tau)}(1+\max_{\tau\leq s\leq t}|\zeta_\delta(\theta_s\omega)|^2)\int_\tau^t\left(\|u^\varepsilon\|^2_{H_g(\mathcal{O})} + \|u^0\|^2_{H_g(\mathcal{O})}\right)ds.\end{aligned} \tag{5.13}$$

By Lemma 5.1 and Lemma 5.2, there exists a positive constant $\varrho = \varrho(\tau, \omega, \rho, T)$ such that for all $t \in [\tau, \tau+T]$ with $T > 0$,

$$\begin{aligned}&\|u^\varepsilon(t) - u^0(t)\|^2_{H_g(\mathcal{O})}\\ &\leq e^{\lambda T}\|u^\varepsilon(\tau) - u^0(\tau)\|^2_{H_g(\mathcal{O})} + \varrho\varepsilon e^{\lambda T}\left[\|u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} + \|u^0_\tau\|^2_{H_g(\mathcal{Q})} + \sum_{i=1}^{3}\int_\tau^{\tau+T}\varphi_i^2(s)ds\right.\\ &\left.+\int_\tau^{\tau+T}\left(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega)\right)ds\right].\end{aligned} \tag{5.14}$$

Utilizing Lemma 5.3 and (5.14), for all $t \in [\tau, \tau+T]$, we have

$$
\begin{aligned}
&\|u^\varepsilon(t,\tau,\omega,u^\varepsilon_\tau) - u^0(t,\tau,\omega,\mathcal{M}u^\varepsilon_\tau)\|^2_{H_g(\mathcal{O})} \\
&\le e^{\lambda T}\|u^\varepsilon_\tau - \mathcal{M}u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} + \varrho\varepsilon e^{\lambda T}\Bigg[\|u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} + \|\mathcal{M}u^\varepsilon_\tau\|^2_{H_g(\mathcal{Q})} + \sum_{i=1}^{3}\int_\tau^{\tau+T}\varphi_i^2(s)ds \\
&\quad + \int_\tau^{\tau+T}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega)\Big)ds\Bigg] \\
&\le c\varepsilon^2\|u^\varepsilon_\tau\|^2_{H^1_\varepsilon(\mathcal{O})} + \varrho\varepsilon e^{\lambda T}\Bigg[\|u^\varepsilon_\tau\|^2_{H_g(\mathcal{O})} + \|\mathcal{M}u^\varepsilon_\tau\|^2_{H_g(\mathcal{Q})} + \sum_{i=1}^{3}\int_\tau^{\tau+T}\varphi_i^2(s)ds \\
&\quad + \int_\tau^{\tau+T}\Big(\|G(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_1(s,\cdot)\|_{L^\infty(\widetilde{\mathcal{O}})} + \|\psi_3(s,\cdot)\|^2_{L^\infty(\widetilde{\mathcal{O}})} + \eta_\delta(\theta_s\omega)\Big)ds\Bigg].
\end{aligned} \tag{5.15}
$$

By (5.14) and the assumption that $\|u^\varepsilon_\tau\|_{H^1_\varepsilon(\mathcal{O})} \le \hat{\eta}(\tau,\omega)$, we get the desired result. □

We finally establish the upper semicontinuity of random attractors as $\varepsilon \to 0$.

Theorem 5.2. *Suppose that* (2.2)–(2.7), (2.23), (2.24) *and* (5.2)–(5.4) *hold. Then, for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} dist_{L^2(\mathcal{O})}(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = 0.$$

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, by the invariance of $\mathcal{A}_\varepsilon$ and (4.1), there exists $\varepsilon_0 > 0$ such that

$$\|u\|^2_{H^1_\varepsilon(\mathcal{O})} \le L(\tau,\omega) \ \text{ for all } 0<\varepsilon<\varepsilon_0 \text{ and } u \in \mathcal{A}_\varepsilon(\tau,\omega), \tag{5.16}$$

where $L(\tau,\omega)$ is the positive constant in (4.1) which is independent of ε. Let $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be the $\mathcal{D}_1$-pullback absorbing set of Ψ_ε obtained in Lemma 4.1 and denote $K_0 = \{K_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ with $K_0(\tau,\omega) = \{\mathcal{M}u : u \in K(\tau,\omega)\}$. Then, K_0 is tempered in $L^2(\mathcal{Q})$ and hence $K_0 \in \mathcal{D}_0$. Since $\mathcal{A}_0$ is the $\mathcal{D}_0$-pullback attractor of Ψ_0 in $L^2(\mathcal{Q})$, given $\eta > 0$, we infer that there exists $T = T(\eta,\tau,\omega) \ge 1$ such that

$$\mathrm{dist}_{L^2(\mathcal{Q})}(\Psi_0(T,\tau-T,\theta_{-T}\omega,K_0(\tau-T,\theta_{-T}\omega)),\mathcal{A}_0(\tau,\omega)) < \frac{1}{2}\eta. \tag{5.17}$$

By the invariance of $\mathcal{A}_\varepsilon(\tau,\omega)$, we obtain that for any $x_\varepsilon \in \mathcal{A}_\varepsilon(\tau,\omega)$, there exists $y_\varepsilon \in \mathcal{A}_\varepsilon(\tau-T,\theta_{-T}\omega)$ such that

$$x_\varepsilon = \Psi_\varepsilon(T,\tau-T,\theta_{-T}\omega,y_\varepsilon). \tag{5.18}$$

By (5.16) and Theorem 5.1, we obtain

$$\lim_{\varepsilon\to 0}\|\Psi_\varepsilon(T,\tau-T,\theta_{-T}\omega,y_\varepsilon) - \Psi_0(T,\tau-T,\theta_{-T}\omega,\mathcal{M}y_\varepsilon)\|_{L^2(\mathcal{O})} = 0.$$

Hence, there exists $\varepsilon_1 \in (0,\varepsilon_0)$ such that for all $\varepsilon < \varepsilon_1$,

$$\|\Psi_\varepsilon(T,\tau-T,\theta_{-T}\omega,y_\varepsilon) - \Psi_0(T,\tau-T,\theta_{-T}\omega,\mathcal{M}y_\varepsilon)\|_{L^2(\mathcal{O})} < \frac{1}{2}\eta. \tag{5.19}$$

Since $y_\varepsilon \in \mathcal{A}_\varepsilon(\tau - T, \theta_{-T}\omega)$ and $\mathcal{A}_\varepsilon(\tau - T, \theta_{-T}\omega) \subseteq K(\tau - T, \theta_{-T}\omega)$, we know $\mathcal{M}y_\varepsilon \in K_0(\tau - T, \theta_{-T}\omega)$, which along with (5.17) implies

$$\mathrm{dist}_{L^2(\mathcal{Q})}(\Psi_0(T, \tau - T, \theta_{-T}\omega, \mathcal{M}y_\varepsilon), \mathcal{A}_0(\tau, \omega)) < \frac{1}{2}\eta. \tag{5.20}$$

By (5.19) and (5.20), one has, for all $\varepsilon < \varepsilon_1$,

$$\mathrm{dist}_{L^2(\mathcal{O})}(\Psi_\varepsilon(T, \tau - T, \theta_{-T}\omega, y_\varepsilon), \mathcal{A}_0(\tau, \omega)) < \eta. \tag{5.21}$$

By (5.18) and (5.21), we deduce, for all $\varepsilon < \varepsilon_1$,

$$\mathrm{dist}_{L^2(\mathcal{O})}(x_\varepsilon, \mathcal{A}_0(\tau, \omega)) < \eta, \text{ for all } x_\varepsilon \in \mathcal{A}_\varepsilon(\tau, \omega).$$

This indicates that for all $\varepsilon < \varepsilon_1$,

$$\mathrm{dist}_{L^2(\mathcal{O})}(\mathcal{A}_\varepsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) \le \eta,$$

as desired. □

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