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Dynamical Analysis for a General Jerky Equation with Random Excitation*

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Abstract A general jerky equation with random excitation is investigated in this paper. Before introducing the random excitation term, the equation is reduced to a two-dimensional model when undergoing a Hopf bifurcation. Then the model with the parametric excitation and external excitation is converted to a stochastic differential equation with singularity based on the stochastic average theory. For the equation, its dynamical behaviors are analyzed in different parameters' spaces, including the stability, stochastic bifurcation and stationary solution. Besides, numerical simulations are given to show the asymptotic behavior of the stationary solution.

Keywords Jerky equation, stochastic stability, stochastic bifurcation, stationary solution

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1. Introduction

In the real world, the motion of objects is inevitably influenced by environmental factors, internal structures and other unknown elements. As a result, stochastic systems can predict the evolution of trends more precisely. Furthermore, it fosters the development of random dynamical systems [2] that have widespread applications in physics [15, 24, 25], economics [4, 8, 12] and ecosystems [5, 9, 10, 13, 17, 26].

The jerky equation, which is a third-order explicit autonomous ordinary differential equation represented as $\dddot{u} = J(u, \dot{u}, \ddot{u})$, describes the motion of objects in terms of displacement u, velocity $\dot{u}$, acceleration $\ddot{u}$ and jerk $\dddot{u}$. In 1998, Eichhorn et al., proposed seven jerky equations $JD_1 - JD_7$, which encompassed nineteen important physical chaotic frameworks (A-S) [6] and Rössler's toroidal (TR) model [21]. Later on, Ren, Yu and Zhu, [20] performed a comprehensive dynamical analysis of discrete-time JD_1 and continuous-time JD_1 with delayed feedback. Correspondingly, Tang, Zhang and Ren [22] systematically investigated the following general jerky equation that comprises $JD_1 - JD_7$

$$\dddot{u} = \alpha_0 + \alpha_1 u + \alpha_2 \dot{u} + \alpha_3 \ddot{u} + \alpha_4 u^2 + \alpha_5 \dot{u}^2 + \alpha_6 u\dot{u} + \alpha_7 u\ddot{u}, \tag{1.1}$$

where α_i are the parameters, and $i = 0, 1, \ldots, 7$. They determined precise bifurcation conditions for Fold, Hopf, Zero-Hopf and Bogdanov-Takens bifurcations. The

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rich dynamical behaviors of equation (1.1) appeal us to investigate its stochastic dynamics, when it is disturbed by the parametric and external excitations. Therefore, we introduce a new stochastic model by incorporating noises into equation (1.1). Before adding the stochasticity, we reduce (1.1) to a two-dimensional equation, when it undergoes Hopf bifurcation by the center manifold theory. Then we add the parametric and external excitations to the two-dimensional equation, and transform it into a stochastic differential equation (SDE) with singularity by using the Khasminskii limit theorem [11,23] and the stochastic averaging method [16,19]. Interestingly, we obtain a nonlinear SDE comprising a singularity term. Following that, we discuss the stochastic stability using the singular boundary theory [14,27], and prove that the SDE without singularity undergoes the stochastic D-bifurcation and stochastic P-bifurcation [3,7,18]. Furthermore, we calculate the stationary solution for SDE with singularity by deriving its probability density function. Finally, we give numerical simulations to show the asymptotic behavior of the stationary solution with respect to various parameters.

2. Preparation

In this section, we reduce equation (1.1) to a two-dimensional system, when it undergoes Hopf bifurcation.

By setting $\dot{u} = v, \dot{v} = w$ in (1.1), the equilibrium $(u^*, 0, 0)$ where Hopf bifurcation occurs in [22] is as follows.

- $u^* = -\frac{\alpha_0}{\alpha_1}$, when $\alpha_1 \neq 0$, $\alpha_4 = 0$;
- $u^* = \frac{-\alpha_1 - \sqrt{\Delta}}{2\alpha_4}$ or $u^* = \frac{-\alpha_1 + \sqrt{\Delta}}{2\alpha_4}$, when $\alpha_4 \neq 0$, $\alpha_1^2 > 4\alpha_4\alpha_0$, where $\Delta = \sqrt{\alpha_1^2 - 4\alpha_4\alpha_0}$.

Making the transformation $\bar{u} \to u - u^*, \bar{v} \to v, \bar{w} \to w$, and still using the original notations u, v, w, system (1.1) becomes

$$\begin{cases} \dot{u} = v, \\ \dot{v} = w, \\ \dot{w} = \alpha_4 u^2 + \alpha_5 v^2 + \alpha_6 uv + \alpha_7 uw \\ \qquad +(2\alpha_4 u^* + \alpha_1)u + (\alpha_6 u^* + \alpha_2)v + (\alpha_7 u^* + \alpha_3)w. \end{cases} \tag{2.1}$$

The Jacobian matrix of (2.1) evaluated at $(0,0,0)$ is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{pmatrix}. \tag{2.2}$$

The characteristic equation of (2.1) at the equilibrium $(0,0,0)$ takes the form $\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0$, where $\alpha = -(\alpha_7 u^* + \alpha_3)$, $\beta = -(\alpha_6 u^* + \alpha_2)$, and $\gamma = -(2\alpha_4 u^* + \alpha_1)$. Substituting $\lambda = i\mu$ into the characteristic equation yields a relation among α, β and γ. If

$$\beta = \frac{\gamma}{\alpha}, \quad \beta = \mu^2,$$

the characteristic equation has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\mu$, where $\mu > 0$. Suppose that $\alpha > 0$. Then another characteristic root is $\lambda_3 = -\alpha < 0$. The next step is to calculate the local center manifold of the equilibrium $(0,0,0)$ for system (2.1). Let $q, p \in \mathbb{C}^3$ be complex eigenvectors which satisfy

$$Aq = i\mu q, \quad A^T p = -i\mu p, \quad \langle p, q\rangle = 1.$$

We obtain

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ i\mu \\ -\mu^2 \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{1}{2\mu(\mu^3 + i\gamma)} \begin{pmatrix} i\gamma\mu \\ i\mu^3 - \gamma \\ -\mu^2 \end{pmatrix}.$$

Let $x = (u, v, w)^T = zq + \bar{z}\bar{q} + y$ and

$$\begin{cases} z = \langle p, x\rangle, \\ y = x - \langle p, x\rangle q - \langle \bar{p}, x\rangle \bar{q}, \end{cases}$$

where $z \in \mathbb{C}^1$, $y \in \mathbb{R}^3$. According to the center manifold theory, we have $y = \frac{1}{2}h_{20}z^2 + h_{11}z\bar{z} + \frac{1}{2}h_{02}\bar{z}^2 + O(|z|^3)$, where $h_{20}, h_{11}, h_{02} \in \mathbb{R}^3$.

Before simplifying $\dot{z}$, we need to calculate the coefficients h_{20}, h_{11}, h_{02} of y. Comparing $\dot{y}$ with the original equation and the representation of the center manifold $y = \frac{1}{2}h_{20}z^2 + h_{11}z\bar{z} + \frac{1}{2}h_{02}\bar{z}^2$, we obtain

$$\begin{cases} (2i\mu I - A)h_{20} = H_{20}, \\ Ah_{11} = H_{11}, \\ (-2i\mu I - A)h_{02} = \bar{H}_{20}, \end{cases}$$

where I is the 3×3 identity matrix and

$$\begin{cases} H_{20} = F(q,q) - \langle p, F(q,q)\rangle q - \langle \bar{p}, F(q,q)\rangle \bar{q}, \\ H_{11} = F(q,\bar{q}) - \langle p, F(q,\bar{q})\rangle q - \langle \bar{p}, F(q,\bar{q})\rangle \bar{q}, \end{cases}$$

$$F(m,n) = \begin{pmatrix} 0 \\ 0 \\ 2\alpha_4 m_1 n_1 + 2\alpha_5 m_2 n_2 + \alpha_6 m_1 n_2 + \alpha_6 m_2 n_1 + \alpha_7 m_1 n_3 + \alpha_7 m_3 n_1 \end{pmatrix},$$

where $m = (m_1, m_2, m_3)^T, n = (n_1, n_2, n_3)^T \in \mathbb{R}^3$. Therefore, we deduce $h_{20} = (h_{201}, h_{202}, h_{203})^T$, $h_{11} = (h_{111}, h_{112}, h_{113})^T$, $h_{02} = (h_{021}, h_{022}, h_{023})^T$, where

$$\begin{aligned} h_{201} =& \frac{3\gamma q_2\bar{p}_3 + \gamma p_3\bar{q}_2 - 2q_3q_2^2\bar{p}_3 - 2p_3q_3q_2\bar{q}_2 - 4q_3^2\bar{p}_3 + 2\gamma p_3q_2 - 4p_3q_3^2 + q_3}{3q_3(2q_2q_3 - \gamma)}\Theta_1, \\ h_{202} =& \frac{2\gamma q_2^2\bar{p}_3 + 2\gamma p_3q_2\bar{q}_2 + \gamma q_3(\bar{p}_3 + p_3) - 6q_3^2q_2\bar{p}_3 - 2p_3q_3^2(2\bar{q}_2 + q_2) + 2q_3q_2}{3q_3(2q_2q_3 - \gamma)}\Theta_1, \\ h_{203} =& \frac{-\gamma q_2\bar{p}_3 + \gamma p_3\bar{q}_2 - 2q_3q_2^2\bar{p}_3 - 2p_3q_3q_2\bar{q}_2 - 4q_3^2\bar{p}_3 - 2\gamma p_3q_2 - 4p_3q_3^2 + 4q_3}{3(2q_2q_3 - \gamma)}\Theta_1, \end{aligned}$$

$$h_{111} =\frac{\bar{p}_3q_2+p_3\bar{q}_2}{q_3}\Theta_2,\ h_{112}=(\bar{p}_3+p_3)\,\Theta_2,\ h_{113}=(\bar{p}_3q_2+p_3\bar{q}_2)\,\Theta_2,$$
$$h_{021} =-\frac{\gamma q_2\bar{p}_3-\gamma p_3\bar{q}_2-2q_3q_2^2\bar{p}_3-2p_3q_3q_2\bar{q}_2+4q_3^2\bar{p}_3+2\gamma p_3q_2+4p_3q_3^2-q_3}{3q_3(2q_2q_3+\gamma)}\bar{\Theta}_1,$$
$$h_{022} =-\frac{2\gamma q_2(q_2\bar{p}_3+p_3\bar{q}_2)-\gamma q_3(\bar{p}_3+p_3)+2q_3^2q_2(\bar{p}_3-p_3)+4p_3q_3^2\bar{q}_2+2q_3q_2}{3q_3(2q_2q_3+\gamma)}\bar{\Theta}_1,$$
$$h_{023} =-\frac{\gamma q_2\bar{p}_3-\gamma p_3\bar{q}_2-2q_3q_2^2\bar{p}_3-2p_3q_3q_2\bar{q}_2-4q_3^2\bar{p}_3+2\gamma p_3q_2-4p_3q_3^2+4q_3}{3(2q_2q_3+\gamma)}\bar{\Theta}_1,$$
$$\Theta_1 =2k_4+2k_3q_2^2+2k_6q_2+2k_5q_3,\quad \Theta_2=2k_4+2k_3q_2\bar{q}_2+2k_5q_3.$$

Finally, equation (2.1) can be reduced into

$$\begin{aligned}\dot{z}&=i\mu z+p_3(\alpha_4x_1^2+\alpha_5x_2^2+\alpha_6x_1x_2+\alpha_7x_1x_3)\\&=i\mu z+g_{20}z^2+g_{11}z\bar{z}+g_{02}\bar{z}^2+g_{30}z^3+g_{21}z^2\bar{z}+g_{12}z\bar{z}^2+g_{03}\bar{z}^3+g_{40}z^4\\&\quad+g_{31}z^3\bar{z}+g_{22}z^2\bar{z}^2+g_{13}z\bar{z}^3+g_{04}\bar{z}^4+O(|z|^5),\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}
g_{20}&=\alpha_4+\alpha_6q_2+\alpha_7q_3+\alpha_5q_2^2,\\
g_{11}&=2\alpha_4+\alpha_6q_2+\alpha_7q_3+2\alpha_5q_2\bar{q}_2+\alpha_6\bar{q}_2+\alpha_7\bar{q}_3,\\
g_{02}&=\alpha_4+\alpha_6\bar{q}_2+\alpha_7\bar{q}_3+\alpha_5\bar{q}_2^2,\\
g_{30}&=\alpha_4h_{201}+\alpha_5h_{202}q_2+\frac{1}{2}\alpha_6h_{202}+\frac{1}{2}\alpha_6h_{201}q_2+\frac{1}{2}\alpha_7h_{203}+\frac{1}{2}\alpha_7h_{201}q_3,\\
g_{21}&=\alpha_4h_{201}+\frac{1}{2}\alpha_6h_{202}+\frac{1}{2}\alpha_7h_{203}+\alpha_5h_{202}\bar{q}_2+\frac{1}{2}\alpha_6h_{201}\bar{q}_2+\frac{1}{2}\alpha_7h_{201}\bar{q}_3\\
&\quad+2\alpha_4h_{111}+2\alpha_5h_{112}q_2+\alpha_6h_{112}+\alpha_6h_{111}q_2+\alpha_7h_{113}+\alpha_7h_{111}q_3,\\
g_{12}&=2\alpha_4h_{111}+\alpha_6h_{112}+\alpha_7h_{113}+2\alpha_5h_{112}\bar{q}_2+\alpha_6h_{111}\bar{q}_2+\alpha_7h_{111}\bar{q}_3\\
&\quad+\alpha_4h_{021}+\alpha_5h_{022}q_2+\frac{1}{2}\alpha_6h_{022}+\frac{1}{2}\alpha_6h_{021}q_2+\frac{1}{2}\alpha_7h_{023}+\frac{1}{2}\alpha_7h_{021}q_3,\\
g_{03}&=\alpha_4h_{021}+\frac{1}{2}\alpha_6h_{022}+\frac{1}{2}\alpha_7h_{023}+\alpha_5h_{022}\bar{q}_2+\frac{1}{2}\alpha_6h_{021}\bar{q}_2+\frac{1}{2}\alpha_7h_{021}\bar{q}_3,\\
g_{40}&=\frac{1}{4}\alpha_4h_{201}^2+\frac{1}{4}\alpha_5h_{202}^2+\frac{1}{4}\alpha_6h_{201}h_{202}+\frac{1}{4}\alpha_7h_{201}h_{203},\\
g_{31}&=\alpha_4h_{111}h_{201}+\alpha_5h_{112}h_{202}+\frac{1}{2}\alpha_6h_{112}h_{201}+\frac{1}{2}\alpha_6h_{111}h_{202}\\
&\quad+\frac{1}{2}\alpha_7h_{113}h_{201}+\frac{1}{2}\alpha_7h_{111}h_{203},\\
g_{22}&=\alpha_4h_{111}^2+\alpha_6h_{112}h_{111}+\alpha_7h_{113}h_{111}+\frac{1}{2}\alpha_4h_{021}h_{201}+\alpha_5h_{112}^2+\frac{1}{2}\alpha_5h_{022}h_{202}\\
&\quad+\frac{1}{4}\alpha_6h_{022}h_{201}+\frac{1}{4}\alpha_6h_{021}h_{202}+\frac{1}{4}\alpha_7h_{023}h_{201}+\frac{1}{4}\alpha_7h_{021}h_{203},\\
g_{13}&=\alpha_4h_{21}h_{111}+\alpha_5h_{22}h_{112}+\frac{1}{2}\alpha_6h_{22}h_{111}+\frac{1}{2}\alpha_6h_{21}h_{112}\\
&\quad+\frac{1}{2}\alpha_7h_{23}h_{111}+\frac{1}{2}\alpha_7h_{21}h_{113},\\
g_{04}&=\frac{1}{4}\alpha_4h_{021}^2+\frac{1}{4}\alpha_5h_{022}^2+\frac{1}{4}\alpha_6h_{021}h_{022}+\frac{1}{4}\alpha_7h_{021}h_{023}.
\end{aligned}$$

Let $z = z_1 + iz_2$ and truncating higher-order terms. Then we have

$$\begin{cases} \dot{z}_1 = -\mu z_2 + m_{20}z_1^2 + m_{11}z_1z_2 + m_{02}z_2^2 + m_{30}z_1^3 + m_{21}z_1^2z_2 + m_{12}z_1z_2^2 \\ \qquad +m_{03}z_2^3 + m_{40}z_1^4 + m_{31}z_1^3z_2 + m_{22}z_1^2z_2^2 + m_{13}z_1z_2^3 + m_{04}z_2^4, \\ \dot{z}_2 = \mu z_1 + n_{20}z_1^2 + n_{11}z_1z_2 + n_{02}z_2^2 + n_{30}z_1^3 + n_{21}z_1^2z_2 + n_{12}z_1z_2^2 \\ \qquad +n_{03}z_2^3 + n_{40}z_1^4 + n_{31}z_1^3z_2 + n_{22}z_1^2z_2^2 + n_{13}z_1z_2^3 + n_{04}z_2^4, \end{cases} \tag{2.4}$$

where

$$\begin{aligned}
&m_{20} = \Re(g_{02} + g_{11} + g_{20}), \quad m_{11} = \Im(2g_{02} - 2g_{20}), \\
&m_{02} = \Re(g_{11} - g_{02} - g_{20}), \quad m_{30} = \Re(g_{03} + g_{12} + g_{21} + g_{30}), \\
&m_{21} = \Im(3g_{03} + g_{12} - g_{21} - 3g_{30}), \quad m_{12} = \Re(g_{12} + g_{21} - 3g_{30} - 3g_{03}), \\
&m_{03} = \Im(g_{12} - g_{21} - g_{03} + g_{30}), \quad m_{40} = \Re(g_{04} + g_{13} + g_{22} + g_{31} + g_{40}), \\
&m_{31} = \Im(4g_{04} + 2g_{13} - 2g_{31} - 4g_{40}), \quad m_{22} = \Re(2g_{22} - 6g_{04} - 6g_{40}), \\
&m_{13} = \Im(4g_{40} + 2g_{13} - 2g_{31} - 4g_{04}), \quad m_{04} = \Re(g_{04} - g_{13} + g_{22} - g_{31} + g_{40}), \\
&n_{20} = \Im(g_{02} + g_{11} + g_{20}), \quad n_{11} = \Re(2g_{20} - 2g_{02}), \\
&n_{02} = \Im(g_{11} - g_{02} - g_{20}), \quad n_{30} = \Im(g_{03} + g_{12} + g_{21} + g_{30}), \\
&n_{21} = \Re(3g_{30} + g_{21} - g_{12} - 3g_{03}), \quad n_{12} = \Im(g_{12} + g_{21} - 3g_{30} - 3g_{03}), \\
&n_{03} = \Re(g_{21} - g_{12} - g_{30} + g_{03}), \quad n_{40} = \Im(g_{04} + g_{13} + g_{22} + g_{31} + g_{40}), \\
&n_{31} = \Re(4g_{40} + 2g_{31} - 2g_{13} - 4g_{04}), \quad n_{22} = \Im(2g_{22} - 6g_{04} - 6g_{40}), \\
&n_{13} = \Re(4g_{04} + 2g_{31} - 2g_{13} - 4g_{40}), \quad n_{04} = \Im(g_{04} - g_{13} + g_{22} - g_{31} + g_{40}),
\end{aligned}$$

where $\Re(\cdot)$ and $\Im(\cdot)$ respectively represent the real part and imaginary part of $(\cdot)$.

3. Modeling

In this section, we mainly propose a stochastic model through transforming the reduced system into an SDE. Under the parametric μ and the external excitations, equation (2.4) becomes

$$\begin{cases} \dot{z}_1 = -\mu z_2 + +m_{20}z_1^2 + m_{11}z_1z_2 + m_{02}z_2^2 + m_{30}z_1^3 + m_{21}z_1^2z_2 + m_{12}z_1z_2^2 + m_{03}z_2^3 \\ \qquad +m_{40}z_1^4 + m_{31}z_1^3z_2 + m_{22}z_1^2z_2^2 + m_{13}z_1z_2^3 + m_{04}z_2^4 + \varepsilon^{\frac{1}{2}}\xi_1(t)z_2 + \varepsilon\xi_2(t), \\ \dot{z}_2 = \mu z_1 + n_{20}z_1^2 + n_{11}z_1z_2 + n_{02}z_2^2 + n_{30}z_1^3 + n_{21}z_1^2z_2 + n_{12}z_1z_2^2 + n_{03}z_2^3 \\ \qquad +n_{40}z_1^4 + n_{31}z_1^3z_2 + n_{22}z_1^2z_2^2 + n_{13}z_1z_2^3 + n_{04}z_2^4 + \varepsilon^{\frac{1}{2}}\xi_3(t)z_1 + \varepsilon\xi_4(t), \end{cases} \tag{3.1}$$

where ε is a small parameter, $\xi_i(t) = \xi_i(\omega, t)$ with $\omega \in \Omega$ and $i = 1, 2, 3, 4$ are the independent stationary stochastic processes with zero mean. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space. Let $z_1 = \varepsilon^{\frac{1}{2}} r \sin\varphi$ and $z_2 = \varepsilon^{\frac{1}{2}} r \cos\varphi$, where $\varphi = \mu t - \phi$. Then equation (3.1) becomes

$$\begin{cases}\dot{r}=\varepsilon^{\frac{1}{2}}(m_{20}r^2\sin^2\varphi+m_{11}r^2\sin^2\varphi\cos\varphi+m_{02}r^2\sin\varphi\cos^2\varphi+\varepsilon^{\frac{1}{2}}r^3(m_{30}\sin^4\varphi\\ \quad +m_{21}\sin^3\varphi\cos\varphi+m_{12}\sin^2\varphi\cos^2\varphi+m_{03}\sin\varphi\cos^3\varphi)+\varepsilon r^4(m_{40}\sin^5\varphi\\ \quad +m_{31}\sin^4\varphi\cos\varphi+m_{22}\sin^3\varphi\cos^2\varphi+m_{13}\sin^2\varphi\cos^3\varphi+m_{04}\sin\varphi\cos^4\varphi)\\ \quad +n_{20}r^2\sin^2\varphi\cos\varphi+n_{11}r^2\sin\varphi\cos^2\varphi+n_{02}r^2\cos^3\varphi+\varepsilon^{\frac{1}{2}}r^3(n_{30}\sin^3\varphi\cos\varphi\\ \quad +n_{21}\sin^2\varphi\cos^2\varphi+n_{12}\sin\varphi\cos^3\varphi+n_{03}\cos^4\varphi)+\varepsilon r^4(n_{40}\sin^4\varphi\cos\varphi\\ \quad +n_{31}\sin^3\varphi\cos^2\varphi+n_{22}\sin^2\varphi\cos^3\varphi+n_{13}\sin\varphi\cos^4\varphi+n_{04}\cos^5\varphi)\\ \quad +(\xi_1(t)+\xi_3(t))r\sin\varphi\cos\varphi+\xi_2(t)\sin\varphi+\xi_4(t)\cos\varphi),\\ \dot{\phi}=\varepsilon^{\frac{1}{2}}(r(m_{20}\sin^2\varphi\cos\varphi+m_{11}\sin\varphi\cos^2\varphi+m_{02}\cos^3\varphi)+\varepsilon^{\frac{1}{2}}r^2(m_{30}\sin^3\varphi\cos\varphi\\ \quad +m_{21}\sin^2\varphi\cos^2\varphi+m_{12}\sin\varphi\cos^3\varphi+m_{03}\cos^4\varphi)+\varepsilon r^3(m_{40}\sin^4\varphi\cos\varphi\\ \quad +m_{31}\sin^3\varphi\cos^2\varphi+m_{22}\sin^2\varphi\cos^3\varphi+m_{13}\sin\varphi\cos^4\varphi+m_{04}\cos^5\varphi)\\ \quad -r(n_{20}\sin^3\varphi+n_{11}\sin^2\varphi\cos\varphi+n_{02}\sin\varphi\cos^2\varphi)-\varepsilon^{\frac{1}{2}}r^2(n_{30}\sin^4\varphi\\ \quad +n_{21}\sin^3\varphi\cos\varphi+n_{12}\sin^2\varphi\cos^2\varphi+n_{03}\sin\varphi\cos^3\varphi)-\varepsilon r^3(n_{40}\sin^5\varphi\\ \quad +n_{31}\sin^4\varphi\cos\varphi+n_{22}\sin^3\varphi\cos^2\varphi+n_{13}\sin^2\varphi\cos^3\varphi+n_{04}\sin\varphi\cos^4\varphi)\\ \quad +\xi_1(t)\cos^2\varphi-\xi_3(t)\sin^2\varphi+\xi_2(t)\frac{\cos\varphi}{r}-\xi_4(t)\frac{\sin\varphi}{r}).\end{cases} \tag{3.2}$$

We have the following theorem.

Theorem 3.1. *For equation (3.2), it can be written as*

$$\frac{dX}{dt}=\varepsilon^{\frac{1}{2}}\Psi(X,t,\xi(t),\varepsilon),\quad X(0)=X_0, \tag{3.3}$$

where $X=(r,\phi)^T$, $\Psi(X,t,\xi(t),\varepsilon)=(\Psi_1(X,t,\xi(t),\varepsilon),\Psi_2(X,t,\xi(t),\varepsilon))^T$, $X_0=(r_0,\phi_0)^T$, $\xi(t)=(\xi_1,\xi_2,\xi_3,\xi_4)^T$ *has piecewise continuous trajectories with probability one and satisfies the strong mixing condition. Then, as* $\varepsilon\to 0$, *the solution to (3.3) weakly converges to a diffusive Markov process* $\bar{X}=(\bar{r},\bar{\phi})^T$ *on a time interval of order* $1/\varepsilon$, *which satisfies the SDE*

$$d\bar{X}=m(\bar{X})dt+\sigma(\bar{X})dW_t, \tag{3.4}$$

where $m(\bar{X})=(m_1,m_2)^T$ *and* $\sigma(\bar{X})=\begin{pmatrix}\sigma_{11}&\sigma_{12}\\ \sigma_{21}&\sigma_{22}\end{pmatrix}$ *with*

$$\begin{aligned} m_i=&\mathcal{M}\left\{G_i^1(X,t)+\frac{\partial G_i^0(X,t)}{\partial X_j}G_j^0(X,t)\right\}\\ &+\mathcal{M}\left\{\int_{-\infty}^0\mathbb{E}\{\frac{\partial F_i^0(X,t,\xi_t)}{\partial X_j}F_j^0(X,t+\tau,\xi_{t+\tau})\}d\tau\right\},\\ \sigma_{kj}=&\mathcal{M}\left\{\int_{-\infty}^{\infty}\mathbb{E}\{F_k^0(X,t,\xi_t)F_j^0(X,t+\tau,\xi_{t+\tau})\}d\tau\right\}, \end{aligned} \tag{3.5}$$

and $W_t=(W_{\bar{r}},W_{\bar{\phi}})^T$ *is a two-dimensional Wiener process,* $\mathbb{E}(W_t)=0$ *and* $\mathbb{E}(W_t^2)=t$. *Here* $\mathbb{E}$ *represents the expectation and* $\mathcal{M}$ *is the averaging operator* $\mathcal{M}(\cdot)=\frac{1}{T}\int_{t_0}^{t_0+T}(\cdot)dt$. *Denote*

$$\Psi_i(X,t,\xi(t),\varepsilon)=F_i^0(X,t,\xi_t)+G_i^0(X,t)+\varepsilon^{\frac{1}{2}}G_i^1(X,t),i=1,2, \tag{3.6}$$

where

$$
\begin{aligned}
F_1^0(X,t,\xi_t) &= (\xi_1(t)+\xi_3(t))r\sin\varphi\cos\varphi+\xi_2(t)\sin\varphi+\xi_4(t)\cos\varphi,\\
G_1^0(X,t) &= m_{20}r^2\sin^3\varphi+m_{11}r^2\sin^2\varphi\cos\varphi+m_{02}r^2\sin\varphi\cos^2\varphi\\
&\quad +n_{20}r^2\sin^2\varphi\cos\varphi+n_{11}r^2\sin\varphi\cos^2\varphi+n_{02}r^2\cos^3\varphi,\\
G_1^1(X,t) &= r^3(m_{30}\sin^4\varphi+m_{21}\sin^3\varphi\cos\varphi+m_{12}\sin^2\varphi\cos^2\varphi+m_{03}\sin\varphi\cos^3\varphi\\
&\quad +n_{30}\sin^3\varphi\cos\varphi+n_{21}\sin^2\varphi\cos^2\varphi+n_{12}\sin\varphi\cos^3\varphi+n_{03}\cos^4\varphi)\\
&\quad +\varepsilon^{\frac{1}{2}}r^4(m_{40}\sin^5\varphi+m_{31}\sin^4\varphi\cos\varphi+m_{22}\sin^3\varphi\cos^2\varphi\\
&\quad +m_{13}\sin^2\varphi\cos^3\varphi+m_{04}\sin\varphi\cos^4\varphi+n_{40}\sin^4\varphi\cos\varphi\\
&\quad +n_{31}\sin^3\varphi\cos^2\varphi+n_{22}\sin^2\varphi\cos^3\varphi+n_{13}\sin\varphi\cos^4\varphi\\
&\quad +n_{04}\cos^5\varphi),\\
F_2^0(X,t,\xi_t) &= \xi_1(t)\cos^2\varphi-\xi_3(t)\sin^2\varphi+\xi_2(t)\frac{\cos\varphi}{r}-\xi_4(t)\frac{\sin\varphi}{r},\\
G_2^0(X,t) &= m_{20}r\sin^2\varphi\cos\varphi+m_{11}r\sin\varphi\cos^2\varphi+m_{02}r\cos^3\varphi\\
&\quad -n_{20}r\sin^3\varphi-n_{11}r\sin^2\varphi\cos\varphi-n_{02}r\sin\varphi\cos^2\varphi,\\
G_2^1(X,t) &= r^2(m_{30}\sin^3\varphi\cos\varphi+m_{21}\sin^2\varphi\cos^2\varphi+m_{12}\sin\varphi\cos^3\varphi+m_{03}\cos^4\varphi\\
&\quad -n_{30}\sin^4\varphi-n_{21}\sin^3\varphi\cos\varphi-n_{12}\sin^2\varphi\cos^2\varphi-n_{03}\sin\varphi\cos^3\varphi)\\
&\quad +\varepsilon^{\frac{1}{2}}r^3(m_{40}\sin^4\varphi\cos\varphi+m_{31}\sin^3\varphi\cos^2\varphi+m_{22}\sin^2\varphi\cos^3\varphi\\
&\quad +m_{13}\sin\varphi\cos^4\varphi+m_{04}\cos^5\varphi-n_{40}\sin^5\varphi-n_{31}\sin^4\varphi\cos\varphi\\
&\quad -n_{22}\sin^3\varphi\cos^2\varphi-n_{13}\sin^2\varphi\cos^3\varphi-n_{04}\sin\varphi\cos^4\varphi).
\end{aligned}
$$

Proof. According to Theorem 2 in [16], we obtain immediately that the solution of equation (3.3) converges weakly to a diffusive Markov process $\bar{X}=(\bar{r},\bar{\phi})^T$, which satisfies the SDE

$$
\begin{cases}
d\bar{r}=m_1dt+\sigma_{11}dW_{\bar{r}}+\sigma_{12}dW_{\bar{\phi}},\\
d\bar{\phi}=m_2dt+\sigma_{21}dW_{\bar{r}}+\sigma_{22}dW_{\bar{\phi}},
\end{cases}
$$

where

$$
\begin{aligned}
m_1 &= \frac{d_1}{\bar{r}}+\frac{d_2}{8}\bar{r}+\frac{d_3}{8}\bar{r}^3, \quad \sigma_{11}^2=d_4+\frac{d_5}{8}\bar{r}^2,\\
m_2 &= d_6+\frac{d_7}{\bar{r}^2}+\frac{d_8}{8}\bar{r}^2, \quad \sigma_{22}^2=d_9+\frac{d_{10}}{\bar{r}^2}, \quad \sigma_{12}=\sigma_{21}=0,\\
d_1 &= \frac{1}{2}(\mathcal{S}_2(\mu)+\mathcal{S}_4(\mu)), \quad d_2=3\mathcal{S}_1(2\mu)-\mathcal{S}_3(2\mu),\\
d_3 &= \frac{1}{2}(6m_{30}+6n_{03}+2n_{21}+2n_{12}+13(m_{20}^2+m_{02}^2)+3(m_{11}^2+n_{11}^2)+5(m_{02}^2\\
&\quad +n_{20}^2)+6(m_{02}m_{20}+n_{02}n_{20})+4(m_{11}n_{20}+m_{02}n_{11}-m_{20}n_{11}-m_{11}n_{02})),\\
d_4 &= 2d_1, \quad d_5=2(\mathcal{S}_1(2\mu)+\mathcal{S}_3(2\mu)),\\
d_6 &= -\frac{1}{4}(\mathcal{H}_1(2\mu)+\mathcal{H}_3(2\mu)), \quad d_7=\mathcal{S}_4(\mu)-\mathcal{H}_2(\mu),\\
d_8 &= 6m_{03}+2m_{21}-2n_{12}-6n_{30}+3m_{11}(m_{20}-m_{02}-n_{11})\\
&\quad +4(m_{02}n_{02}-n_{20}n_{11}-m_{20}n_{20}),\\
d_9 &= \frac{1}{2}(\mathcal{S}_1(0)+\mathcal{S}_3(0))+\frac{1}{4}(\mathcal{S}_1(2\mu)+\mathcal{S}_3(2\mu)), \quad d_{10}=\mathcal{S}_2(\mu)+\mathcal{S}_4(\mu).
\end{aligned}
$$

Here

$$\begin{cases} \mathcal{S}_i(\zeta) = \int_{-\infty}^{0} \mathbb{E}(\xi_i(t)\xi_i(t+\tau)\cos(\zeta\tau))d\tau, \\ \mathcal{H}_i(\zeta) = \int_{-\infty}^{0} \mathbb{E}(\xi_i(t)\xi_i(t+\tau)\sin(\zeta\tau))d\tau. \end{cases}$$

□

4. Stochastic dynamics

In this section, we focus on the stochastic dynamical behavior of the averaging amplitude $\bar{r}$, which satisfies the following SDE

$$d\bar{r} = (\frac{d_1}{\bar{r}} + \frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3)dt + (d_4 + \frac{d_5}{8}\bar{r}^2)^{\frac{1}{2}} dW_{\bar{r}}. \tag{4.1}$$

Definition 4.1. (Stochastic stability [1]) The equilibrium solution $X(t)$ of the stochastic differential equation

$$dX(t) = f(t, X(t))dt + G(t, X(t))dW(t), \quad t \geq t_0, \quad X(t_0) = c, \tag{4.2}$$

is called stochastically stable (stable in probability), if for every $\epsilon > 0$,

$$\lim_{c \to 0} P(\sup_{t_0 \leq t < \infty} |X(c)| \geq \epsilon) = 0,$$

where $f(t, X(t))$ is the function mapping $[t_0, \infty) \times \mathbb{R}^n$ into $\mathbb{R}^n$ with $f(t, 0) = 0$, $G(t, X(t))$ is the function mapping $[t_0, \infty) \times \mathbb{R}^n$ into $\mathbb{R}^n \times \mathbb{R}^m$ with $G(t, 0) = 0$, and $W(t)$ is an m-dimensional Brownian motion.

Before getting the global stochastic stability of the system, we need to introduce the singular boundary theory. The SDE (4.1) can be written as

$$d\bar{r} = m(\bar{r})dt + \sigma(\bar{r})dW_{\bar{r}}$$

with the boundary 0 and $+\infty$, where

$$m(\bar{r}) = \frac{d_1}{\bar{r}} + \frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3, \quad \sigma(\bar{r}) = (d_4 + \frac{d_5}{8}\bar{r}^2)^{\frac{1}{2}}.$$

If $\sigma(\bar{r}) = 0$ at the boundary, this boundary is called the first kind singular boundary. If $m(\bar{r})$ is unbounded at the boundary, this boundary is called the second kind singular boundary. For the first kind, the diffusion exponent ρ_1, the drifting exponent ρ_2 and the characteristic value ρ_3 are defined as follows:

$$\begin{aligned} &\sigma^2(\bar{r}) = O|\bar{r} - r|^{\rho_1}, \bar{r} \to r, \quad m(\bar{r}) = O|\bar{r} - r|^{\rho_2}, \bar{r} \to r, \\ &\rho_3(\bar{r}) = \lim_{\bar{r} \to r^+} \frac{2m(\bar{r})(\bar{r} - r^+)^{\rho_1 - \rho_2}}{\sigma^2(\bar{r})}, \quad \rho_3(\bar{r}) = -\lim_{\bar{r} \to r^-} \frac{2m(\bar{r})(r^- - \bar{r})^{\rho_1 - \rho_2}}{\sigma^2(\bar{r})}. \end{aligned}$$

For the second kind, the diffusion exponent ρ_1, the drifting exponent ρ_2 and the characteristic value ρ_3 are defined as follows:

$$\begin{aligned} &\sigma^2(\bar{r}) = O|\bar{r} - r|^{-\rho_1}, \bar{r} \to r, \quad m(\bar{r}) = O|\bar{r} - r|^{-\rho_2}, \bar{r} \to r, \\ &\rho_3(\bar{r}) = \lim_{\bar{r} \to r^+} \frac{2m(\bar{r})(\bar{r} - r^+)^{\rho_2 - \rho_1}}{\sigma^2(\bar{r})}, \quad \rho_3(\bar{r}) = -\lim_{\bar{r} \to r^-} \frac{2m(\bar{r})(r^- - \bar{r})^{\rho_2 - \rho_1}}{\sigma^2(\bar{r})}. \end{aligned}$$

There are four classifications of boundaries: regular boundary, exceeded boundary, accessible boundary and natural boundary (abbreviated as RB, EB, AB and NB). Furthermore, the natural boundary has three kinds: attractively natural (ANB), repulsively natural (RNB) and strictly natural (SNB). Different kinds of singular boundaries and specific ρ_1, ρ_2, ρ_3 correspond to different boundaries (see [27] for details).

Next, we analyze the stochastic dynamics for equation (4.1) in three subsections.

4.1. The first case

If $d_1 = 0$, $d_3 = 0$ and $d_4 = 0$, equation (4.1) becomes a linear SDE

$$d\bar{r} = (\frac{d_2}{8}\bar{r})dt + (\frac{d_5}{8}\bar{r}^2)^{\frac{1}{2}} dW_{\bar{r}}. \tag{4.3}$$

We have the following theorem on its local stability.

Theorem 4.1. *If $2d_2 < d_5$, then the trivial solution to (4.3) is stable in probability; if $2d_2 > d_5$, then the trivial solution to (4.3) is unstable in probability.*

Proof. Let $F(\bar{r}, t) = \ln(\bar{r}(t))$. Applying Itô's formula, we have

$$dF(\bar{r}, t) = (\frac{d_2}{8} - \frac{d_5}{16})dt + (\frac{d_5}{8})^{\frac{1}{2}} dW_{\bar{r}}.$$

Integrating from 0 to t yields

$$\ln(\frac{\bar{r}(t)}{\bar{r}(0)}) = (\frac{d_2}{8} - \frac{d_5}{16})t + (\frac{d_5}{8})^{\frac{1}{2}} W_{\bar{r}}.$$

Hence, the solution to equation (4.3) can be solved as

$$\bar{r}(t) = \bar{r}(0) \exp\left((\frac{d_2}{8} - \frac{d_5}{16})t + (\frac{d_5}{8})^{\frac{1}{2}} W_{\bar{r}} \right). \tag{4.4}$$

Define $||\bar{r}(t)|| = (\bar{r}(t))^{\frac{1}{2}}$ and the Lyapunov exponent $\lambda = \lim_{t\to+\infty} \frac{1}{t} \ln ||\bar{r}(t)||$. Then

$$\begin{aligned}
\lambda &= \lim_{t\to+\infty} \frac{1}{t} \ln(\bar{r}(t))^{\frac{1}{2}} = \lim_{t\to+\infty} \frac{1}{2t} \ln(\bar{r}(t)) \\
&= \lim_{t\to+\infty} \frac{1}{2t} [\ln(\bar{r}(0)) + (\frac{d_2}{8} - \frac{d_5}{16})t + (\frac{d_5}{8})^{\frac{1}{2}} W_{\bar{r}}] \\
&= \frac{d_2}{16} - \frac{d_5}{32}.
\end{aligned}$$

Therefore, the trivial solution to (4.3) is stable for $\lambda < 0$ and unstable for $\lambda > 0$. □

According to the singular boundary theory, we obtain the following theorem from Table 1.

Theorem 4.2. *(Global stochastic stability)*

(i) *If $2d_2 > d_5$, the trivial solution of (4.3) is unstable;*

(ii) *If $2d_2 < d_5$, the trivial solution of (4.3) is stable.*

4.2. The second case

If $d_1 = d_4 = 0$ and $d_3 \neq 0$, equation (4.1) becomes a nonlinear SDE

$$d\bar{r} = (\frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3)dt + (\frac{d_5}{8}\bar{r}^2)^{\frac{1}{2}} dW_{\bar{r}}. \tag{4.5}$$

The classification of boundary refers to Table 2, and we have the following theorem.

Theorem 4.3. *(Global stochastic stability)*

(*i*) *If* $2d_2 > d_5$ *and* $d_3 > 0$, *the trivial solution of (4.5) is unstable;*

(*ii*) *If* $2d_2 < d_5$ *and* $d_3 < 0$, *the trivial solution of (4.5) is stable.*

In the following part, we analyze the stochastic bifurcation for equation (4.5). Letting $\bar{r} \to \sqrt{-\frac{d_3}{8}}\bar{r}$ and by Itô's formula, we have

$$\begin{aligned} d\left(\sqrt{-\frac{d_3}{8}}\bar{r}\right) &= \sqrt{-\frac{d_3}{8}}\left(\frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3\right)dt + \sqrt{-\frac{d_3}{8}}\left(\frac{d_5}{8}\bar{r}^2\right)^{\frac{1}{2}} dW_{\bar{r}} \\ &= \left(\frac{d_2}{8}\frac{1}{\sqrt{-\frac{d_3}{8}}}\sqrt{-\frac{d_3}{8}}\bar{r} + \frac{d_3}{8}\left(\sqrt{-\frac{d_3}{8}}\right)^3 \frac{1}{\left(\sqrt{-\frac{d_3}{8}}\right)^3}\bar{r}^3\right)\sqrt{-\frac{d_3}{8}}dt \\ &\quad + \sqrt{-\frac{d_3}{8}}\left(\frac{d_5}{8}\bar{r}^2\right)^{\frac{1}{2}} dW_{\bar{r}}. \end{aligned}$$

Still writing $\sqrt{-\frac{d_3}{8}}\bar{r}$ as $\bar{r}$, then (4.5) becomes

$$d\bar{r} = (\frac{d_2}{8}\bar{r} - \bar{r}^3)dt + (\frac{d_5}{8}\bar{r}^2)^{\frac{1}{2}} dW_{\bar{r}}.$$

Through the transformation from an Itô SDE to a Stratonovich SDE, we get

$$d\bar{r} = (\frac{d_2}{8}\bar{r} - \frac{d_5}{16}\bar{r} - \bar{r}^3)dt + (\frac{d_5}{8}\bar{r}^2)^{\frac{1}{2}} \circ dW_{\bar{r}}. \tag{4.6}$$

From the theory of stochastic pitchfork bifurcation in [2], we deduce the invariant measures for D-bifurcation and P-bifurcation.

Table 1. The indices at $\bar{r} = 0$ and $\bar{r} = \infty$ respectively.

$\bar{r}$	ρ_1	ρ_2	ρ_3	Condition	Type
∞	2	3	$-\frac{2d_2}{d_5}$	$2d_2 > d_5$	ANB
				$2d_2 < d_5$	RNB
				$2d_2 = d_5$	SNB
0	2	1	$\frac{2d_2}{d_5}$	$2d_2 > d_5$	RNB
				$2d_2 < d_5$	ANB
				$2d_2 = d_5$	SNB

Table 2. The indices at $\bar{r}=0$ and $\bar{r}=\infty$ respectively.

$\bar{r}$	ρ_1	ρ_2	ρ_3	Condition	Type
∞	2	3	$-\frac{2d_3}{d_5}$	$d_3>0$	EB
				$d_3<0$	AB
0	2	1	$\frac{2d_2}{d_5}$	$2d_2>d_5$	RNB
				$2d_2<d_5$	ANB
				$2d_2=d_5$	SNB

Theorem 4.4. *The stochastic dynamical behaviors for (4.6) are as follows.*

(i) It undergoes a D-bifurcation of the trivial reference measure δ_0 at $2d_2=d_5$, where δ_0 is the random Dirac measure.

(ii) It undergoes a P-bifurcation of the invariant measures $\nu^{\pm}$ at $d_2=d_5$, and the corresponding stationary measures of $\nu^{\pm}$ are $\mu^{\pm}(r)$, where

$$\nu^{\pm}=\delta_{\pm d(\omega)},\quad d(\omega)=\left(2\int_{-\infty}^{0}e^{(\frac{d_2}{4}-\frac{d_5}{8})t+\frac{\sqrt{d_5}}{\sqrt{2}}W_{\bar{r}}(s)}ds\right)^{-\frac{1}{2}},$$

$$\mu^{+}(\bar{r})=\begin{cases}\dfrac{(\frac{8}{d_5})^{\frac{1}{2}-\frac{d_2}{d_5}}}{\Gamma(\frac{d_2}{d_5}-\frac{1}{2})}\bar{r}^{\frac{2d_2}{d_5}-2}e^{-\frac{8\bar{r}^2}{d_5}}, & \bar{r}>0,\\ 0, & \bar{r}\le 0,\end{cases}\qquad \mu^{-}(\bar{r})=\mu^{+}(-\bar{r}).$$

Proof. The solution to equation (4.6) is

$$\varphi(t,\omega)\bar{r}=\frac{\bar{r}e^{(\frac{d_2}{8}-\frac{d_5}{16})t+\sqrt{\frac{d_5}{8}}W_{\bar{r}}}}{\left(1+2\bar{r}^2\int_0^t e^{(\frac{d_2}{8}-\frac{d_5}{16})s+\sqrt{\frac{d_5}{2}}W_{\bar{r}}(s)}ds\right)^{\frac{1}{2}}}.$$

The random domain $D(t,\omega)$ of $\varphi(t,\omega)$ is

$$D(t,\omega)=\begin{cases}\mathbb{R}, & t\ge 0,\\ (-\kappa(t,\omega)^{-1},\kappa(t,\omega)^{-1}), & t<0,\end{cases}$$

where $\kappa(t,\omega)=\sqrt{2|\int_0^t e^{(\frac{d_2}{4}-\frac{d_5}{8})s+\sqrt{\frac{d_5}{2}}W_{\bar{r}}(s)}ds|}$. Then the random range $R(t,\omega)$ of $\varphi(t,\omega)$ is

$$R(t,\omega)=\begin{cases}\mathbb{R}, & t\le 0,\\ (-\bar{r}(t,\omega),\bar{r}(t,\omega)), & t>0,\end{cases}$$

where $\bar{r}(t,\omega)=e^{(\frac{d_2}{8}-\frac{d_5}{16})t+\sqrt{\frac{d_5}{8}}W_{\bar{r}}}\kappa(t,\omega)^{-1}$.

(i) $D(\omega)=\bigcap\limits_{t\in\mathbb{R}}D(t,\omega)$, then

$$D(\omega)=\begin{cases}0, & 2d_2\le d_5,\\ [-\kappa(-\infty,\omega)^{-1},\kappa(-\infty,\omega)^{-1}], & 2d_2>d_5.\end{cases}$$

If $2d_2 \le d_5$, the unique invariant measure is the random Dirac measure δ_0. The linearized SDE is

$$dx = (\frac{d_2}{8} - \frac{d_5}{16} - 3\bar{r}^2)xdt + (\frac{d_5}{8}x^2)^{\frac{1}{2}} \circ dW_{\bar{r}}.$$

For $\bar{r} = 0$, the Lyapunov exponent is $\lambda(\delta_0) = \frac{d_2}{8} - \frac{d_5}{16} \le 0$. If $2d_2 > d_5$, there are two ergodic invariant measures $\nu^{\pm} = \delta_{\pm d(\omega)}$ in addition to the trivial reference measure $\delta_0(\lambda(\delta_0) > 0)$, where $d(\omega) = \kappa(-\infty, \omega)^{-1}$. The Lyapunov exponent is $\lambda(\nu) = \frac{d_5}{8} - \frac{d_2}{4}$. Hence, we have a D-bifurcation of the trivial reference measure δ_0 at $2d_2 = d_5$.

(ii) The measures $\nu^{\pm} = \delta_{\pm d(\omega)}$ correspond to the stationary measures

$$\mu^+(\bar{r}) = \begin{cases} \frac{(\frac{8}{d_5})^{\frac{1}{2} - \frac{d_2}{d_5}}}{\Gamma(\frac{d_2}{d_5} - \frac{1}{2})} \bar{r}^{\frac{2d_2}{d_5} - 2} e^{-\frac{8\bar{r}^2}{d_5}}, & \bar{r} > 0, \\ 0, & \bar{r} \le 0, \end{cases}$$

and $\mu^-(\bar{r}) = \mu^+(-\bar{r})$. This can be solved by the Fokker-Planck equation. Combining with $\mu^{\pm}(\bar{r})$, we have a P-bifurcation of the invariant measures $\nu^{\pm}$ at $d_2 = d_5$. □

4.3. The third case

If d_1, d_3 and d_4 are nonzero parameters, we investigate the dynamics of equation (4.1).

When $\bar{r} = \infty$, the diffusion exponent, the drifting exponent and characteristic value are 2, 3 and $-\frac{2d_3}{d_5}$, respectively. If $d_3 > 0$, then $\bar{r} = \infty$ is an EB; if $d_3 < 0$, then $\bar{r} = \infty$ is an AB. For $\bar{r} = 0$, it is neither the first kind singular boundary nor the second kind singular boundary. As a consequence, its global stability is unclear. However, we can analyze the stationary solution to the probability density function for equation (4.1). The probability density function is governed by the corresponding Fokker-Planck equation

$$\frac{\partial p(\bar{r}, t)}{\partial t} = -\frac{\partial}{\partial \bar{r}} \left((\frac{d_1}{\bar{r}} + \frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3)p(\bar{r}, t) \right) + \frac{1}{2}\frac{\partial^2}{\partial \bar{r}^2} \left((d_4 + \frac{d_5}{8}\bar{r}^2)p(\bar{r}, t) \right) \quad (4.7)$$

with the initial value $p(\bar{r}, t)_{t \to t_0} = \delta(\bar{r} - \bar{r}_0)$, where δ is the Dirac delta function. When $\partial p(\bar{r}, t)/\partial t = 0$, we acquire the stationary solution by solving the following second-order differential equation

$$0 = -\frac{\partial}{\partial \bar{r}} \left((\frac{d_1}{\bar{r}} + \frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3)p(\bar{r}) \right) + \frac{1}{2}\frac{\partial^2}{\partial \bar{r}^2} \left((d_4 + \frac{d_5}{8}\bar{r}^2)p(\bar{r}) \right). \quad (4.8)$$

Performing a straightforward integration on (4.8), we obtain a first-order equation

$$\int_{\bar{r}_0}^{\bar{r}} \frac{\partial}{\partial \bar{r}} \left((\frac{d_1}{\bar{r}} + \frac{d_2}{8}\bar{r} + \frac{d_3}{8}\bar{r}^3)p(\bar{r}) \right) d\bar{r} = \int_{\bar{r}_0}^{\bar{r}} \frac{1}{2}\frac{\partial^2}{\partial r^2} \left((d_4 + \frac{d_5}{8}\bar{r}^2)p(\bar{r}) \right) d\bar{r},$$

$$(\frac{d_1}{\bar{r}} + \frac{d_2}{8}\bar{r} + \frac{d_3}{8}r^3)p(\bar{r}) - C_1 = \frac{1}{2}\frac{\partial}{\partial \bar{r}} \left((d_4 + \frac{d_5}{8}\bar{r}^2)p(\bar{r}) \right) - C_2,$$

where $C_1 = (\frac{d_1}{\bar{r}_0} + \frac{d_2}{8}\bar{r}_0 + \frac{d_3}{8}\bar{r}_0^3)p(\bar{r}_0)$ and $C_2 = \frac{1}{2}\frac{\partial}{\partial \bar{r}}\left((d_4 + \frac{d_5}{8}\bar{r}^2)p(\bar{r})\right)_{\bar{r}=\bar{r}_0}$. Simplifying it as

$$\frac{\partial p(\bar{r})}{\partial \bar{r}} + P(\bar{r})p(\bar{r}) = Q(\bar{r}), \quad p(r_0) = p_0,$$

where

$$P(\bar{r}) = \frac{-2(8\frac{d_1}{\bar{r}} + d_2\bar{r} - d_5\bar{r} + d_3\bar{r}^3)}{8d_4 + d_5\bar{r}^2}, \quad Q(\bar{r}) = \frac{16(C_2 - C_1)}{8d_4 + d_5\bar{r}^2}.$$

Then the stationary solution is

$$\begin{aligned} p(\bar{r}) =& e^{-\int_{\bar{r}_0}^{\bar{r}} P(x)dx} \left(\int_{\bar{r}_0}^{\bar{r}} Q(s) e^{\int_{\bar{r}_0}^{s} P(x)dx} ds + p_0 \right), \\ =& (\Xi(\bar{r}) - \Xi(\bar{r}_0)) \left(p_0 + \int_{\bar{r}_0}^{\bar{r}} Q(s)(\Xi^{-1}(s) - \Xi^{-1}(\bar{r}_0)) ds \right), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \Xi(\bar{r}) =& e^{\frac{d_3\bar{r}^2}{d_5}} \bar{r}^{\frac{2d_1}{d_4}} (8d_4 + d_5\bar{r}^2)^{-1 - \frac{d_1}{d_4} + \frac{d_2 d_5 - 8 d_3 d_4}{d_5^2}}, \\ C_2 =& \frac{1}{8} d_5 \bar{r}_0 p_0 + \frac{1}{2}(d_4 + \frac{d_5}{8}\bar{r}_0^2) \frac{\partial \Xi(\bar{r}_0)}{\partial \bar{r}} p_0. \end{aligned}$$

Next, we give the numerical simulation of $p(\bar{r})$ to show its changes with respect to parameters $d_1(d_4 = 2d_1)$, d_2 and d_5 (see Figure 1, Figure 2 and Figure 3). We do not give the figure between d_3 and $p(\bar{r})$, because $p(\bar{r})$ has little change with respect to d_3.

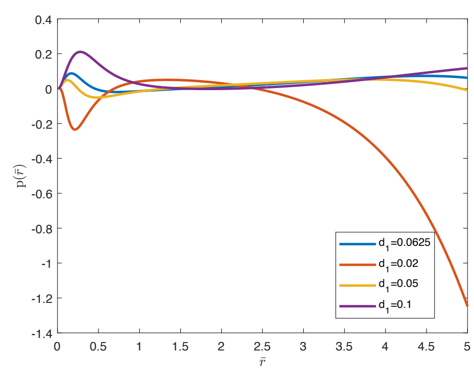

Figure 1. The stationary density function $p(\bar{r})$ with respect to $d_1(d_4 = 2d_1)$, when the other parameters are $d_2 = -1.2$, $d_3 = -0.25$ and $d_5 = 8$.

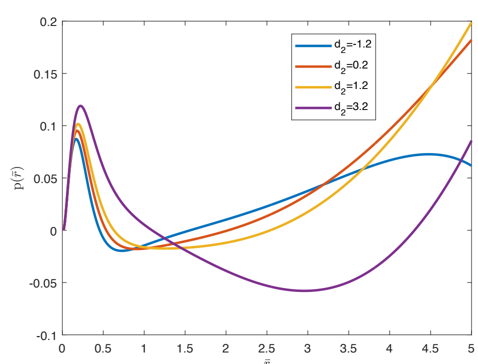

Figure 2. The stationary density function $p(\bar{r})$ with respect to d_2, when the other parameters are $d_4 = 2d_1 = 0.125$, $d_3 = -0.25$ and $d_5 = 8$.

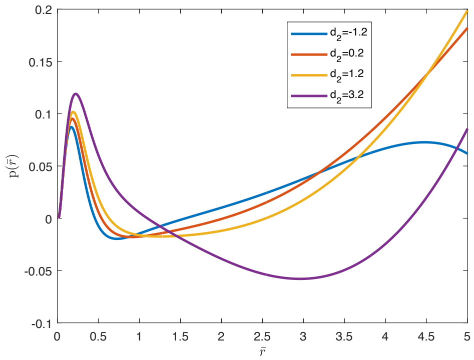

Figure 3. The stationary density function $p(\bar{r})$ with respect to d_5, when the other parameters are $d_4 = 2d_1 = 0.125$, $d_2 = -1.2$ and $d_3 = -0.25$.

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