

# Group-Invariant Solutions and Conservation Laws of One-Dimensional Nonlinear Wave Equation

Ben Yang<sup>1</sup>, Yunjia Song<sup>1</sup>, Yanzhi Ma<sup>1</sup> and Xinxue Zhang<sup>1,†</sup>

**Abstract** Based on classical Lie symmetry method, the one-dimensional nonlinear wave equation is investigated. By using four-dimensional subalgebras of the equation, the invariant groups and commutator table are constructed. Furthermore, optimal system of the equation is obtained, and the exact solutions can be gained by solving reduced equations. Finally, a complete derivation of the conservation law is given by using conservation multipliers.

**Keywords** One-dimensional nonlinear wave equation, Lie symmetry, optimal system, conservation law

**MSC(2010)** 35A08, 35C08, 35Q51.

## 1. Introduction

Wave equations describe various wave phenomena and have a wide range of applications in the fields of physics [19], biology and engineering [22,25], making the solution of wave equations indispensable. The methods of solving partial differential equations (PDEs) mainly include ( $G'/G$ ) expansion method [6,14,35], extended hyperbolic method [26], inverse scattering method [7], exponential function method [32], generalized exp-function method [13], Bäcklund transformation method [33], Jacobi elliptic method [2,4,15], hyperbolic tangent method [1],  $F$ -expansion method [11], homogeneous equilibrium method [24], Lie symmetry analysis method [12,20,21,29] and so on [3,9,10,34].

The Lie symmetry method can solve PDEs efficiently. In this article, we consider a one-dimensional nonlinear wave equation

$$u_{tt} = \left( (1+u)^{2a} u_x \right)_x, \quad (1.1)$$

where  $u$  is a function of  $x, t$  and  $a > 0$  is a constant. In [5], Ames, Lohner and Adams proposed a general nonlinear fluctuation equation

$$u_{tt} = [\mathcal{B}(u)u_x]_x, \quad (1.2)$$

where  $\mathcal{B}$  is expressed as a function of  $u$ . Then, they discussed equation (1.2) with Lie symmetric analysis, and derived explicit invariant solutions to wave propagation and

---

<sup>†</sup>The corresponding author.

Email address: yangben09@163.com (B. Yang), zhangxinxue01@163.com (X. Zhang)

<sup>1</sup>School of Mathematical Sciences, Liaocheng University, Liaocheng, Shandong 252000, China

transonic equation in gases. Furthermore, Sophocleous and Kingston [27] attempted the following three special cases

$$u_{tt} = F(u_x) u_{xx}, \quad u_{tt} = F(u_{xx}), \quad u_{xt} = F(u). \quad (1.3)$$

Equation (1.3) exists in the discrete symmetries groups which form finite order cyclic groups. In [16], Hu studied the degenerate initial-boundary value problem of equation (1.1), and obtained the global existence of the solution by using the eigendecomposition method under relaxed conditions. The global existence of the solution to a more general  $2 \times 2$  conservation system of equation (1.1) was proven in [18, 30]. In [8, 31], the conservation system of equation (1.1) has been studied. In [28], Sugiyama introduced the large-time behavior of the solution of the equation under the Cauchy problem, obtained the sufficient conditions for the degradation of the equation in finite time and derived a threshold for the global existence and degradation of the separated solution.

This article mainly includes the following sections. In the second section, the concepts of Lie symmetry and prolongation method are introduced, followed by a study on Lie point symmetry and one-dimensional optimal system of the equation. Investigating the group invariant solutions of the equation by the optimal system, we obtain the exact solutions through symmetry reduction. The third section discusses the conservation laws of the equation. Finally, a simple summary is drawn.

## 2. Lie symmetry analysis and optimal system of equation (1.1)

### 2.1. Definition introduction

Based on the conclusions of Sophus Lie, some concepts of Lie symmetry [23] have been set up.

Assume that the  $s$ -order partial differential equation system  $Q$  with  $q$  independent variable and  $m$  dependent variable is

$$\Delta q(x, u^{(n)}) = 0, \quad Q = 1, 2, 3, \dots, k, \quad (2.1)$$

in which  $x = (x^1, x^2, \dots, x^q)$ ,  $u = (u^1, u^2, \dots, u^m)$  and  $u^{(n)}$  represents arbitrary order derivative of  $u$ , and its range of value is from 0 to  $n$ . Now, let us discuss the infinitesimal one-parameter Lie group transformation of the system

$$\bar{x}^k = x^k + \varepsilon \xi^k(x, u) + o(\varepsilon^2), \quad \bar{u}^p = u^p + \varepsilon \phi^p(x, u) + o(\varepsilon^2), \quad (2.2)$$

where  $\varepsilon$  is an arbitrary, and  $\xi^k$ ,  $\phi^p$  represent the infinitesimal transformations of function independent variables and dependent variables respectively.

Considering the  $n$ -order differential equations for  $u$

$$\Delta(x, u^{(n)}) = 0, \quad (2.3)$$

in which  $\Delta$  denotes a smooth mapping from  $X \times U^{(n)}$  to  $\mathbb{R}$ :

$$\Delta : X \times U^{(n)} \rightarrow \mathbb{R}.$$

The following subset can be obtained by equation (2.3)

$$S' = \left\{ (x, u^n) : \Delta \left( x, u^{(n)} \right) = 0 \right\} \subset X \times U^{(n)}.$$

Assume that  $S'$  is an open subset of  $X \times U^{(2)}$ , and  $\Delta \left( x, u^{(2)} \right) = u_{tt} - \left( (1+u)^{2a} u_x \right)_x = 0$  is the  $n$ -order equation defined on  $S'$ . Then, the vector  $v$  on the open subset  $S'$  is

$$v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u}, \quad (2.4)$$

in which  $\xi$ ,  $\tau$ ,  $\phi$  are infinitesimal generators. The second-order prolongation for equation (1.1) is

$$Pr^{(2)}v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}, \quad (2.5)$$

where

$$\begin{aligned} \phi^x &= D_x \phi - u_x D_x \xi - u_t D_x \tau, \\ \phi^{xx} &= D_x^2 (\phi - \xi u_x - \tau u_t), \\ \phi^{tt} &= D_t^2 (\phi - \xi u_x - \tau u_t), \end{aligned} \quad (2.6)$$

where  $D_x$  and  $D_t$  are fully differentiable operators with respect to  $x$ ,  $t$ .

## 2.2. Lie symmetry analysis

First, we consider a one-parameter Lie group of point transformation:

$$\tilde{x} = x + \varepsilon \xi + O(\varepsilon^2),$$

$$\tilde{t} = t + \varepsilon \tau + O(\varepsilon^2),$$

$$\tilde{u} = u + \varepsilon \phi + O(\varepsilon^2),$$

in which  $\varepsilon$  is the group parameter, and  $\xi$ ,  $\tau$ ,  $\phi$  are infinitesimal variables of  $x$ ,  $t$ ,  $u$ .

Substituting (2.6) into (2.5) and using Maple software to solve the determining equations, the infinitesimal can be deduced as follows.

$$\xi = k_1 x + k_2, \quad \tau = k_3 t + k_4, \quad \phi = \frac{(k_1 - k_3)(1+u)}{a}, \quad (2.7)$$

where  $k_i (i = 1, \dots, 4)$  are constants.

Thus, the Lie algebra of equation (1.1) is generated by four generators

$$v_1 = x \frac{\partial}{\partial x} + \frac{1}{a} (1+u) \frac{\partial}{\partial u}, \quad v_2 = \frac{\partial}{\partial x}, \quad v_3 = t \frac{\partial}{\partial t} - \frac{1}{a} (1+u) \frac{\partial}{\partial u}, \quad v_4 = \frac{\partial}{\partial t}. \quad (2.8)$$

Substituting (2.8) into (2.4), we obtain

$$v = k_1 \left( x \frac{\partial}{\partial x} + \frac{1}{a} (1+u) \frac{\partial}{\partial u} \right) + k_2 \frac{\partial}{\partial x} + k_3 \left( t \frac{\partial}{\partial t} + \frac{1}{a} (-1-u) \frac{\partial}{\partial u} \right) + k_4 \frac{\partial}{\partial t}. \quad (2.9)$$

Next, in order to get the one-parameter transformation group of equation (1.1), the system of ODEs with initial values needs to be solved

$$\frac{d}{d\varepsilon} (\tilde{x}, \tilde{t}, \tilde{u}) = \psi (\tilde{x}, \tilde{t}, \tilde{u}), (\tilde{x}, \tilde{t}, \tilde{u})|_{\varepsilon=0} = (x, t, u).$$

The corresponding one-parameter transformation group of the equation is

$$\begin{cases} K_1 : (e^\varepsilon x, t, (-1 + (1 + u) e^{\frac{\varepsilon}{a}})), \\ K_2 : (x + \varepsilon, t, u), \\ K_3 : (x, e^\varepsilon t, (-1 + (1 + u) e^{-\frac{\varepsilon}{a}})), \\ K_4 : (x, t + \varepsilon, u). \end{cases}$$

If  $u = f(x, t)$  is the solution of equation (1.1), then the following function is also the solution of equation (1.1)

$$\begin{cases} u^{(1)} = -1 + (1 + f(xe^{-\varepsilon}, t)) e^{\frac{\varepsilon}{a}}, \\ u^{(2)} = f(x - \varepsilon, t), \\ u^{(3)} = -1 + (1 + f(x, te^{-\varepsilon})) e^{-\frac{\varepsilon}{a}}, \\ u^{(4)} = f(x, t - \varepsilon). \end{cases}$$

### 2.3. Construction of the optimal system

In the following, we construct the optimal system of equation (1.1) using the commutator table and adjoint representation table. According to the definition of Lie bracket and adjoint representation,

$$[v_m, v_n] = v_m v_n - v_n v_m,$$

$$Ad(\exp(\varepsilon) v_m) v_n = v_n - \varepsilon [v_m, v_n] + \frac{1}{2} \varepsilon^2 [v_m, [v_m, v_n]] - \dots.$$

We can get the following two tables respectively.

Table 1. Commutator table				
$[v_i, v_j]$	$v_1$	$v_2$	$v_3$	$v_4$
$v_1$	0	$-v_2$	0	0
$v_2$	$v_2$	0	0	0
$v_3$	0	0	0	$-v_4$
$v_4$	0	0	$v_4$	0

Using the adjoint table to give the classification of the subalgebras of the vector fields (2.8), consider the vector

$$V = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4. \quad (2.10)$$

First, assume that  $a_1 \neq 0$ , and take  $a_1 = 1$ ,

$$V = v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4.$$

**Table 2.** Adjoint representation table

$Ad$	$v_1$	$v_2$	$v_3$	$v_4$
$v_1$	$v_1$	$e^\varepsilon v_2$	$v_3$	$v_4$
$v_2$	$v_1 - \varepsilon v_2$	$v_2$	$v_3$	$v_4$
$v_3$	$v_1$	$v_2$	$v_3$	$e^\varepsilon v_4$
$v_4$	$v_1$	$v_2$	$v_3 - \varepsilon v_4$	$v_4$

We act  $Ad(\exp(a_2v_2))$  on  $V$ ,

$$V^{(1)} = Ad(\exp(a_2v_2))v = v_1 + a_3v_3 + a_4v_4.$$

Then, acting  $Ad(\exp(\frac{a_4}{a_3}v_4))$  to  $V^{(1)}$ ,

$$V^{(2)} = Ad\left(\exp\left(\frac{a_4}{a_3}v_4\right)\right)V^{(1)} = v_1 + a_3v_3. \quad (2.11)$$

Therefore, if  $a_1 \neq 0$ , every one-dimensional subalgebra generated by  $V$  is equal to  $v_1 + a_3v_3$ .

Secondly, assuming that  $a_1 = 0$ ,  $a_3 \neq 0$ , then we assume that  $a_3 = 1$ . That is,

$$V = a_2v_2 + v_3 + a_4v_4.$$

Then act  $Ad(\exp(a_4v_4))$  to  $V$ ,

$$V^{(3)} = Ad(\exp(a_4v_4))V = a_2v_2 + v_3. \quad (2.12)$$

In the third case, when  $a_1 = 0$ ,  $a_3 = 0$ , and  $V = a_2v_2 + a_4v_4$ , there are four cases

$$v_2, v_2 \pm v_4, v_4. \quad (2.13)$$

According to (2.11), (2.12) and (2.13), we obtain that the one-dimensional optimal system of the four subalgebras (2.8) is spanned by: (a)  $v_1 + a_3v_3$ , (b)  $a_2v_2 + v_3$ , (c)  $v_2 + v_4$ , (d)  $v_2 - v_4$ , (e)  $v_2$ , (f)  $v_4$ .

## 2.4. Exact solutions for equation (1.1)

In this section, the symmetry reductions and exact solutions are studied for equation (1.1).

**Case 1.** For generator

$$V = v_1 + a_3v_3 = x \frac{\partial}{\partial x} + \frac{1}{a}(1+u) \frac{\partial}{\partial u} + a_3 \left( t \frac{\partial}{\partial t} - \frac{1}{a}(1+u) \frac{\partial}{\partial u} \right),$$

the characteristic equation satisfies

$$\frac{dx}{x} = \frac{dt}{a_3t} = \frac{du}{\frac{1-a_3}{a}(1+u)}. \quad (2.14)$$

The group invariant solution is

$$u = -1 + t^{\frac{1-a_3}{a_3}} f(h), \quad (2.15)$$

in which  $h = \frac{x^{a_3}}{t}$ . Assuming that  $a_3 = 1$ ,  $a = 1$  and substituting (2.15) into equation (1.1), we obtain

$$-2f(f')^2 + f''\xi^2 - f^2 f'' + 2f'\xi = 0. \quad (2.16)$$

Integrating equation (2.16),

$$(f^2 - \xi^2) f' = 0, \quad (2.17)$$

and then we have

$$f = -\xi, f = \xi, f = c_1.$$

Therefore, equation (1.1) has self-similar solutions as  $u = -1 - \frac{x}{t}$ ,  $u = -1 + \frac{x}{t}$  and  $u = -1 + k_1$ .

**Case 2.** For generator

$$V = a_2 v_2 + v_3 = a_2 \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{1}{a} (1 + u) \frac{\partial}{\partial u},$$

the characteristic equation satisfies

$$\frac{dx}{a_2} = \frac{dt}{t} = \frac{du}{-\frac{1}{a}(1+u)}.$$

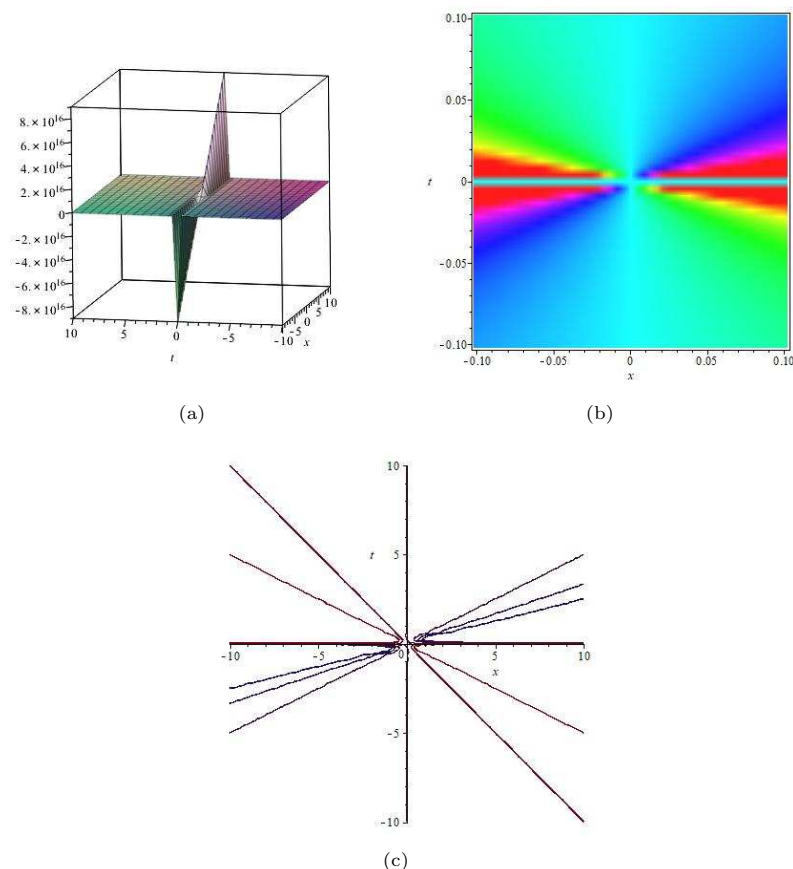
The group invariant solution is

$$u = -1 + t^{-\frac{1}{a}} g(\xi),$$

where the invariant  $\xi = ce^{\frac{x}{t}}$ . Taking it into equation (1.1),

$$g^2 g'' \xi^2 + 2g(g')^2 \xi^2 + g^2 g' \xi - g'' \xi^2 - 4g' \xi - 2g = 0. \quad (2.18)$$

Equation (2.18) is a completely nonlinear ordinary differential equation, which is not easy to solve.



**Figure 1.** The dynamical structures of  $u = -1 + \frac{x}{t}$ : (a) singularity profile of  $u = -1 + \frac{x}{t}$ ; (b) density plot; (c) contour plot

**Case 3.** When  $V = v_2 + v_4 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ , the characteristic equation is  $\frac{dx}{1} = \frac{dt}{1}$ . The traveling wave solution is  $\xi = \xi(x-t)$ , where  $u$  can be expressed as  $u = f(x-t)$ . Then, taking it into equation (1.1),

$$f' = (1 + f)^{2a} f' - k_1, \quad (2.19)$$

in which  $a, k_1$  are arbitrary constants. Integrating (2.19), we get

$$\frac{(1 + f)^{2a+1}}{2a + 1} - f = \frac{k_1 \xi + k_2}{2a + 1}, \quad (2.20)$$

$$(1 + f)^{2a+1} - (2a + 1)f = k_1 \xi + k_2, \quad (2.21)$$

where  $k_1$  and  $k_2$  are arbitrary constants.

**Case 4.** When  $V = v_2 = \frac{\partial}{\partial x}$ , the solution of equation (1.1) can be expressed as  $u = f(t)$ . For this equation,  $f'' = 0$ , we can obtain  $f = k_1 t + k_2$ , where  $k_1$  and  $k_2$  are arbitrary constants.

**Case 5.** When  $V = v_4 = \frac{\partial}{\partial t}$ , the invariant is  $\xi = x$ , and then  $u = f(x)$ . Taking it into equation (1.1), we obtain

$$(1 + f)^{2a} f' = k_1, \quad (2.22)$$

in which  $k_1$  is an arbitrary constant. Then, integrating equation (2.22), we get

$$(1 + f)^{2a+1} = (2a + 1) \cdot k_1 x.$$

### 3. Analysis of conservation laws of equation (1.1)

In this section, we discuss the conservation laws of this one-dimensional nonlinear wave equation. In [17], some definitions of conservation laws are given.

**Definition 3.1** Let  $x, t$  be the independent variables,  $u = u(x, t)$  be the dependent variable, and  $u_x, u_t, u_x x, u_t t$ , etc, be its partial derivative. Next, we introduce the conjugate equation and the definition of the multiplier method.

Assume that the  $s$ -order partial differential equation with  $m$  independent variable can be expressed as

$$F(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)}) = 0, \quad (3.1)$$

where  $x = (x_1, x_2, \dots, x_m)$ ,  $u_{(i)}$  represents be derivatives of  $u$  with respect to  $x_1, x_2, \dots, x_n$ .

The conjugate equation of equation (3.1) is  $F^* = (x, t, u, v, u_x, v_x, u_t, v_t, u_{xx}, \dots)$ ,

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \dots, u_{(s)}, v_{(s)}) = \frac{\delta(\Lambda F)}{\delta u}. \quad (3.2)$$

The operator  $\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_{ij} \frac{\partial}{\partial u_{ij}} - D_{ijk} \frac{\partial}{\partial u_{ijk}} + D_{ijkl} \frac{\partial}{\partial u_{ijkl}} - \dots$  is the Euler-Lagrange operator, and  $D_i$  represents the total differentiation of  $x_i$ .

**Definition 3.2** Equation (3.1) and equation (3.2) have a Lagrange

$$\mathcal{L} = F = \Lambda(F(x, u, u_{(1)}, u_{(2)}, u_{(3)} \dots, u_{(s)})), \quad (3.3)$$

where  $\mathcal{L}$  meets  $\frac{\delta \mathcal{L}}{\delta u} = F^*$ ,  $\frac{\delta \mathcal{L}}{\delta v} = F$ . For equation (3.1), its conservation laws can be expressed as

$$\begin{aligned} M^i = & \xi^i \mathcal{L} + W^\alpha \left[ \frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left( \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) - D_j D_k D_m \left( \frac{\partial \mathcal{L}}{\partial u_{ijkm}^\alpha} \right) \right. \\ & + \dots \left. \right] + D_j (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left( \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + D_k D_m \left( \frac{\partial \mathcal{L}}{\partial u_{ijkm}^\alpha} \right) - \dots \right] \\ & + D_j D_k (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - D_m \left( \frac{\partial \mathcal{L}}{\partial u_{ijkm}^\alpha} \right) + \dots \right] \\ & + D_j D_k D_m (W^\alpha) \left[ \frac{\partial \mathcal{L}}{\partial u_{ijkm}^\alpha} - D_n \left( \frac{\partial \mathcal{L}}{\partial u_{ijkmn}^\alpha} \right) + \dots \right], \end{aligned} \quad (3.4)$$



in which  $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ .

The formal Lagrangian of equation (1.1) is

$$\mathcal{L} = \Lambda(x, t, u, u_x, u_t) \left( u_{tt} - \left( (1+u)^{2a} u_x \right)_x \right). \quad (3.5)$$

Assuming that  $\Lambda = k_1 x + k_2 t + k_3 xt$ , we obtain the following four cases.

**Case 1.** For generator  $v_1 = x \frac{\partial}{\partial x} + \frac{1}{a} (1+u) \frac{\partial}{\partial u}$ , the Lie characteristic function is  $W = \frac{1}{a} (1+u) - x u_x$ ,

$$\begin{aligned} M^x &= \left( \frac{1+u}{a} - x u_x \right) \left( -4av u_x (1+u)^{2a-1} + v_x (1+u)^{2a} + \frac{2av u_x (1+u)^{2a}}{1+u} \right) \\ &\quad - v (1+u)^{2a} \left( \frac{u_x}{a} - x u_{xx} - u_x \right), \\ M^t &= -v_t \left( \frac{1+u}{a} - x u_x \right) + v \left( \frac{u_t}{a} - x u_{xt} \right). \end{aligned}$$

**Case 2.** For generator  $v_2 = \frac{\partial}{\partial x}$ , the Lie characteristic function is  $W = -u_x$ ,

$$\begin{aligned} M^x &= -u_x \left( -\frac{2av u_x (1+u)^{2a}}{1+u} + v_x (1+u)^{2a} \right) + u_{xx} v (1+u)^{2a}, \\ M^t &= u_x v_t - v u_{xt}. \end{aligned}$$

**Case 3.** For generator  $v_3 = t \frac{\partial}{\partial t} + \frac{1}{a} (-1-u) \frac{\partial}{\partial u}$ , the Lie characteristic function is  $W = -\frac{1}{a} (1+u) - t u_t$ ,

$$\begin{aligned} M^x &= \left( -\frac{1+u}{a} - t u_t \right) \left( -\frac{2av (1+u)^{2a} u_x}{1+u} + v_x (1+u)^{2a} \right) \\ &\quad - v (1+u)^{2a} \left( -\frac{u_x}{a} - t u_{xt} \right), \\ M^t &= -v_t \left( -\frac{1+u}{a} - t u_t \right) + v \left( -\frac{u_t}{a} - t u_{tt} - u_t \right). \end{aligned}$$

**Case 4.** For generator  $v_4 = \frac{\partial}{\partial t}$ , the Lie characteristic function is  $W = -u_t$ ,

$$\begin{aligned} M^x &= \frac{2av u_x u_t (1+u)^{2a}}{1+u} + v u_{xt} (1+u)^{2a}, \\ M^t &= -v u_{tt}. \end{aligned}$$

## 4. Conclusions

We analyze the symmetry of one-dimensional nonlinear wave equation by classical Lie symmetry method, and then the optimal system of the symmetry are derived. By solving the reduced equation, we can calculate the solutions of the equation. Finally, the conservation laws have been established through the use of conservation law multiplier.

## Acknowledgements

The authors are grateful to the reviewers and editors for their helpful comments and suggestions that have helped improve our paper.

## References

- [1] A. R. Adem, *Symbolic computation on exact solutions of a coupled Kadomtsev-Petviashvili equation: Lie symmetry analysis and extended tanh method*, Computers & Mathematics with Applications, 2017, 74(8), 1897–1902.
- [2] H. M. Ahmed, W. B. Rabie and M. A. Ragusa, *Optical solitons and other solutions to Kaup-Newell equation with Jacobi elliptic function expansion method*, Analysis and Mathematical Physics, 2021, 11(1), 23, 16 pages.
- [3] A. O. Akdemir, A. Karaođlan, M. A. Ragusa and E. Set, *Fractional Integral Inequalities via Atangana-Baleanu Operators for Convex and Concave Functions*, Journal of Function Spaces, 2021, Article ID 1055434, 10 pages.
- [4] K. K. Al-Kalbani, K. S. Al-Ghafri, E. V. Krishnan and A. Biswas, *Pure-cubic optical solitons by Jacobis elliptic function approach*, Optik, 2021, 243, Article ID 167404, 13 pages.
- [5] W. F. Ames, R. J. Lohner and E. Adams, *Group properties of  $u_{tt} = [f(u)u_x]x$* , International Journal of Non-Linear Mechanics, 1981, 16(5–6), 439–447.
- [6] A. Aniqah and J. Ahmad, *Soliton solution of fractional Sharma-Tasso-Olevers equation via an efficient  $(G'/G)$ -expansion method*, Ain Shams Engineering Journal, 2022, 13(1), Article ID 101528, 23 pages.
- [7] Y. Chen, B. Feng and L. Ling, *The robust inverse scattering method for focusing Ablowitz-Ladik equation on the non-vanishing background*, Physica D: Nonlinear Phenomena, 2021, 424, Article ID 132954, 27 pages.
- [8] N. D. Cristescu, *Dynamic Plasticity*, North-Holland, Amsterdam, 1967.
- [9] A. O. El-Badary, S. M. Helal and M. S. Abdel, *Some New Exact Solutions for Time fractional Thin-film Equation*, Journal of Nonlinear Modeling and Analysis, 2020, 2(3), 375–391.
- [10] B. Elma and E. Misirli, *New Exact Solutions of Some Non-linear Evolution Equations via Functional Variable Method*, Filomat, 2021, 35(13), 4267–4273.
- [11] A. Filiz, M. Ekici and A. Sonmezoglu, *F-expansion method and new exact solutions of the Schrödinger-KdV equation*, The Scientific World Journal, 2014, Article ID 534063, 15 pages.
- [12] B. Ghanbari, S. Kumar, M. Niwas and D. Baleanu, *The Lie symmetry analysis and exact Jacobi elliptic solutions for the Kawahara-KdV type equations*, Results in Physics, 2021, 23, Article ID 104006, 15 pages.
- [13] B. Ghanbari, M. S. Osman and D. Baleanu, *Generalized exponential rational function method for extended Zakharov-Kuznetsov equation with conformable derivative*, Modern Physics Letters A, 2019, 34(20), Article ID 1950155, 16 pages.
- [14] A. K. M. K. S. Hossain and M. A. Akbar, *Traveling wave solutions of Benny Luke equation via the enhanced  $(G'/G)$ -expansion method*, Ain Shams Engineering Journal, 2021, 12(4), 4181–4187.
- [15] K. Hosseini, S. Salahshour, M. Mirzazadeh, et al., *The  $(2+1)$ -dimensional Heisenberg ferromagnetic spin chain equation: its solitons and Jacobi elliptic function solutions*, The European Physical Journal Plus, 2021, 136(2), 206, 9 pages.

- [16] Y. Hu, *On the existence of solutions to a one-dimensional degenerate nonlinear wave equation*, Journal of Differential Equations, 2018, 265(1), 157–176.
- [17] N. H. Ibragimov, *A new conservation theorem*, Journal of Mathematical Analysis and Applications, 2007, 333(1), 311–328.
- [18] J. L. Johnson, *Global continuous solutions of hyperbolic systems of quasi-linear equations*, University of Michigan, Michigan, 1967.
- [19] N. A. Kudryashov and E. V. Antonova, *Solitary waves of equation for propagation pulse with power nonlinearities*, Optik, 2020, 217, Article ID 164881, 5 pages.
- [20] S. Kumar and D. Kumar, *Solitary wave solutions of (3+1)-dimensional extended Zakharov–Kuznetsov equation by Lie symmetry approach*, Computers & Mathematics with Applications, 2019, 77(8), 2096–2113.
- [21] S. Kumar, D. Kumar and A. Kumar, *Lie symmetry analysis for obtaining the abundant exact solutions, optimal system and dynamics of solitons for a higher-dimensional Fokas equation*, Chaos, Solitons & Fractals, 2021, 142, Article ID 110507, 21 pages.
- [22] A. W. Leung, *Systems of Nonlinear Partial Differential Equations: Applications to Biology and Engineering*, Springer Science & Business Media, Berlin, 2013.
- [23] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer Science & Business Media, Berlin, 2000.
- [24] A. S. A. Rady, E. S. Osman and M. Khalfallah, *The homogeneous balance method and its application to the Benjamin–Bona–Mahoney (BBM) equation*, Applied Mathematics and Computation, 2010, 217(4), 1385–1390.
- [25] M. Raissi, P. Perdikaris and G. E. Karniadakis, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations*, Journal of Computational Physics, 2019, 378, 686–707.
- [26] Y. Shang, Y. Huang and W. Yuan, *The extended hyperbolic functions method and new exact solutions to the Zakharov equations*, Applied Mathematics and Computation, 2008, 200(1), 110–122.
- [27] C. Sophocleous and J. G. Kingston, *Cyclic symmetries of one-dimensional nonlinear wave equations*, International Journal of Non-Linear Mechanics, 1999, 34(3), 531–543.
- [28] Y. Sugiyama, *Degeneracy in Finite Time of 1D Quasilinear Wave Equations*, SIAM Journal on Mathematical Analysis, 2016, 48(2), 847–860.
- [29] S. Tian, *Lie symmetry analysis, conservation laws and solitary wave solutions to a fourth-order nonlinear generalized Boussinesq water wave equation*, Applied Mathematics Letters, 2020, 100, Article ID 106056, 8 pages.
- [30] M. Yamaguti and T. Nishida, *On some global solution for quasilinear hyperbolic equations*, Funkcialaj Ekvacioj, 1968, 11(1), 51–57.
- [31] N. J. Zabusky, *Exact Solution for the Vibrations of a Nonlinear Continuous Model String*, Journal of Mathematical Physics, 1962, 3(5), 1028–1039.

- [32] S. Zhang, J. Tong and W. Wang, *Exp-function method for a nonlinear ordinary differential equation and new exact solutions of the dispersive long wave equations*, Computers & Mathematics with Applications, 2009, 58(11–12), 2294–2299.
- [33] Z. Zhao and L. He, *Bäcklund transformations and Riemann-Bäcklund method to a (3+1)-dimensional generalized breaking soliton equation*, The European Physical Journal Plus, 2020, 135(8), 639, 21 pages.
- [34] Y. Zhou and J. Zhuang, *Bifurcations and Exact Solutions of the Raman Soliton Model in Nanoscale Optical Waveguides with Metamaterials*, Journal of Nonlinear Modeling and Analysis, 2021, 3(1), 145–165.
- [35] J. Zuo and Y. Zhang, *Application of the  $(G'/G)$ -expansion method to solve coupled MKdV equations and coupled Hirota-Satsuma coupled KdV equations*, Applied mathematics and computation, 2011, 217(12), 5936–5941.