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Bifurcations and Exact Solutions of the Gerdjikov-Ivanov Equation*

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Abstract For the Gerdjikov-Ivanov equation, by using the method of dynamical system, this paper investigates the exact explicit solutions with the form $q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \xi = x - ct$. In the given parameter regions, more than 14 explicit exact parametric representations are presented.

Keywords Bifurcation, exact solution, planar Hamiltonian system, Gerdjikov-Ivanov equation

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1. Introduction

In [8], Fang stated that "The nonlinear Schrödinger (NLS) equation is one of the most generic soliton equations, and arises from a wide variety of fields such as quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied. Among them, there are three celebrated equations with derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger (DNLS) equations. One is the Kaup-Newell equation [17]:

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0, \tag{1.1}$$

which is usually called DNLSI. The second type is the Chen-Lee-Liu equation [4]:

$$iq_t + q_{xx} + i|q|^2 q_x = 0, \tag{1.2}$$

which is called DNLSII. The last one takes the form [13]:

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 (q^*)^2 = 0, \tag{1.3}$$

which is called the Gerjikov–Ivanov (GI) equation or DNLSIII. In equation (1.3), q^* denotes the complex conjugation of q.

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Equation (1.3) has been studied by many authors (see [1, 2, 5–7, 9–12, 14–16, 18, 21–32]). The purpose of this paper is to study some new exact traveling wave solutions in explicit form of the GI equation by using the bifurcation theory of dynamical system. We assume that the exact solutions of equation (1.3) take the form:

$$q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \quad \xi = x - ct, \tag{1.4}$$

where c is the wave velocity, and $\phi(\xi), \theta(\xi)$ are two functions with variable ξ, κ and ω are two constant parameters. Substituting (1.4) into equation (1.3) and separating the real and imaginary parts respectively, we have

$$\phi'' + (c-2\kappa)\theta'\phi - (\kappa + \theta')\phi^3 + (\omega - \kappa^2)\phi - (\theta')^2\phi = 0, \tag{1.5}$$

$$\phi'\phi^2 - \theta''\phi - 2\phi'\theta' + (c-2\kappa)\phi' = 0, \tag{1.6}$$

where "$'$" is the derivative with respect to ξ. Integrating (1.6), it follows that $(c-2\kappa)\phi + \frac{1}{3}\phi^3 = C_1 + \theta'\phi + \int \theta' d\phi$, where C_1 is an integral constant. Thus, we obtain $C_1 = 0$ and

$$\theta' = \frac{1}{2}(c-2\kappa) + \frac{1}{4}\phi^2, \quad \theta(\xi) = \frac{1}{2}(c-2\kappa)\xi + \frac{1}{4}\int \phi^2(\xi)d\xi. \tag{1.7}$$

Substituting (1.7) into (1.5), we obtain the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\left(\left(\frac{1}{4}c^2 - c\kappa + \omega\right)\phi - \frac{1}{2}c\phi^3 + \frac{3}{16}\phi^5\right). \tag{1.8}$$

System (1.8) has the first integral:

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{1}{2}\alpha\phi^2 - \frac{c}{8}\phi^4 + \frac{1}{32}\phi^6 = h, \tag{1.9}$$

where $\alpha = \frac{1}{4}c^2 - c\kappa + \omega$.

System (1.8) is a three-parameter planar dynamical system depending on the parameter group (c, κ, ω). Since the parametric representations of the phase orbits defined by the vector fields of system (1.8) give rise to all exact solutions with the form (1.4) of equation (1.3), we need to investigate the bifurcations of phase portraits for system (1.8) in the $(\phi, y)-$phase plane as the parameters are changed (see [19, 20, 33]).

The main result in the present paper is summarized as follows.

Theorem 1.1. *Assume that the parameter $c > 0$ of system* (1.8) *is fixed. Consider the solutions of equation* (1.3) *with the form $q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right]$. Then, the following conclusions hold.*

(i) For any $(\kappa, \omega) \in R^2$ in Figure 1 (a), corresponding to the families of the periodic orbits of system (1.8)*, equation* (1.1) *always has the exact explicit solutions with the parametric representations given by* (4.1) *or* (4.2).

(ii) When $(\kappa, \omega) \in (II), (III), (IV)$ and $(L_2), (L_3)$ in Figure 1 (a), corresponding to the families of the periodic orbits of system (1.8)*, equation* (1.1) *has the exact explicit solutions with the parametric representations given by* (4.3)*,* (4.4)*,* (4.5)*,* (4.6) *and* (4.7).

(iii) When $(\kappa, \omega) \in (II), (III)$ and $(L_1), (L_2)$ in Figure 1 (a), corresponding to the heteroclinic orbits of system (1.8)*, equation* (1.1) *has the exact explicit solutions with the parametric representations given by* (4.8)*,* (4.9) *and* (4.10).

(iv) When $(\kappa,\omega)\in(II),(III),(IV)$ *and* $(L_2),(L_3)$ *in Figure 1 (a), corresponding to the homoclinic orbits of system* (1.8), *equation* (1.1) *has the exact explicit solutions with the parametric representations given by* (4.11), (4.12), (4.13) *and* (4.14).

The proof of this theorem is given in the following Sections 2, 3 and 4.

The rest of this paper is organized as follows: in Section 2, the bifurcations of the phase portraits of system (1.8) are studied. In Section 3, all exact explicit parameter representations of bounded phase orbits of system (1.8) are given in given parameter regions. In Section 4, corresponding to the solutions of system (1.8), all exact explicit parameter representations of equation (1.3) are derived.

2. Bifurcations of phase portraits of system (1.8)

To find the equilibrium points of system (1.8), write that $f(\phi)=\alpha-\frac{1}{2}c\phi^2+\frac{3}{16}\phi^4$. Clearly, if $\Delta=c^2+12(c\kappa-\omega)>0$, then when $\phi^2=\frac{2}{3}(2c\mp\sqrt{\Delta}), f(\phi)=0$. Thus, for a fixed $c>0$, we have the following conclusions.

(i) When $\omega>c\kappa+\frac{1}{12}c^2, \Delta<0$, system (1.8) has only one singular point $O(0,0)$.

(ii) When $\omega=c\kappa+\frac{1}{12}c^2, \Delta=0$, system (1.8) has one simple singular point $O(0,0)$ and two double equilibrium points $E_{1,2}\left(\mp\frac{2}{3}\sqrt{3c},0\right)$.

(iii) When $c\kappa-\frac{1}{4}c^2<\omega<c\kappa+\frac{1}{12}c^2$, we have $\Delta>0, 2c-\sqrt{\Delta}>0$. Hence, on the ϕ-axis, there exist five equilibrium points of system (1.8) at $E_1(-\phi_2,0), E_2(-\phi_1,0)$, $O(0,0)$, $E_3(\phi_1,0)$ and $E_4(\phi_2,0)$, where $\phi_{1,2}=\left(\frac{2}{3}(2c\mp\sqrt{\Delta})\right)^{\frac{1}{2}}$.

(iv) When $\omega=c\kappa-\frac{1}{4}c^2$, we have $2c-\sqrt{\Delta}=0$. system (1.8) has a high-order equilibrium point $O(0,0)$ and two simple equilibrium points $E_1(-\phi_2,0)$ and $E_2(\phi_2,0)$.

(v) When $\omega<c\kappa-\frac{1}{4}c^2$, system (1.8) has three simple equilibrium points at $O(0,0), E_1(-\phi_2,0)$ and $E_2(\phi_2,0)$.

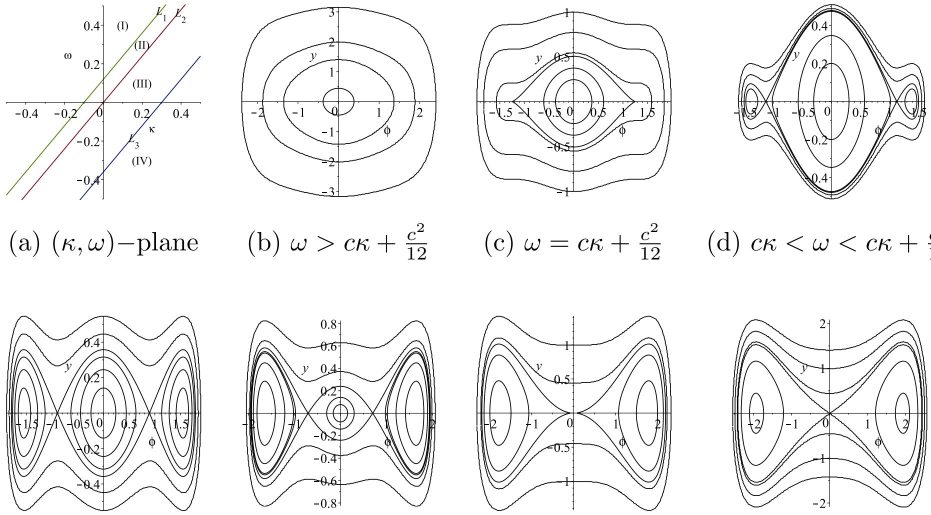
Let $M(\phi_j,0)$ be the coefficient matrix of the linearized system of system (1.8) at the equilibrium point E_j. Let $J(\phi_j,0)$ be its Jacobin determinant. Then, one has

$$J(0,0)=\alpha,\quad J(\phi_j,0)=\phi_j f'(\phi_j).$$

By the theory of planar dynamical system, for an equilibrium point of a planar Hamiltonian system, if $J<0$, then the equilibrium point is a saddle point; if $J>0$, then it is a center point; if $J=0$ and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

Now, write that $h_j=H(\phi_j,0), h_0=H(0,0)=0$. We have that $h_1=\frac{\Delta}{54}(c+\sqrt{\Delta}), h_2=\frac{\Delta}{54}(c-\sqrt{\Delta})$. Obviously, when $\omega=c\kappa$, we have $h_2=0$.

Based on the above results, for a fixed parameter $c>0$, in the $(\kappa,\omega)-$parameter plane, there exist three bifurcation straight lines $(L_1):\omega=c\kappa+\frac{1}{12}c^2; (L_2):\omega=c\kappa; and (L_3):\omega=c\kappa-\frac{1}{4}c^2$. These straight lines divide the $(\kappa,\omega)-$parameter plane into the four regions: (I), (II), (III), (IV) (see Figure 1 (a)). As ω is varied, we obtain the bifurcations of the phase portraits of system (1.8), as shown in Figure 1(b)-(h).



(a) $(\kappa, \omega)-$plane (b) $\omega > c\kappa + \frac{c^2}{12}$ (c) $\omega = c\kappa + \frac{c^2}{12}$ (d) $c\kappa < \omega < c\kappa + \frac{c^2}{12}$

(e) $\omega = c\kappa$ (f) $c\kappa - \frac{c^2}{4}\omega < c\kappa$ (g) $\omega = c\kappa - \frac{c^2}{4}$ (h) $\omega < c\kappa - \frac{c^2}{4}$

Figure 1. Bifurcations of phase portraits of system (1.8) for a fixed $c > 0$

3. Exact parametric representations of the solutions defined by all bounded orbits of system (1.8)

We see from (1.9) that

$$y^2 = 2h - \alpha\phi^2 + \frac{c}{4}\phi^4 - \frac{1}{16}\phi^6. \tag{3.1}$$

By using the first equation of (1.8), we obtain

$$\xi = \int_{\phi_0}^{\phi} \frac{4d\phi}{\sqrt{16h - 8\alpha\phi^2 + 4c\phi^4 - \phi^6}} = \int_{\psi_0}^{\psi} \frac{2d\psi}{\sqrt{\psi(16h - 8\alpha\psi + 4c\psi^2 - \psi^3)}}. \tag{3.2}$$

Thus, we can calculate the exact parametric representations defined by the bounded orbits of system (1.8).

3.1. Exact periodic solutions of system (1.8) when $(\kappa, \omega) \in (I)$ in Figure 1(a)

In this case, corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a family of periodic orbits of system (1.8) (see Figure 1(b)). Now, (3.2) can be written as $\frac{1}{2}\xi = \int_0^{\psi} \frac{d\psi}{\sqrt{(\psi_1-\psi)\psi(\psi+\psi_3)(\psi+\psi_4)}}$ or $\frac{1}{2}\xi = \int_0^{\psi} \frac{d\psi}{\sqrt{(\psi_1-\psi)\psi[(\psi-b_1)^2+a_1^2]}}$. Hence, we have the following two exact parametric representations of the periodic family of system (1.8):

$$\phi(\xi) = \frac{\sqrt{\psi_3}\hat{\alpha}_1 \mathrm{sn}(\Omega_1\xi, k)}{(1 - \hat{\alpha}_1^2 \mathrm{sn}^2(\Omega_1\xi, k))^{\frac{1}{2}}}, \tag{3.3}$$

where $\hat{\alpha}_1^2 = \frac{\psi_1}{\psi_1+\psi_3}, k^2 = \frac{\hat{\alpha}_1^2(\psi_4-\psi_3)}{\psi_4}, \Omega_1 = \frac{1}{4}\sqrt{(\psi_1+\psi_3)\psi_4}, \mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$ are the Jacobian elliptic functions (see Byrd and Fridman [3]).

$$\phi(\xi) = \left(\alpha_1 + \frac{\beta_1}{1+\hat{\alpha}_2 \mathrm{cn}(\Omega_2\xi, k)}\right)^{\frac{1}{2}}, \tag{3.4}$$

where $A_1^2 = (\psi_1 - b_1)^2 + a_1^2, B_1^2 = b_1^2 + a_1^2, k^2 = \frac{(\psi_1^2-(A_1-B_1)^2}{4A_1B_1}, \alpha_1 = \frac{-\psi_1 B_1}{A_1-B_1}, \beta_1 = \frac{2A_1B_1}{A_1^2-B_1^2}, \hat{\alpha}_2 = \frac{A_1-B_1}{A_1+B_1}, \Omega_2 = \frac{1}{2}\sqrt{A_1B_1}$.

3.2. Exact periodic solutions of system (1.8) when $(\kappa, \omega) \in (L_1)$ in Figure 1(a).

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, h_1), (h_1, \infty)$, $h_1 = h_2 = \frac{2c^3}{27}$, there exist two families of periodic orbits of system (1.8) (see Figure 1(c)). These periodic orbits have the same exact parametric representations as (3.4).

(ii) Corresponding to the level curves defined by $H(\phi, y) = \frac{2c^3}{27}$, there exist two heteroclinic orbits connecting two double equilibrium points $E_{1,2}\left(\mp\frac{2}{3}\sqrt{3c}, 0\right)$ of system (1.8) (see Figure 1(c)). Now, (3.2) can be written as $\xi = \int_0^\phi \frac{12\sqrt{3}d\phi}{(4c-3\phi^2)^{\frac{3}{2}}} = \frac{3\sqrt{3}\phi}{c\sqrt{4c-3\phi^2}}$. Thus, it follows the exact parametric representations:

$$\phi(\xi) = \mp\frac{4c^{\frac{3}{2}}\xi}{\sqrt{3(9+c^2\xi^2)}}. \tag{3.5}$$

3.3. Exact periodic solutions and homoclinic, heteroclinic solutions of system (1.8) when $(\kappa, \omega) \in (II)$ in Figure 1(a).

In this case, we have $0 < h_2 < h_1$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, h_2]$, there exists a family of periodic orbits of system (1.8), enclosing the origin $O(0,0)$ (see Figure 1(d)). When $0 < h < h_2$, these periodic orbits have the same exact parametric representations as (3.4).

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in [h_2, h_1)$, there exist three families of periodic orbits of system (1.8), enclosing the origin $O(0,0)$, $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively (see Figure 1(d)). Now, (3.2) can be written as $\frac{1}{2}\xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi_b-\psi)(\psi_c-\psi)\psi}}$ and $\frac{1}{2}\xi = \int_{\psi_b}^\psi \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi-\psi_b)(\psi-\psi_c)\psi}}$. These periodic orbits have the exact parametric representations as follows:

$$\phi(\xi) = \frac{\sqrt{\psi_a}|\hat{\alpha}_3|\mathrm{sn}(\Omega_3\xi, k)}{(1-\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi, k))^{\frac{1}{2}}}, \tag{3.6}$$

where $\hat{\alpha}_3^2 = \frac{-\psi_c}{\psi_a-\psi_c}, k^2 = \frac{-\hat{\alpha}_3^2(\psi_a-\psi_b)}{\psi_b}, \Omega_3 = \frac{1}{4}\sqrt{(\psi_a-\psi_c)\psi_b}$ and

$$\phi(\xi) = \mp\left(\psi_c + \frac{\psi_b-\psi_c}{1-\hat{\alpha}_4^2\mathrm{sn}^2(\Omega_3\xi, k)}\right)^{\frac{1}{2}}, \tag{3.7}$$

where $\hat{\alpha}_4^2 = \frac{\psi_a-\psi_b}{\psi_a-\psi_c}, k^2 = \frac{\hat{\alpha}_4^2\psi_c}{\psi_b}$.

Notice that the level curve defined by $H(\phi, y) = h_2$ contain a periodic orbit enclosing the origin $O(0,0)$ and two singular points $E_1(-\phi_2, 0), E_4(\phi_2, 0)$. In this case, in (3.6), $\psi_a = \psi_b = \phi_2^2, k^2 = 0, \hat{\alpha}_3^2 = \frac{-\psi_c}{\phi_2^2 - \psi_c}$ and $\Omega_3 = \frac{1}{4}\sqrt{(\phi_2^2 - \psi_c)\phi_2^2}$. The periodic orbit has the exact parametric representation:

$$\phi(\xi) = \frac{\phi_2 |\hat{\alpha}_3| \sin(\Omega_3 \xi)}{(1 - \hat{\alpha}_3^2 \sin^2(\Omega_3 \xi)^{\frac{1}{2}}}. \tag{3.8}$$

(iii) Corresponding to the level curves defined by $H(\phi, y) = h_1$, there exist two homoclinic orbits enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively, and two heteroclinic orbits enclosing the origin $O(0,0)$ (see Figure 1 (d)). In this case, (3.2) becomes that $\frac{1}{2}\xi = \int_0^\psi \frac{d\psi}{(\psi_1 - \psi)\sqrt{(\psi_M - \psi)\psi}}$ and $\frac{1}{2}\xi = \int_\psi^{\psi_M} \frac{d\psi}{(\psi - \psi_1)\sqrt{(\psi_M - \psi)\psi}}$, where $\psi_1 = \phi_1^2 = \frac{2}{3}\left(2c - \sqrt{\Delta}\right), \psi_M = \frac{4}{3}\left(c + \sqrt{\Delta}\right)$. Thus, the two heteroclinic orbits have the exact parametric representations:

$$\phi(\xi) = \mp \frac{\sqrt{\psi_M \psi_1} \tanh(\omega_1 \xi)}{\left((\psi_M - \psi_1) + \psi_1 \tanh^2(\omega_1 \xi)\right)^{\frac{1}{2}}}, \tag{3.9}$$

where $\omega_1 = \frac{1}{2}\sqrt{(\psi_M - \psi_1)\psi_1}$. The two homoclinic orbits have the exact parametric representations:

$$\phi(\xi) = \mp \left(\psi_1 + \frac{2(\psi_M - \psi_1)\psi_1}{\psi_M \cosh(2\omega_1 \xi) + 2\psi_1 - \psi_M}\right)^{\frac{1}{2}}. \tag{3.10}$$

(iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (1.8), enclosing six equilibrium points (see Figure 1(d)). These periodic orbits have the same exact parametric representations as (3.4).

3.4. Exact periodic solutions and homoclinic, heteroclinic solutions of system (1.8) when $(\kappa, \omega) \in (L_2)$ in Figure 1(a).

In this case, we have $\phi_1 = \frac{1}{3}\sqrt{6c}, \phi_2 = \sqrt{2c}, h_2 = 0$ and $h_1 = \frac{c^3}{27}$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, h_1)$, there exist three families of periodic orbits of system (1.8), enclosing the origin $O(0,0)$, $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively (see Figure 1(e)). These periodic orbits have the same exact parametric representations as (3.6) and (3.7).

(ii) Corresponding to the level curves defined by $H(\phi, y) = \frac{c^3}{27}$, there exist two homoclinic orbits enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively, and two heteroclinic orbits enclosing the origin $O(0,0)$ (see Figure 1(d)). In this case, (3.2) becomes that $\frac{1}{2}\xi = \int_0^\psi \frac{d\psi}{\left(\frac{2}{3}c - \psi\right)\sqrt{\left(\frac{8}{3}c - \psi\right)\psi}}$ and $\frac{1}{2}\xi = \int_\psi^{\frac{8}{3}c} \frac{d\psi}{\left(\psi - \frac{2}{3}c\right)\sqrt{\left(\frac{8}{3}c - \psi\right)\psi}}$. Therefore, the two heteroclinic orbits have the exact parametric representations:

$$\phi(\xi) = \mp \frac{4\sqrt{c} \tanh(\omega_2 \xi)}{\left(18 + 6 \tanh^2(\omega_2 \xi)\right)^{\frac{1}{2}}}, \tag{3.11}$$

where $\omega_2 = \frac{c}{\sqrt{3}}$. The two homoclinic orbits have the exact parametric representations:

$$\phi(\xi) = \mp \left(\frac{2}{3}c + \frac{2c}{2\cosh(2\omega_2 \xi) - 1}\right)^{\frac{1}{2}}. \tag{3.12}$$

(iii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (1.8), enclosing six equilibrium points (see Figure 1(e)). These periodic orbits have the same exact parametric representations as (3.4).

3.5. Exact periodic solutions and homoclinic, heteroclinic solutions of system (1.8) when $(\kappa, \omega) \in (III)$ in Figure 1(a).

In this case, we have $h_2 < 0 < h_1$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, 0]$, there exist two families of periodic orbits of system (1.8), enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively (see Figure 1(f)). (3.2) can be written as $\frac{1}{2}\xi = \int_{\psi_b}^{\psi} \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi-\psi_b)\psi(\psi+\psi_d)}}$. It gives rise to the following exact parametric representations:

$$\phi(\xi) = \mp \frac{\sqrt{\psi_b}}{(1 - \hat{\alpha}_5^2 \mathrm{sn}^2(\Omega_5 \xi, k))^{\frac{1}{2}}}, \tag{3.13}$$

where $\hat{\alpha}_5^2 = 1 - \frac{\psi_b}{\psi_a}, k^2 = \frac{\hat{\alpha}_5^2 \psi_d}{\psi_b + \psi_d}$ and $\Omega_5 = \frac{1}{4}\sqrt{\psi_a(\psi_b + \psi_d)}$.

Especially, the level curves defined by $H(\phi, y) = 0$ are two periodic orbits enclosing the singular points $E_1(-\phi_2, 0)$ and $E_2(\phi_2, 0)$ respectively. In this case, in (3.13), we have $\psi_a = 2(c + 2\sqrt{c\kappa - \omega}), \psi_b = 2(c - 2\sqrt{c\kappa - \omega}), \psi_d = 0, \hat{\alpha}_5^2 = \frac{4\sqrt{c\kappa-\omega}}{c+2\sqrt{c\kappa-\omega}}, k^2 = 0, \Omega_5 = \frac{1}{2}\sqrt{c^2 - 4c\kappa + 4\omega}$. Hence, the two periodic orbits have the following exact parametric representations:

$$\phi(\xi) = \mp \frac{\sqrt{\psi_b}}{\left(1 - \hat{\alpha}_5^2 \sin^2(\Omega_5 \xi)\right)^{\frac{1}{2}}}, \tag{3.14}$$

(ii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, h_1)$, there exist three families of periodic orbits of system (1.8), enclosing the origin $O(0, 0)$ and $E_1(-\phi_2, 0), E_4(\phi_2, 0)$ respectively (see Figure 1(f)). These periodic orbits have the same exact parametric representations as (3.6) and (3.7).

(iii) Corresponding to the level curves defined by $H(\phi, y) = h_1$, there exist two homoclinic orbits enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively, and two heteroclinic orbits enclosing the origin $O(0, 0)$ (see Figure 1(f)). These orbits have the same exact parametric representations as (3.9) and (3.10).

(iv) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (1.8), enclosing six equilibrium points (see Figure 1(f)). These periodic orbits have the same exact parametric representations as (3.4).

3.6. Exact periodic solutions and homoclinic, heteroclinic solutions of system (1.8) when $(\kappa, \omega) \in (L_3)$ in Figure 1(a).

In this case, we have $\phi_1 = 0, \phi_2 = \frac{2}{3}\sqrt{6c}, h_1 = 0$ and $h_2 = -\frac{8c^3}{27}$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$, there exist two families of periodic orbits of system (1.8), enclosing $E_1(-\phi_2, 0)$

and $E_4(\phi_2, 0)$ respectively (see Figure 1(g)). These periodic orbits have the same exact parametric representations as (3.13).

(ii) Corresponding to the level curves defined by $H(\phi, y) = 0, h \in (h_2, 0)$, there exist two homoclinic orbits of system (1.8), enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively (see Figure 1(g)). In this case, (3.2) becomes that $\frac{1}{4}\xi = \int_\psi^{4c} \frac{d\psi}{\psi\sqrt{(4c-\psi)\psi}}$. Thus, we have the following exact parametric representations:

$$\phi(\xi) = \mp \frac{4\sqrt{c}}{\sqrt{4 + c^2\xi^2}}. \tag{3.15}$$

(iii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a family of periodic orbits of system (1.8), enclosing three equilibrium points (see Figure 1(g)). These periodic orbits have the same exact parametric representations as (3.4).

3.7. Exact periodic solutions and homoclinic solutions of system (1.8) when $(\kappa, \omega) \in (IV)$ in Figure 1(a).

In this case, we have $h_2 < 0$.

(i) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$, there exist two families of periodic orbits of system (1.8), enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively (see Figure 1(h)). These periodic orbits have the same exact parametric representations as (3.13).

(ii) Corresponding to the level curves defined by $H(\phi, y) = 0, h \in (h_2, 0)$, system (1.8) has two homoclinic orbits, enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$ respectively (see Figure 1(h)). In this case, (3.2) becomes that $\frac{1}{2}\xi = \int_\psi^{\psi_M} \frac{d\psi}{\psi\sqrt{(\psi_M-\psi)(\psi+\psi_i)}}$, where $\psi_M = 2(c + 2\sqrt{c\kappa - \omega}), \psi_i = 2(-c + 2\sqrt{c\kappa - \omega})$. Hence, we obtain the exact parametric representations of two homoclinic orbits:

$$\phi(\xi) = \mp \left(\frac{2\psi_M\psi_i}{(\psi_M + \psi_i)\cosh(\omega_0\xi) - (\psi_M - \psi_i} \right)^{\frac{1}{2}} = \mp \left(\frac{8(c\kappa - \omega - \frac{1}{4}c^2)}{2\sqrt{c\kappa - \omega}\cosh(\omega_0\xi) - c} \right)^{\frac{1}{2}}, \tag{3.16}$$

where $\omega_0 = \frac{1}{2}\sqrt{\psi_M\psi_i} = 2\sqrt{c\kappa - \omega - \frac{1}{4}c^2}$.

(iii) Corresponding to the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a family of periodic orbits of system (1.8), enclosing three equilibrium points (see Figure 1(g)). These periodic orbits have the same exact parametric representations as (3.4).

4. Exact parametric representations of the solutions of equation (1.3) with the form (1.4)

In this section, we use the exact solutions of system (1.8) derived in Section 3 to give the exact parametric representations of the solutions of equation (1.3) with the form (1.4). Notice that the function $\theta(\xi) = \frac{1}{2}(c - 2\kappa)\xi + \frac{1}{4}\int \phi^2(\xi)d\xi$. Therefore, we first need to calculate the integral $\int \phi^2(\xi)d\xi$, where $\phi(\xi)$ is given by (3.3)-(3.16).

4.1. Exact solutions of equation (1.3) given by the periodic solutions of system (1.8).

(i) Corresponding to the family of the periodic orbits of system (1.8) enclosing the origin $O(0,0)$ or the global family of periodic orbits enclosing five or three equilibrium points, given by (3.3) or (3.4), we have

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \tfrac{\psi_3\hat{\alpha}_1^2\mathrm{sn}^2(\Omega_1\xi,k)d\xi}{1-\hat{\alpha}_1^2\mathrm{sn}^2(\Omega_1\xi,k)}\\ &= \tfrac{\psi_3}{\hat{\Omega}_1}\left[\Pi\left(\arcsin(\mathrm{sn}(\Omega_1\xi,k)),\hat{\alpha}_1^2,k\right)-F(\arcsin(\mathrm{sn}(\Omega_1\xi,k)),k)\right],\end{aligned}$$

where F and Π are normal elliptic integrals of the first kind and the third kind respectively, and

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int\left(\alpha_1+\tfrac{\beta_1}{1+\hat{\alpha}_2\mathrm{cn}(\Omega_2\xi,k)}\right)d\xi\\ &= \alpha_1\xi+\tfrac{\beta_1}{1-\hat{\alpha}_2^2}\left[\Pi\left(\arccos(\mathrm{sn}(\Omega_1\xi,k)),\tfrac{\hat{\alpha}_2^2}{\hat{\alpha}_2^2-1},k\right)-\hat{\alpha}_2 f_1\right],\end{aligned}$$

where the function f_1 can be seen in (Byrd and Fridman [3], pp.215).

Similarly, corresponding to the family of the periodic orbits of system (1.8) enclosing the origin $O(0,0)$ given by (3.6), we have

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \tfrac{\psi_a\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k)d\xi}{1-\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k)}\\ &= \tfrac{\psi_a}{\hat{\Omega}_3}\left[\Pi\left(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),\hat{\alpha}_3^2,k\right)-F(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),k)\right].\end{aligned}$$

Especially, corresponding to the periodic orbits of system (1.8) enclosing the origin $O(0,0)$ given by (14_a), we have

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \tfrac{\psi_2|\hat{\alpha}_3|\sin(\Omega_3\xi)d\xi}{(1-\hat{\alpha}_3^2\sin^2(\Omega_3\xi)}\\ &= -\psi_2\xi+2\arctan\left(\tfrac{(2-\hat{\alpha}_3^2)\tan(\omega_3\xi)+\hat{\alpha}_3^2)}{2\sqrt{1-\hat{\alpha}_3^2}}\right).\end{aligned}$$

Write that

$$\theta_1(\xi)=\frac{1}{2}(c-2\kappa)\xi+\frac{\psi_3}{4\Omega_1}\left[\Pi\left(\arcsin(\mathrm{sn}(\Omega_1\xi,k)),\hat{\alpha}_1^2,k\right)-F(\arcsin(\mathrm{sn}(\Omega_1\xi,k)),k)\right];$$

$$\theta_2(\xi)=\frac{1}{2}(c-2\kappa)\xi+\frac{1}{4}\alpha_1\xi+\frac{\beta_1}{4(1-\hat{\alpha}_2^2)}\left[\Pi\left(\arccos(\mathrm{sn}(\Omega_1\xi,k)),\frac{\hat{\alpha}_2^2}{\hat{\alpha}_2^2-1},k\right)-\hat{\alpha}_2 f_1\right];$$

$$\theta_3(\xi)=\frac{1}{2}(c-2\kappa)\xi+\frac{\psi_a}{4\Omega_3}\left[\Pi\left(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),\hat{\alpha}_3^2,k\right)-F(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),k)\right];$$

$$\theta_4(\xi)=\frac{1}{2}(c-2\kappa)\xi-\frac{1}{4}\psi_2\xi+\frac{1}{2}\arctan\left(\frac{(2-\hat{\alpha}_3^2)\tan(\omega_3\xi)+\hat{\alpha}_3^2)}{2\sqrt{1-\hat{\alpha}_3^2}}\right).$$

Thus, we obtain the following exact solutions of equation (1.3):

$$q(x,t)=q_{_1}(x,t)=\frac{\sqrt{\psi_3}\hat{\alpha}_1\mathrm{sn}(\Omega_1\xi,k)}{(1-\hat{\alpha}_1^2\mathrm{sn}^2(\Omega_1\xi,k))^{\frac{1}{2}}}\exp\left(i\left[\kappa x-\omega t+\theta_1(\xi)\right]\right); \tag{4.1}$$

$$q(x,t)=q_2(x,t)=\left(\alpha_1+\frac{\beta_1}{1+\hat{\alpha}_2\mathrm{cn}(\Omega_2\xi,k)}\right)^{\frac{1}{2}}\exp\left(i\left[\kappa x-\omega t+\theta_2(\xi)\right]\right); \tag{4.2}$$

$$q(x,t)=q_3(x,t)=\frac{\sqrt{\psi_a}|\hat{\alpha}_3|\mathrm{sn}(\Omega_3\xi,k)}{(1-\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k))^{\frac{1}{2}}}\exp\left(i\left[\kappa x-\omega t+\theta_3(\xi)\right]\right); \tag{4.3}$$

$$q(x,t)=q_4(x,t)=\frac{\phi_2|\hat{\alpha}_3|\sin(\Omega_3\xi)}{(1-\hat{\alpha}_3^2\sin^2(\Omega_3\xi)^{\frac{1}{2}}}\exp\left(i\left[\kappa x-\omega t+\theta_4(\xi)\right]\right). \tag{4.4}$$

(ii) Corresponding to the family of the periodic orbits of system (1.8) enclosing the equilibrium points $E_1(-\phi_2,0)$ and $E_4(\phi_2,0)$ given by (3.7), (3.13) and (3.14), we have

$$\begin{aligned}\int\phi^2(\xi)d\xi&=\int\left(\psi_c+\tfrac{\psi_b-\psi_c}{1-\hat{\alpha}_4^2\mathrm{sn}^2(\Omega_3\xi,k)}\right)d\xi\\&=\psi_c\xi+(\psi_b-\psi_c)\Pi\left(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),\hat{\alpha}_4^2,k\right);\end{aligned}$$

$$\int\phi^2(\xi)d\xi=\int\left(\frac{\psi_b}{1-\hat{\alpha}_5^2\mathrm{sn}^2(\Omega_5\xi,k)}\right)d\xi=\psi_b\Pi\left(\arcsin(\mathrm{sn}(\Omega_5\xi,k)),\hat{\alpha}_5^2,k\right);$$

and

$$\begin{aligned}\int\phi^2(\xi)d\xi&=\int\left(\tfrac{\psi_b}{1-\hat{\alpha}_5^2\sin^2(\Omega_5\xi,k)}\right)d\xi\\&=\tfrac{\psi_b}{\Omega_5\sqrt{1-\hat{\alpha}_5^2}}\arctan\left(\tfrac{\sqrt{2-\hat{\alpha}_5^2}\tan(\Omega_5\xi)+\hat{\alpha}_5^2}{2\sqrt{1-\hat{\alpha}_5^2}}\right).\end{aligned}$$

Write that

$$\theta_5(\xi)=\frac{1}{2}(c-2\kappa+2\psi_c)\xi+\frac{1}{4}(\psi_b-\psi_c)\Pi\left(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),\hat{\alpha}_4^2,k\right)$$

and

$$\theta_6(\xi)=\frac{1}{2}(c-2\kappa)\xi+\frac{1}{4}\psi_b\Pi\left(\arcsin(\mathrm{sn}(\Omega_5\xi,k)),\hat{\alpha}_5^2,k\right),$$

$$\theta_7(\xi)=\frac{1}{2}(c-2\kappa)\xi+\frac{\psi_b}{4\Omega_5\sqrt{1-\hat{\alpha}_5^2}}\arctan\left(\frac{\sqrt{2-\hat{\alpha}_5^2}\tan(\Omega_5\xi)+\hat{\alpha}_5^2}{2\sqrt{1-\hat{\alpha}_5^2}}\right).$$

Therefore, we obtain the following exact solutions of equation (1.3):

$$q(x,t)=q_5(x,t)=\left(\psi_c+\frac{\psi_b-\psi_c}{1-\hat{\alpha}_4^2\mathrm{sn}^2(\Omega_3\xi,k)}\right)^{\frac{1}{2}}\exp\left(i\left[\kappa x-\omega t+\theta_5(\xi)\right]\right); \tag{4.5}$$

$$q(x,t)=q_6(x,t)=\frac{\sqrt{\psi_b}}{(1-\hat{\alpha}_5^2\mathrm{sn}^2(\Omega_5\xi,k))^{\frac{1}{2}}}\exp\left(i\left[\kappa x-\omega t+\theta_6(\xi)\right]\right); \tag{4.6}$$

$$q(x,t)=q_7(x,t)=\left(\frac{\psi_b}{1-\hat{\alpha}_5^2\sin^2(\Omega_5\xi,k)}\right)^{\frac{1}{2}}\exp\left(i\left[\kappa x-\omega t+\theta_7(\xi)\right]\right). \tag{4.7}$$

(iii) Corresponding to the two heteroclinic orbits of system (1.8) enclosing the equilibrium points $O(0,0)$ given by (3.5), (3.9) and (3.11), we have

$$\int\phi^2(\xi)d\xi=\frac{16c^3}{3}\int\frac{\xi^2}{9+c^2\xi^2}=16\left[\frac{1}{3}c\xi-\arctan\left(\frac{1}{3}c\xi\right)\right];$$

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$$\int \phi^2(\xi)d\xi = \int \frac{\psi_M\psi_1\tanh^2(\omega_1\xi)d\xi}{(\psi_M-\psi_1)+\psi_1\tanh^2(\omega_1\xi)}$$
$$= \psi_1\xi - 2\text{arctan}\left(\sqrt{\frac{\psi_1}{\psi_M-\psi_1}}\tanh\left(\tfrac{1}{2}\xi\right)\right)$$

and specially, for (3.11),

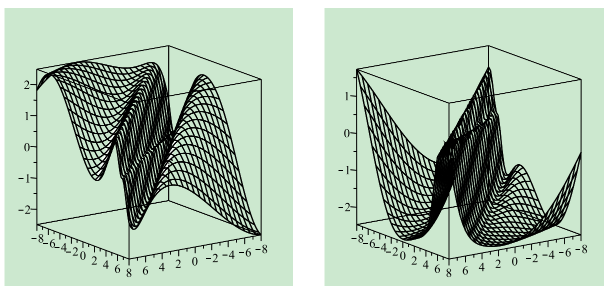
$$\int \phi^2(\xi)d\xi = \int \frac{16c\tanh^2(\omega_2\xi)d\xi}{18+6\tanh^2(\omega_2\xi)} = \frac{2}{3}c\xi - 2\arctan\left(\frac{1}{\sqrt{3}}\tanh\left(\frac{1}{2}\xi\right)\right).$$

Write that

$$\theta_8(x,t) = \left(\frac{11}{6}c-\kappa\right)\xi - 4\arctan\left(\frac{1}{3}c\xi\right);$$
$$\theta_9(x,t) = \frac{1}{2}\left(c-2\kappa+\frac{1}{2}\psi_1\right)\xi - \frac{1}{2}\text{arctan}\left(\sqrt{\frac{\psi_1}{\psi_M-\psi_1}}\tanh\left(\frac{1}{2}\xi\right)\right);$$
$$\theta_{10}(x,t) = \frac{1}{2}\left(\frac{4}{3}c-2\kappa\right)\xi - \frac{1}{2}\arctan\left(\frac{1}{\sqrt{3}}\tanh\left(\frac{1}{2}\xi\right)\right).$$

These give rise to the following exact solutions of equation (1.3):

$$q(x,t) = q_8(x,t) = \frac{4c^{\frac{3}{2}}\xi}{\sqrt{3(9+c^2\xi^2)}}\exp\left(i\left[\kappa x-\omega t+\theta_8(\xi)\right]\right); \tag{4.8}$$



(a) (b)

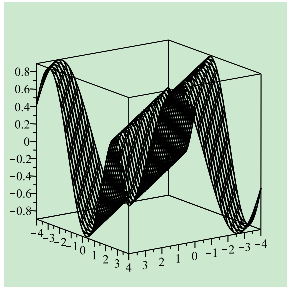
Figure 2. The graphs of the real part and imaginary part of solution $q_8(x,t)$

$$q(x,t) = q_9(x,t) = \frac{\sqrt{\psi_M\psi_1}\tanh(\omega_1\xi)}{\left((\psi_M-\psi_1)+\psi_1\tanh^2(\omega_1\xi)\right)^{\frac{1}{2}}}\exp\left(i\left[\kappa x-\omega t+\theta_9(\xi)\right]\right); \tag{4.9}$$

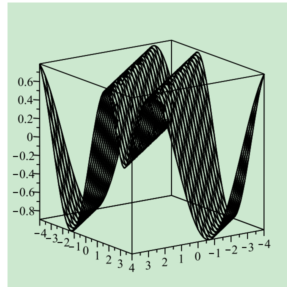
$$q(x,t) = q_{10}(x,t) = \frac{4\sqrt{c}\tanh(\omega_2\xi)}{\left(18+6\tanh^2(\omega_2\xi)\right)^{\frac{1}{2}}}\exp\left(i\left[\kappa x-\omega t+\theta_{10}(\xi)\right]\right). \tag{4.10}$$

(iv) Corresponding to the two homoclinic orbits of system (1.8) enclosing the equilibrium points $E_1(-\phi_2,0)$ and $E_4(\phi_2,0)$ given by (3.10), (3.12), (3.15) and (3.16), we have

$$\int \phi^2(\xi)d\xi = \int\left(\psi_1 + \frac{2(\psi_M-\psi_1)\psi_1}{\psi_M\cosh(2\omega_1\xi)+2\psi_1-\psi_M}\right)d\xi$$
$$= 2\arctan\left(\sqrt{\frac{\psi_M-\psi_1}{\psi_1}}\tanh\left(\tfrac{1}{2}\omega_1\xi\right)\right);$$



(a)



(b)

Figure 3. The graphs of the real part and imaginary part of solution $q_{10}(x,t)$

$$\int \phi^2(\xi)d\xi = \int \left(\frac{2}{3}c + \frac{2c}{2\cosh(2\omega_2\xi)-1}\right)d\xi = \frac{2}{3}c\xi + 2\arctan\left(\sqrt{3}\tanh\left(\frac{c}{2\sqrt{3}}\xi\right)\right);$$

$$\int \phi^2(\xi)d\xi = 16c\int \frac{d\xi}{4+c^2\xi^2} = 8\arctan\left(\frac{1}{2}c\xi\right);$$

$$\int \phi^2(\xi)d\xi = \int \left(\frac{8(c\kappa-\omega-\frac{1}{4}c^2)}{2\sqrt{c\kappa-\omega}\cosh(\omega_0\xi)-c}\right)d\xi$$
$$= 4\arctan\left(\left(\frac{2\sqrt{c\kappa-\omega}+c}{2\sqrt{c\kappa-\omega}-c}\right)^{\frac{1}{2}}\tanh\left(\sqrt{c\kappa-\omega-\frac{1}{4}c^2}\xi\right)\right).$$

Write that

$$\theta_{11}(\xi) = \frac{1}{2}(c-2\kappa)\xi + \frac{1}{2}\arctan\left(\sqrt{\frac{\psi_M-\psi_1}{\psi_1}}\tanh\left(\frac{1}{2}\omega_1\xi\right)\right);$$

$$\theta_{12}(\xi) = \frac{1}{2}\left(\frac{4}{3}c-2\kappa\right)\xi + \frac{1}{2}\arctan\left(\sqrt{3}\tanh\left(\frac{c}{2\sqrt{3}}\xi\right)\right);$$

$$\theta_{13}(\xi) = \frac{1}{2}(c-2\kappa)\xi + 4\arctan\left(\frac{1}{2}c\xi\right);$$

$$\theta_{14}(\xi) = \frac{1}{2}(c-2\kappa)\xi + \arctan\left(\left(\frac{2\sqrt{c\kappa-\omega}+c}{2\sqrt{c\kappa-\omega}-c}\right)^{\frac{1}{2}}\tanh\left(\sqrt{c\kappa-\omega-\frac{1}{4}c^2}\xi\right)\right).$$

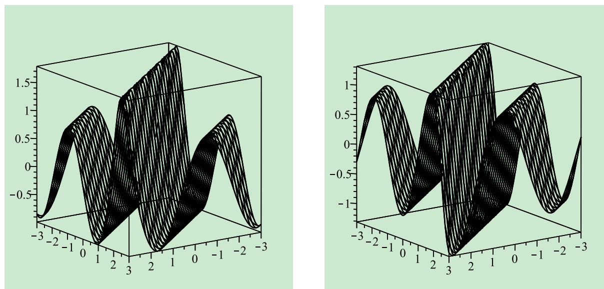
Hence, we obtain the exact solutions of equation (1.3) as follows:

$$q(x,t) = q_{11}(x,t) = \left(\psi_1 + \frac{2(\psi_M-\psi_1)\psi_1}{\psi_M\cosh(2\omega_1\xi)+2\psi_1-\psi_M}\right)^{\frac{1}{2}}\exp\left(i\left[\kappa x-\omega t+\theta_8(\xi)\right]\right); \tag{4.11}$$

$$q(x,t) = q_{12}(x,t) = \left(\frac{2}{3}c + \frac{2c}{2\cosh(2\omega_2\xi)-1}\right)^{\frac{1}{2}}\exp\left(i\left[\kappa x-\omega t+\theta_9(\xi)\right]\right); \tag{4.12}$$

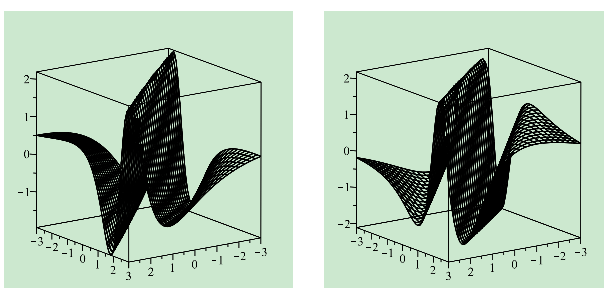
$$q(x,t) = q_{13}(x,t) = \frac{4\sqrt{c}}{\sqrt{4+c^2\xi^2}}\exp\left(i\left[\kappa x-\omega t+\theta_{10}(\xi)\right]\right); \tag{4.13}$$

$$q(x,t) = q_{14}(x,t) = \left(\frac{8(c\kappa-\omega-\frac{1}{4}c^2)}{2\sqrt{c\kappa-\omega}\cosh(\omega_0\xi)-c}\right)^{\frac{1}{2}}\exp\left(i\left[\kappa x-\omega t+\theta_{11}(\xi)\right]\right). \tag{4.14}$$



(a) (b)

Figure 4. The graphs of the real part and imaginary part of solution $q_{12}(x,t)$



(a) (b)

Figure 5. The graphs of the real part and imaginary part of solution $q_{13}(x,t)$

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