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Zero-Hopf Bifurcation at the Origin and Infinity for a Class of Generalized Lorenz System*

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Abstract In this paper, the zero-Hopf bifurcations are studied for a generalized Lorenz system. Firstly, by using the averaging theory and normal form theory, we provide sufficient conditions for the existence of small amplitude periodic solutions that bifurcate from zero-Hopf equilibria under appropriate parameter perturbations. Secondly, based on the Poincaré compactification, the dynamic behavior of the generalized Lorenz system at infinity is described, and the zero-Hopf bifurcation at infinity is investigated. Additionally, for the above theoretical results, some related illustrations are given by means of the numerical simulation.

Keywords Generalized Lorenz system, zero-Hopf bifurcation, averaging theory, normal form theory, Poincaré compactification

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1. Introduction

Due to the great application of chaotic systems in the real world, more and more scholars have focused on the complex dynamic properties of the chaotic models and the generation mechanism of chaos. Many different types of chaotic models have been found or constructed after Lorenz model was presented [16], for instance, Chua's circuit system [4], Chen system [3], Lü system [18], Yang-Chen system [26] and other Lorenz-type systems [9,13]. Then a great number of results on theoretical analysis have been obtained for the practical chaotic models above mentioned in the past several decades.

Here it is worth mentioning the study on Hopf bifurcation for these chaotic models. Its early results put emphasis on the existence and stability of only a single Hopf bifurcation, see e.g., [7, 10]. And then the multiple Hopf bifurcations began to be investigated for some models, for example, the Lü system [19], the Lorenz system [23], and the Maxwell-Bloch system [11]. The readers can also refer to the

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recent literatures [5, 17, 21]. However, for many chaotic models as general high-dimensional systems, the upper bounds on the cyclicity in the vicinity of a Hopf singular point is a very challenging problem [24, 28].

As for the zero-Hopf bifurcation, recently it is getting more and more attention in the researches on chaotic models. One important reason is that the zero-Hopf singular point of the high-dimensional system may reflect the emergence of chaotic behavior [1]. The hallmark of zero-Hopf bifurcation is that the linear part of the system has a zero eigenvalue and a pair of pure conjugates complex eigenvalues. The common tool for investigating this problem is the averaging theory, and many chaotic systems were considered, e.g., [14, 20]. Furthermore, the authors of [30] applied the normal form theory to investigate Rössler system, and showed that the method of normal forms is applicable for all types of zero-Hopf bifurcations, while the averaging method is successful only for a certain type of zero-Hopf singular points.

In addition, the study on the zero-Hopf bifurcation in some chaotic models can extend to two aspects: one is its multiplicity and cyclicity, and the other is zero-Hopf bifurcation problem at infinity. For the former, it is less fully studied though multiple limit cycles have been discovered in the application of averaging theory of second order, see e.g. [20]. For the latter, it has also been rarely considered though many dynamic behaviors at infinity have begun to be intensively analyzed [6, 12, 15, 25].

In this paper, we will consider a generalized Lorenz system which the following form

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - y - xz + dyz + ey^2 + fxy, \\ \dot{z} = xy - bz, \end{cases} \tag{1.1}$$

where $a, b, c, d, e, f \in \mathbb{R}$. When $d = e = f = 0$, system (1.1) becomes the Lorenz system. It is different from the generalized Lorenz system early proposed in [2], but system (1.1) also contains all the information of the Lorenz system. Here we will study the zero-Hopf bifurcation at finite equilibria and infinity for this generalized Lorenz system (1.1), and try to determine the zero-Hopf cyclicity only in the sense of first order averaging.

The rest of the article is organized as follows. In Section 2, we study the zero-Hopf bifurcations at the finite equilibria by using the averaging theory and normal form theory successively for system (1.1). In Section 3, the dynamical behaviors at infinity are discussed via the Poincaré compactification of the polynomial vector field in $\mathbb{R}^3$. In particular, the zero-Hopf bifurcation at infinity is investigated by applying the normal form theory. For the theoretical results obtained, the corresponding numerical simulations are given respectively in the above process of analysis.

2. Zero-Hopf bifurcation for the equilibria

In this section, we will study zero-Hopf bifurcation at finite equilibrium points. System (1.1) always has the equilibrium point $O = (0, 0, 0)$ for any parameter value. If $b(c-1)(d-1)\neq 0$, there exist two symmetrical equilibria

$$\begin{aligned} E_+ &= \left(\tfrac{2b(1-c)}{b(e+f)+\kappa}, \tfrac{2b(1-c)}{b(e+f)+\kappa}, \tfrac{4b(1-c)^2}{(b(e+f)+\kappa)^2}\right), \\ E_- &= \left(\tfrac{2b(1-c)}{b(e+f)-\kappa}, \tfrac{2b(1-c)}{b(e+f)-\kappa}, \tfrac{4b(1-c)^2}{(b(e+f)-\kappa)^2}\right), \end{aligned} \tag{2.1}$$

where we have let

$$d = \frac{b^2(e+f)^2 + 4b(c-1) - \kappa^2}{4b(c-1)}, \ \kappa \geq 0.$$

2.1. The case at the origin

Firstly, we apply the averaging theory [14, 20] to investigate the zero-Hopf bifurcation at the origin. It is easy to get the following characteristic polynomial at the origin O,

$$p(\lambda) = \lambda^3 + (1+a+b)\lambda^2 + (a+b+ab-ac)\lambda + ab(-1+c). \tag{2.2}$$

To guarantee that its eigenvalues are 0 and $\pm\omega i$ with $\omega > 0$, which means the origin is a zero-Hopf bifurcation point, we quickly get the conditions, which are given in the following Lemma.

Lemma 2.1. *The origin of system* (1.1) *is a zero-Hopf equilibrium if and only if the following condition is satisfied:*

$$a = -1, \quad b = 0, \quad c = 1 + \omega^2. \tag{2.3}$$

Next, we briefly introduce the averaging theory of first order. The following two initial value problems are considered:

$$\dot{\mathrm{x}} = \varepsilon F_1(t, \mathrm{x}) + \varepsilon^2 F_2(t, \mathrm{x}, \varepsilon), \qquad \mathrm{x}(0) = \mathrm{x}_0, \tag{2.4}$$

and

$$\dot{\mathrm{y}} = \varepsilon f(\mathrm{y}), \qquad \mathrm{y}(0) = \mathrm{x}_0, \tag{2.5}$$

where x, y and x_0 are in some open Ω of $\mathbb{R}^n$, $t \in [0,\infty)$, $\varepsilon \in (0, \varepsilon_0]$, F_1 and F_2 are periodic functions of period T in the variable t, and $f(\mathrm{y})$ is the averaged function of $F_1(t, \mathrm{y})$ with respect to t, namely,

$$f(\mathrm{y}) = \frac{1}{T}\int_0^T F_1(t, \mathrm{y})\mathrm{d}t. \tag{2.6}$$

And we give the two notations for the following Lemma, $D_{\mathrm{x}}F$ means the Jacobian matrix of F with respect to the components of x, and $D_{\mathrm{xx}}F$ means the Hessian matrix of F.

Lemma 2.2 ([14, 20]). *Assume that: (i) the functions F_1, $D_{\mathrm{x}}F_1, D_{\mathrm{xx}}F_1$ and $D_{\mathrm{x}}F_2$ are continuous, bounded by a constant, independent of ε in $[0,\infty)\times\Omega\times(0,\varepsilon_0]$;*
(ii) For $t \in [0, 1/\varepsilon]$, it follows that $\mathrm{x}(t) - \mathrm{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$;
(iii) If $\mathrm{p} \neq 0$ is an equilibrium point of system (2.5) and $\det D_{\mathrm{y}} f(\mathrm{p}) \neq 0$.
Then there exists a periodic solution $\Phi(t, \varepsilon)$ of period T for system (2.4) *which is close to* p *and such that $\Phi(t,\varepsilon) \to \mathrm{p}$ as $\varepsilon \to 0$.*

Remark 2.1. If there are m equilibrium points of system (2.5) satisfying all the conditions in the above Lemma 2.2, then system (2.4) has m corresponding periodic solutions, i.e., m limit cycles via zero-Hopf bifurcation. We call this the zero-Hopf cyclicity in the sense of first order averaging, and will consciously find its upper bounds without repeating this description in the following discussions.

Theorem 2.1. *Let*

$$(a, b, c) = (-1 + \alpha\varepsilon, \beta\varepsilon, 1 + \omega^2 + \gamma\varepsilon), \tag{2.7}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and ε is a sufficiently small positive parameter, for system (1.1) *a zero-Hopf bifurcation can occur around the origin. And under certain parameter conditions, one stable limit cycle can bifurcate.*

Proof. Substituting (2.7) into system (1.1), one gets the following system

$$\begin{cases} \dot{x} = y(\alpha\varepsilon - 1) - x(\alpha\varepsilon - 1), \\ \dot{y} = x(1 + \omega^2 + \gamma\varepsilon) - y - xz + dyz + ey^2 + fxy, \\ \dot{z} = xy - \beta\varepsilon z. \end{cases} \tag{2.8}$$

Via the rescaling of the variables (x, y, z)=$(\varepsilon X, \varepsilon Y, \varepsilon Z)$, system (2.8) in the new variables (X, Y, Z) becomes

$$\begin{cases} \dot{X} = Y(\alpha\varepsilon - 1) - X(\alpha\varepsilon - 1), \\ \dot{Y} = (1 + \omega^2 + \gamma\varepsilon)X - Y - \varepsilon XZ + \varepsilon(dYZ + eY^2 + fXY), \\ \dot{Z} = \varepsilon XY - \beta\varepsilon Z. \end{cases} \tag{2.9}$$

For the linear part at the origin O of system (2.9), when $\varepsilon = 0$, by the means of the linear substitution

$$(X, Y, Z) = \left(\frac{2(u - \omega v)}{\omega^2 + 1}, 2u, w\right).$$

we can transform it into the following real Jordan normal form, i.e.

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

Under the new variable (u, v, w) and time rescaling $t \to \omega t$, system (2.9) becomes the following form:

$$\begin{cases} \dot{u} = -v + \dfrac{\varepsilon f_u}{\omega\left(\omega^2 + 1\right)}, \\ \dot{v} = u - \dfrac{\varepsilon f_v}{\omega^2\left(\omega^2 + 1\right)}, \\ \dot{w} = -\dfrac{\varepsilon f_w}{\omega\left(\omega^2 + 1\right)}, \end{cases} \tag{2.11}$$

where $f_u = \gamma u - \gamma\omega v + uw\left(d\omega^2 + d - 1\right) + 2u^2\left(e\omega^2 + e + f\right) - 2fuv\omega + vw\omega$, $f_v = -uw\left(d\omega^2 + d - 1\right) - 2u^2\left(e\omega^2 + e + f\right) + 2fuv\omega + u\left(\alpha\omega^2\left(\omega^2 + 1\right) - \gamma\right) + v\omega\left(\alpha\omega^2 + \alpha + \gamma\right) - vw\omega$, and $f_w = \beta w\left(\omega^2 + 1\right) - 4u^2 + 4uv\omega$.

Furthermore, via the cylindrical coordinate substitution $(u,v,w) \rightarrow (r\cos\theta, r\sin\theta, w)$, we obtain the following form:

$$\begin{cases} \dot{r} = \varepsilon\dfrac{r}{\omega^2(1+\omega^2)}(\omega\cos\theta-\sin\theta)(-w\cos\theta+dw\cos\theta+\gamma\cos\theta \\ \qquad +dw\omega^2\cos\theta-w\omega\sin\theta+\beta\omega\sin\theta+\gamma\omega\sin\theta+\beta\omega^3\sin\theta, \\ \dot{\theta} = -1-\varepsilon\dfrac{\cos\theta^2}{\omega^2+\omega^4}(\gamma-\beta\omega^2-\beta\omega^4+w(-1+d+d\omega^2) \\ \qquad +\omega(\beta+2\gamma+\beta\omega^2+w(-2+d+d\omega^2))\tan\theta-(w-\gamma)\omega^2\tan\theta^2, \\ \dot{w} = \varepsilon\dfrac{-w\alpha(1+\omega^2)-r^2\cos\theta(\cos\theta+\omega\sin\theta)}{\omega(1+\omega^2)}. \end{cases} \tag{2.12}$$

By taking θ as the new independent variable, the differential system (2.12) becomes

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \varepsilon\frac{-r}{\omega^2(1+\omega^2)(1+\tan\theta^2)}(\omega-\tan\theta)(-w+dw+\gamma+dw\omega^2 \\ &\quad -w\omega\tan\theta+\beta\omega\tan\theta+\gamma\omega\tan\theta+\beta\omega^3\tan\theta)+O(\varepsilon^2) \\ &= \varepsilon F_1(r,\theta,w)+O(\varepsilon^2), \\ \frac{\mathrm{d}w}{\mathrm{d}\theta} &= \varepsilon\frac{(-r^2+w\alpha\sec\theta^2+w\alpha\omega^2\sec\theta-r^2\omega\tan\theta)}{\omega(1+\omega^2)(1+\tan\theta^2)}+O(\varepsilon^2) \\ &= \varepsilon F_2(r,\theta,w)+O(\varepsilon^2). \end{aligned} \tag{2.13}$$

According to the averaging theory described in Lemma 2.2 for system (2.13), using the notation introduced in Lemma 2.2, we have $t=\theta$, $T=2\pi$, $\mathrm{x}=(r,w)^T$, and

$$F(r,\theta,w) = \begin{pmatrix} F_1(r,\theta,w) \\ F_2(r,\theta,w) \end{pmatrix}, \quad f(r,w) = \begin{pmatrix} f_1(r,w) \\ f_2(r,w) \end{pmatrix},$$

where

$$\begin{aligned} f_1(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta,w)d\theta, \\ f_2(r,w) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta,w)d\theta. \end{aligned}$$

It is easy to know that system (2.13) satisfies all the conditions in Lemma 2.2. Then, we obtain

$$\begin{aligned} f_1(r,w) &= \frac{r(dw-\alpha)}{2\omega}, \\ f_2(r,w) &= \frac{2r^2-w\beta(1+\omega^2)}{\omega(1+\omega^2)}, \end{aligned} \tag{2.14}$$

where $\omega=\sqrt{c-1}$. Clearly, when $\alpha\beta d>0$ and $c>1$, the two equations $f_1(r,w)=f_2(r,w)=0$ have three common solutions $(r,w)=(0,0)$ and $(\pm\sqrt{\frac{\alpha\beta(1+\omega^2)}{2d}}, \frac{\beta}{d})$. Since r represents the amplitude of periodic solutions, it should be positive. Then we choose the unique positive solution (r_*, w_*), namely,

$$r_* = \sqrt{\frac{\alpha\beta c}{2d}}, \qquad w_* = \frac{\beta}{d},$$

and know that the Jacobian at the point (r_*, w_*) is

$$\frac{\partial(f_1, f_2)}{\partial(r, w)}\,|_{(r,w)=(r_*,w_*)}= \begin{pmatrix} 0 & \frac{2\sqrt{2}\sqrt{\alpha\beta}}{\sqrt{c-1}\sqrt{cd}} \\ \frac{\sqrt{\alpha\beta}\sqrt{cd}}{2\sqrt{2}\sqrt{c-1}} & -\frac{\alpha}{\sqrt{c-1}} \end{pmatrix}. \tag{2.15}$$

And more the determinant of Jacobian matrix (2.15) at (r_*, w_*) takes the following value:

$$-\frac{\alpha\beta}{c-1} \neq 0.$$

By Lemma 2.2, we establish the existence of a 2π periodic solution $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ of system (2.13) for the sufficiently small ε, such that $(r(0,\varepsilon), w(0,\varepsilon)) \to (r_*, w_*)$ when $\varepsilon \to 0$. Hence, we obtain only one periodic solution of system (2.11)

$$u(\theta,\varepsilon) = \varepsilon r_* \cos\theta + O(\varepsilon^2), v(\theta,\varepsilon) = \varepsilon r_* \sin\theta + O(\varepsilon^2), w(\theta,\varepsilon) = \varepsilon w_* + O(\varepsilon^2),$$

namely, for $\varepsilon > 0$ sufficiently small, system (2.8) has one periodic solution. Furthermore, we figure out two eigenvalues of the Jacobian at the point (r_*, w_*) in (2.15) as follows:

$$\lambda_{1,2} = \frac{-\beta \pm \sqrt{\beta(4\alpha+\beta)}}{2\sqrt{c-1}}.$$

It is easy to know that if and only if $\beta > 0$, $c > 1$ and $-\frac{\beta}{4} \leq \alpha < 0$, $\lambda_{1,2}$ are all negative, namely the steady-state solution (r_*, w_*) is stable.

Therefore, there is a stable periodic solution around the origin when $\varepsilon = 0$. The proof is completed. □

Here we give a numerical example of one stable limit cycle via zero-Hopf bifurcation around the origin of system (1.1) as shown in Figure1. In this example, we have set $d = e = f = 0$, and $a = -1 + \alpha\varepsilon, b = \beta\varepsilon, c = 1 + \omega^2 + \gamma\varepsilon$ with $\alpha = -0.125, \beta = 1, \gamma = -1, \varepsilon = 0.01, \omega = 1$.

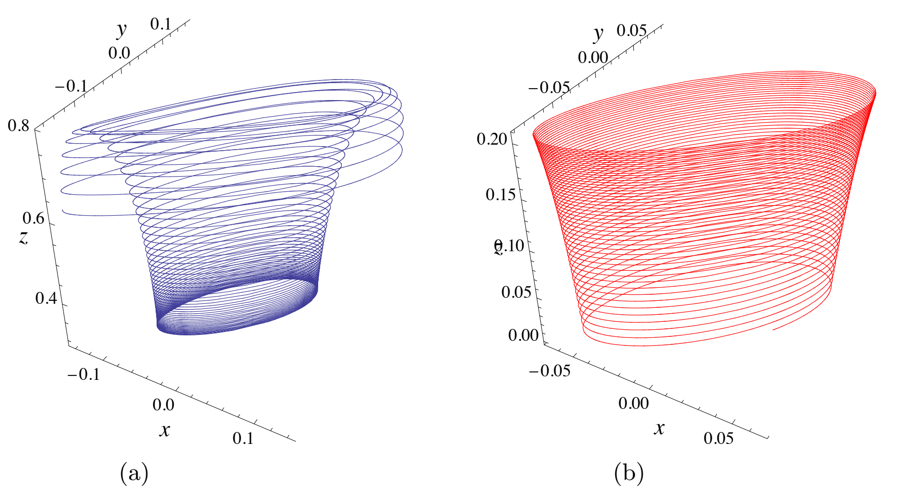


Figure 1. Simulations of system (1.1) for $a = -1.00125, b = 0.01, c = 1.99, d = e = f = 0$ converging to the stable limit cycle around the origin with the initial conditions: (a) $(x_0, y_0, z_0) = (0.092, 0, 0)$ and (b) $(x_0, y_0, z_0) = (0.04, 0, 0)$.

2.2. The case at the equilibrium points $E_\pm$

Now, we consider the equilibrium points $E_\pm$. For the convenience of calculation, we let $e+f=1$, i.e., $e=1-f$. Calculating the characteristic polynomial at E_+ or E_- to guarantee that its eigenvalues are 0 and $\pm\omega i$ with $\omega>0$, we get one necessary condition:

$$\kappa=0,\ \text{i.e.,}\ d=\frac{b+4(c-1)}{4(c-1)}.$$

At this time, from (2.1) the two equilibrium points $E_\pm$ merge into one point $E=(2(1-c),2(1-c),\frac{4(1-c)^2}{b})$. And the other necessary conditions are also determined:

$$\begin{aligned} a&=\tfrac{b^2+bc-b+\omega^2}{2(c-1)}:=a_0,\\ f&=\tfrac{b^3+3b^2(c-1)+b\left(6c^2-10c+\omega^2+4\right)-8(c-1)^3}{4b(c-1)^2}:=f_0. \end{aligned} \tag{2.16}$$

Thus the equilibrium point E is a zero-Hopf bifurcation point. Under these conditions (2.16), via the translation $\mathrm{x}=(x,y,z)\mapsto \mathrm{x}+E$, and through one non-degenerate linear transformation $\mathrm{x}'=T\mathrm{u}'$ where the transposed vector $\mathrm{u}'=(u,v,w)'$ and the transformation matrix

$$T=\begin{pmatrix} -\frac{(b+c-1)\left(\omega^2+b(b+c-1)\right)}{2(c-1)((b+c-1)^2+\omega^2)} & -\frac{-2\omega^3-2b(b+c-1)\omega}{4(c-1)((b+c-1)^2+\omega^2)} & \frac{b}{4-4c}\\ -\frac{2b(b+c-1)^2+2(b-c+1)\omega^2}{4(c-1)((b+c-1)^2+\omega^2)} & -\frac{-2\omega^3-2(b+c-1)(b+2c-2)\omega}{4(c-1)((b+c-1)^2+\omega^2)} & \frac{b}{4-4c}\\ 2 & 0 & 1 \end{pmatrix},$$

we can obtain the system with the linear part of real Jordan normal form as follows:

$$\begin{cases} \dot u=-\omega\, v+P_u(u^2,v^2,w^2,uv,uw,vw),\\ \dot v=\omega\, u+P_v(u^2,v^2,w^2,uv,uw,vw),\\ \dot w=P_w(u^2,v^2,w^2,uv,uw,vw), \end{cases} \tag{2.17}$$

where P_u,P_v and P_u are quadratic polynomials in u,v,w.

Remark 2.2. With the same process in the last subscetion, applying the averaging theory to investigate the zero-Hopf bifurcation around the origin of system (2.17), we discover that there exists no positive solution of r for the corresponding equations $f_1(r,w)=f_2(r,w)=0$. In other words, using the averaging theory of first order we cannot find a periodic solution bifurcating from the zero-Hopf equilibrium point E. Similar to the case in [30], it also reveals that the time-averaging method of first order fails for some type of zero-Hopf bifurcation.

Now, we shall use the normal form theory to investigate the zero-Hopf bifurcation at the equilibrium E of system (1.1). For this purpose, we first give the following Lemma.

Lemma 2.3. *By perturbing slightly the critical values $a=a_0,f=f_0$ and $\kappa=0$, for system* (1.1) *with $e+f=1$, the characteristic polynomial at the equilibrium $E_\pm$ or E can have one real eigenvalue λ and a pair of complex conjugates $\mu\pm\omega i$ where $\omega>0$ and $0<(|\lambda|,|\mu|)\ll 1$.*

Proof. Without losing generality, taking E_- as an example, we calculate its characteristic polynomial $P(\sigma)$, then set $P(\sigma) = (\sigma-\lambda)((\sigma-\mu)^2+\omega^2)$. Comparing the corresponding coefficients of the polynomials yields

$$\begin{cases} a = a_0 + \frac{k_1\lambda}{2b(c-1)(b^2+b(c-1)+\omega^2)} + \frac{\mu}{1-c} + o(|\lambda|,|\mu|), \\ f = f_0 + \frac{k_2\lambda}{4b^2(c-1)^2(b^2+b(c-1)+\omega^2)} - \frac{\mu}{2(c-1)^2} + o(|\lambda|,|\mu|), \\ \kappa = \frac{\lambda\,\omega^2}{b^2+b(c-1)+\omega^2} + o(|\lambda|,|\mu|), \end{cases} \tag{2.18}$$

where $k_1 = (b^2+\omega^2)(b^2+b(c-1)-\omega^2)$, $k_2 = b^5+3b^4(c-1)+b^3(2c^2-4c-\omega^2+2) - 2b\omega^2(c^2-c+\omega^2) - 8(c-1)^3\omega^2$, and $0 < (|\lambda|,|\mu|) \ll 1$. Therefore, the conclusion of this Lemma is obtained. □

Furthermore, we have the following Theorem.

Theorem 2.2. *For system* (1.1)*, the zero-Hopf bifurcation can occur around the equilibrium E at the critical values: $a = a_0, f = f_0$ and $\kappa = 0$. And by setting appropriate parameter values under the perturbing conditions* (2.18)*, one stable limit cycle can bifurcate.*

Proof. By perturbing the critical value $a = a_0, f = f_0$ and $\kappa = 0$ with the same forms as (2.18), and making the translation $\mathrm{x} = (x,y,z) \mapsto \mathrm{x} + E$ and the non-degenerate linear transformation $\mathrm{x}' = \tilde{T}\mathrm{u}'$ where $\tilde{T}$ is the perturbing form of T with respect to λ and μ. According to Lemma 2.3, from system (1.1) we can obtain

$$\begin{cases} \dot{u} = \mu u - \omega\, v + P_u(u^2,v^2,w^2,uv,uw,vw), \\ \dot{v} = \omega\, u + \mu v + P_v(u^2,v^2,w^2,uv,uw,vw), \\ \dot{w} = \lambda w + P_w(u^2,v^2,w^2,uv,uw,vw), \end{cases} \tag{2.19}$$

where μ and λ are the perturbation parameters (this is also called unfolding). Note that P_u, P_v and P_u are given in system (2.17), and higher-order terms involving μ, λ are ignored. Now, applying the Maple program in [22, 27], for system (2.19) with the unfolding added, we obtain the following normal form expressed in cylindrical coordinates [29] (for convenience, the notation w is still used in the normal form),

$$\begin{cases} \dot{w} = \lambda w + b_{20} r^2, \\ \dot{r} = \mu r + r_{11} r w + r_{30} r^3, \\ \dot{\theta} = \omega + a_{01} w + a_{20} a r^2, \end{cases} \tag{2.20}$$

where all a_{ij}, b_{ij} and r_{ij} are functions in b, c, ω, which can also be found in the website: https://github.com/lujingping/Liuhongpu.git. The first two equations in the normal form (2.20) can be used for the bifurcation analysis, while the third equation can be used to determine the frequency of periodic solutions.

Next, we will search for the steady-state solutions by setting $\dot{w} = \dot{r} = 0$ in (2.20), and all the steady-state solutions are obtained as follows:

$$(w_0, r_0) = (0,0),\ (w^*, r^*) = \left(\frac{b_{20}\mu}{r_{30}\lambda - b_{20}r_{11}}, \pm\sqrt{\frac{\lambda\mu}{b_{20}r_{11} - r_{30}\lambda}}\right).$$

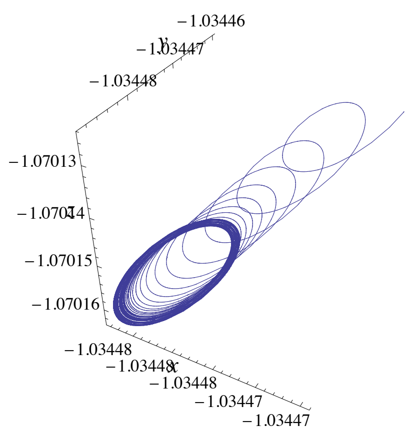
Note that just the positive solution $r_+^* = \sqrt{\frac{\lambda\mu}{b_{20}r_{11}-r_{30}\lambda}}$ represents a periodic orbit in the original three-dimensional space, and the periodic orbit is only one here.

The stability of the two steady-state solutions can be determined by the Jacobian of the first two equations of (2.20). Since the Jacobian evaluated at $(w_0, r_0) = (0, 0)$ results in two eigenvalues λ and μ, we can determine the solution, i.e., the origin of (2.20) is stable (unstable) if $\mu < 0, \lambda < 0$ ($\mu > 0$ or $\lambda > 0$). Furthermore, evaluating the Jacobian matrix at $(w, r) = (w^*, r_+^*)$ yields its determinant and trace

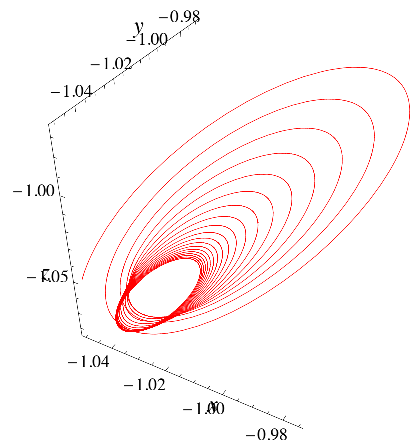
$$\mathrm{Det} = -2\mu\lambda, \quad \mathrm{Tr} = \frac{\lambda(b_{20}r_{11} - \lambda r_{30} + 2\mu r_{30})}{b_{20}r_{11} - \lambda r_{30}}.$$

Then its two eigenvalues are all negative if and only if $\mathrm{Det} > 0, \mathrm{Tr} < 0$, namely the stability conditions of the periodic orbit, which can be satisfied by selecting appropriate parameter values. Therefore, the proof of the theorem has been completed. □

Here we give a numerical example of one stable limit cycle via zero-Hopf bifurcation around the equilibrium E of system (1.1), as shown in Figure 2. In this example, we have set $e = 1 - f$ and $c = \frac{3}{2}, b = -1, \omega = 1$, then $\kappa = \frac{2}{3}\lambda, a = \frac{3}{2} - 2\mu + \frac{2}{3}\lambda$ and $f = 4 - 2\mu + \frac{7}{3}\lambda$ from (2.18). Furthermore, we choose small perturbation parameter value $\lambda = -0.05$ and $\mu = 0.01$ without losing generality.



(a)



(b)

Figure 2. Simulations of system (1.1) for $a = 1.4467, b = 4, c = 1.5, d = 2.99986, e = -2.863, f = 3.863$, converging to the stable limit cycle around the equilibrium E with the initial conditions: (a) $(x_0, y_0, z_0) = (-0.97, -0.97, -0.94)$ and (b) $(x_0, y_0, z_0) = (-1.05, -1.05, -1.05)$.

3. Zero-Hopf bifurcation at infinity

In this section, we will use the Poincaré compactification technique to analyze the dynamic behavior of system (1.1) at infinity. This technique can extend a polynomial vector field χ in $\mathbb{R}^n$ to a unique analytic vector field on the unit sphere $\mathbb{S}^n$. The extension is called the Poincaré compactification, which was described in [15], and one can also see [12, 25] for details. Let the Poincaré ball

$\mathbb{S}^3 = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4, ||z|| = 1\}$ be the unit sphere, $\mathbb{S}_+ = \{z \in \mathbb{S}^3, z_4 > 0\}$ and $\mathbb{S}_- = \{z \in \mathbb{S}^3, z_4 < 0\}$ be the northern and southern hemispheres, and denote the tangent hyperplanes at the points $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, $(0, 0, 0, \pm 1)$ respectively by the charts U_i, V_i where $U_i = \{z \in \mathbb{S}^3, z_i > 0\}$ and $V_i = \{z \in \mathbb{S}^3, z_i < 0\}$ for $i = 1, 2, 3, 4$. To analyze the dynamic behavior at infinity on x, y, and z, we will investigate the local charts U_i and V_i via the central projections and the diffeomorphisms $F_i : U_i \to \mathbb{R}^3$ and $G_i : V_i \to \mathbb{R}^3$, $i = 1, 2, 3$. Furthermore, the ow in the local chart V_i is the same as that in the local chart U_i reversing the time. Hence, we need only consider the local chart U_i, $i = 1, 2, 3$.

3.1. In the local chart U_1

Under the transformation of the variables $(x, y, z) = (z_3^{-1}, z_1 z_3^{-1}, z_2 z_3^{-1})$, and $t = z_3 \tau$, system (1.1) becomes

$$\begin{cases} \dfrac{\mathrm{d}z_1}{\mathrm{d}\tau} = f z_1 - z_2 + c z_3 + (a-1) z_1 z_3 + d z_1 z_2 + e z_1^2 - a z_1^2 z_3, \\ \dfrac{\mathrm{d}z_2}{\mathrm{d}\tau} = z_1 + (a-b) z_2 z_3 - a z_1 z_2 z_3, \\ \dfrac{\mathrm{d}z_3}{\mathrm{d}\tau} = -a z_1 z_3^2 + a z_3^2. \end{cases} \tag{3.1}$$

It is easy to find that it has one variant manifold $z_3 = 0$, and when $z_3 = 0$, namely on the variant manifold, system (3.1) can be reduced to

$$\begin{cases} \dfrac{\mathrm{d}z_1}{\mathrm{d}\tau} = f z_1 - z_2 + d z_1 z_2 + e z_1^2, \\ \dfrac{\mathrm{d}z_2}{\mathrm{d}\tau} = z_1. \end{cases} \tag{3.2}$$

Then $(0, 0)$ is the only equilibrium point of the system (3.2), and the two eigenvalues of the linear part at the origin are $\frac{1}{2}(f \pm \sqrt{f^2 - 4})$. We know that when $|f| \geq 2$, the origin is a stable or unstable node; when $0 < |f| < 2$, the origin is a stable or unstable focus; while $f = 0$, the origin is a center-focus or Hopf bifurcation point. Further, we have the following conclusions.

Proposition 3.1. *(i) When $f = 0$, the origin of system* (3.2) *has the first order fine focus quantity $V_3 = \pi de$.*

(ii) When $f = d = 0$ or $f = e = 0$, the origin is a center, and system (3.2) *has the corresponding first integrals respectively as follows:*

$$\begin{aligned} H(z_1, z_2) &= \frac{1}{2} \mathrm{Exp}[2 e z_2](z_1^2 - z_2^2), \\ H(z_1, z_2) &= d z_1 + \ln|1 - d z_1| - \frac{1}{2} d^2 z_2^2. \end{aligned} \tag{3.3}$$

(iii) For system (3.2)*, namely system* (3.1) *restricted to the variant manifold $z_3 = 0$, there exists one limit cycle at most via Hopf bifurcation.*

For Proposition 3.1 (iii), we obtain one stable limit cycle at the origin of system (3.2) by numerical simulation, as shown in Figure 3 (a). At the same time, we also give the corresponding phase portrait in the Poincaré disc for system (3.2), as shown

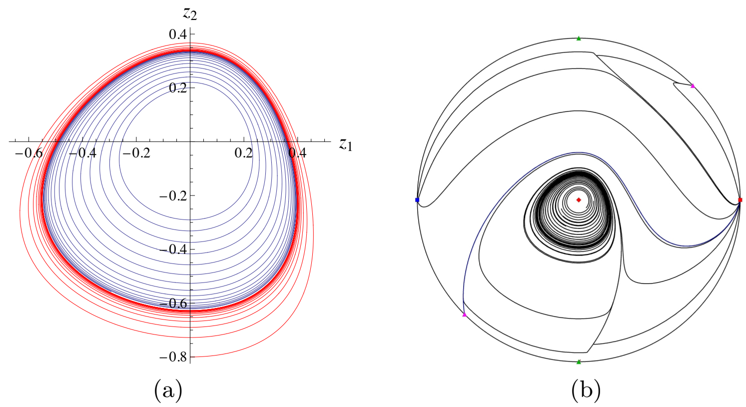


Figure 3. (a) One stable limit cycle of system (3.1) restricted to the invariant algebraic surface: $z_3 = 0$; (b) Phase portrait of the differential system (3.2) in the Poincaré disc, where $d = 1, e = -1, f = 0.05$.

in Figure 3 (b). Thus, we can comprehend the phase portrait of system (1.1) near the sphere at infinity in the local charts U_1 under the corresponding conditions.

We can know the origin $(0, 0, 0)$ is the only equilibrium point of system (3.1) on the plane $z_3 = 0$. That is to say, there exists the only equilibrium point related to infinity in the chart U_1, and the Jacobian at the origin of (3.1) has one zero eigenvalue and two non-zero eigenvalues: $\frac{1}{2}(f \pm \sqrt{f^2 - 4})$, which implies that when $f = 0$, the origin $(0, 0, 0)$ is a zero-Hopf bifurcation point. Thus we can analyze the corresponding zero-Hopf bifurcation at infinity of system (1.1) for a sufficiently small $z_3 > 0$, and the following conclusion is obtained.

Theorem 3.1. *For system* (3.1)*, when* $0 < f \ll 1$ *and* $ed < 0$*, the zero-Hopf bifurcation can occur around the origin. Correspondingly for system* (1.1)*, there exists the zero-Hopf bifurcation at infinity* $(x, y, z) = (+\infty, 0, 0)$*, then one stable bifurcating limit cycle at infinity* $(x, y, z) = (+\infty, 0, 0)$.

Proof. By perturbing the critical value $f = 0$, we let $f = \mu$, $0 < |\mu| \ll 1$. At the same time, for the convenience of calculating the norm form, we introduce one auxiliary linear perturbation of the third equation for (3.1) as follows:

$$\frac{\mathrm{d}z_3}{\mathrm{d}\tau} = -\lambda z_3 - a z_1 z_3^2 + a z_3^2, \tag{3.4}$$

where $0 \le \lambda \ll 1$. Then we make the nondegenerate linear transformation

$$(z_1, z_2, z_3) = (-2v - \frac{c\lambda}{\lambda^2 + 1} v, 2u + \frac{c}{\lambda^2 + 1} w, w).$$

From system (3.1), we obtain

$$\begin{cases} \dot{w} = -\lambda w - a w^2 (v - 1), \\ \dot{u} = v - w(a(v - 1)u + b(cw + u)), \\ \dot{v} = -u + \mu v + v(ev + w(a + cd - av - 1) + du), \end{cases} \tag{3.5}$$

where μ and λ are the perturbation parameters. Similarly, the higher-order terms involving μ, λ are ignored, and by applying the Maple program in [22,27], for system

(3.5) we obtain the following normal form with the unfolding added (the notation w is still used in the normal form),

$$\begin{cases} \dot{w} = -\lambda w + a w^2, \\ \dot{r} = \frac{1}{2}\mu r + \frac{1}{2}(2a - b + cd - 1) r w + \frac{1}{2} d e r^3, \\ \dot{\theta} = 1 - \frac{1}{8}\mu^2 - \frac{1}{6}(d^2 + 4e^2) r^2. \end{cases} \tag{3.6}$$

Next, we search for the steady-state solutions by setting $\dot{w} = \dot{r} = 0$ in (3.6). Since there can be neither the linear term nor the linear perturbation in the third equation of original system (3.5), namely λ always vanishes as an auxiliary parameter, and only yields $w = 0$. Hence, all the steady-state solutions are given as follows:

$$(w, r) = (0, 0), (0, \pm\sqrt{-\frac{\mu}{de}}).$$

Noting that the positive solution $r = \sqrt{-\frac{\mu}{de}}$ represents a periodic orbit in the original three dimensional space, and the periodic orbit is only one here.

By calculating the Jacobian of the first two equations of (3.6), evaluated at $(w, r) = (0, 0)$ with $\lambda = 0$, and resulting in two eigenvalues 0 and $\frac{\mu}{2}$, we can determine the solution $(w, r) = (0, 0)$, i.e., the origin of (3.6) is stable (unstable) if $\mu < 0$ ($\mu > 0$). Furthermore, evaluating the Jacobian at $(w, r) = (0, \sqrt{-\frac{\mu}{de}})$ yields its two eigenvalues 0 and $-\mu$. This indicates that the stability conditions of the periodic orbit $w = 0, r = \sqrt{-\frac{\mu}{de}}$ are given as follows:

$$\mu > 0,\ ed < 0. \tag{3.7}$$

Therefore, the proof of the theorem has been completed. □

Noting that in the vicinity of this zero-Hopf bifurcation point, i.e., the origin of system (3.5), the periodic orbit obtained has been restricted on the invariant manifold $w = 0$, thus it is actually a limit cycle via Hopf bifurcation. This is also consistent with the conclusion of Proposition 3.1 (iii).

3.2. In the local chart U_2

Under the transformation of the variables $(x, y, z) = (z_1 z_3^{-1}, z_3^{-1}, z_2 z_3^{-1})$, and $t = z_3 \tau$, the system (1.1) becomes

$$\begin{cases} \dfrac{dz_1}{d\tau} = a z_3 - e z_1 + (1 - a) z_1 z_3 - d z_1 z_2 - f z_1^2 - c z_1^2 z_3 + z_1^2 z_2, \\ \dfrac{dz_2}{d\tau} = z_1 - e z_2 - b z_2 z_3 - c z_1 z_2 z_3 + z_2 z_3 + z_1 z_2^2 - d z_2^2, \\ \dfrac{dz_3}{d\tau} = -z_3(e + f z_1 - z_3 + c z_1 z_3 - z_1 z_2). \end{cases} \tag{3.8}$$

Obviously, $z_3 = 0$ is its invariant algebraic surface, and on the variant manifold, the system (3.8) can be reduced into

$$\begin{cases} \dfrac{dz_1}{d\tau} = -z_1(e + f z_1 + d z_2 - z_1 z_2), \\ \dfrac{dz_2}{d\tau} = z_1 - z_2(e + f z_1 + d z_2 - z_1 z_2). \end{cases} \tag{3.9}$$

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If $ed = 0$, then $(0, 0)$ is the only equilibrium point of system (3.9), while $ed \neq 0$, system (3.9) has another equilibrium point $(0, -\frac{e}{d})$. Furthermore, when $e \neq 0$, the origin is a node with two eigenvalues: $\{-e, -e\}$, and the equilibrium point $(0, -\frac{e}{d})$ is a semi-hyperbolic singular point with two eigenvalues: $\{0, e\}$. When $e = 0$, the origin is a nilpotent node around which there does not exist one small amplitude limit cycle, one can see [8] for more details. In addition, system (3.9) always has one first integral

$$H(z_1, z_2, \tau) = \frac{z_2 \text{Exp}[-\tau]}{z_1}. \tag{3.10}$$

Especially, when $e = f = 0$, system (3.9) has one first integral as follows:

$$H(z_1, z_2) = -\frac{1}{dz_1} + \frac{z_2^2}{2z_1^2} - \text{Log}[\frac{z_1}{d - z_1}]. \tag{3.11}$$

Its phase portraits are as shown in Figure 4 (a), and under these conditions, we have the corresponding phase portrait in the Poincaré disc of (3.9), as shown in Figure 4 (b).

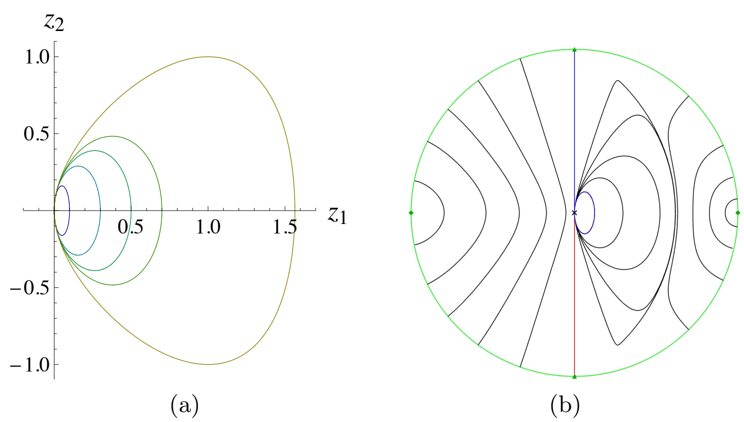

Figure 4. (a) Phase portraits for the first integrals in (3.11); (b) Phase portrait of the differential system (3.9) in the Poincaré disc, where $e = f = 0, d = 2$.

Furthermore, for system (3.8) related to the local chart U_2, we figure out that the Jacobian at the origin has three eigenvalues: $\{-e, -e, -e\}$. Then the origin is a node or high order singularity, and yields that the origin cannot be a zero-Hopf bifurcation point. And for another equilibrium point $(0, -\frac{e}{d}, 0)$ of system (3.8) with $ed \neq 0$, its Jacobian has three eigenvalues: $\{0, 0, e\}$, hence it cannot be a zero-Hopf bifurcation point, either.

From these, we give the following proposition.

Proposition 3.2. *In the local chart* U_2 *of system* (1.1)*, i.e., for system* (3.8)*, there does not exist zero-Hopf bifurcation at all the equilibria related to infinity.*

3.3. In the local chart U_3

Finally, we analyze infinity concerning z, i.e., $z = \infty$. Under the transformation of the variables $(x, y, z) = (z_1 z_3^{-1}, z_2 z_3^{-1}, z_3^{-1})$, and $t = z_3\tau$, system (1.1) becomes

$$\begin{cases} \dfrac{dz_1}{d\tau} = (b-a)z_1 z_3 + a z_2 z_3 - z_1^2 z_2, \\ \dfrac{dz_2}{d\tau} = d z_2 - z_1 + (b-1) z_2 z_3 + c z_1 z_3 + e z_2^2 + f z_1 z_2 - z_1 z_2^2, \\ \dfrac{dz_3}{d\tau} = z_3(b z_3 - z_1 z_2). \end{cases} \tag{3.12}$$

One quickly knows that $z_3 = 0$ is its invariant algebraic surface, and on the variant manifold, system (3.12) can be reduced into

$$\begin{cases} \dfrac{dz_1}{d\tau} = -z_1^2 z_2, \\ \dfrac{dz_2}{d\tau} = -z_1 + z_2(d + e z_2 + f z_1 - z_1 z_2). \end{cases} \tag{3.13}$$

If $ed = 0$, then $(0, 0)$ is the only equilibrium point of system (3.13), while $ed \neq 0$, system (3.13) has another equilibrium point $(0, -\frac{d}{e})$. Furthermore, in the local chart U_3, we know that the Jacobian at the origin of system (3.12) has three eigenvalues: $\{0, 0, d\}$, which implies that the origin cannot be a zero-Hopf bifurcation point. And for another equilibrium point $(0, -\frac{d}{e}, 0)$ of system (3.12) with $ed \neq 0$, its Jacobian has three eigenvalues: $\{0, 0, -d\}$, hence it cannot be a zero-Hopf bifurcation point, either.

From these, we have the following proposition.

Proposition 3.3. *In the local chart U_3 of system* (1.1)*, i.e., for system* (3.12)*, there does not exist zero-Hopf bifurcation at all the equilibria related to infinity.*

4. Conclusions

In this paper, we have studied the zero-Hopf bifurcation of a class of generalized Lorenz systems. Applying the averaging theory of first order, we have obtained only one limit cycle around the origin. When the averaging theory of first order fails, the normal form theory is put to use in investigating the zero-Hopf bifurcation at the other equilibria. Then only one limit cycle can be found. Further, through Poincaré compactification, we have discussed the zero-Hopf bifurcation at infinity, and also have obtained such a limit cycle. The similar results at infinity have not been seen in the existing references.

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