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Global Bifurcation for a Class of Lotka-Volterra Competitive Systems*

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Abstract For a class of Lotka-Volterra competitive systems including both diffusion and advection, a global bifurcation result of positive steady states is established via a bifurcation approach. Also, the phenomenon of multiple positive steady states is discussed.

Keywords Competitive system, bifurcation, positive steady state, multiple solutions

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1. Introduction

Over the past several decades, the Crandall-Rabinowitz local bifurcation theorem [4] and the global bifurcation theorem [24] have been widely utilized to understand the solution set of nonlinear equations and to reveal critical roles played by physical or biological parameters (see, e.g., [5,17] for a class of nonlocal elliptic equations, [1,13, 19,33,34] for the two-species reaction-diffusion competition models and [18,28,31] for the predator-prey-taxis models). For more investigations, we refer the interested readers to [16,20,21,29,30,35,39] (to mention but a few).

In this paper, we are mainly interested in the following general competitive parabolic system including both diffusion and advection

$$\begin{cases} u_t = \mathcal{L}_1 u + u[r_1(x) - u - bv], & x \in \Omega,\ t > 0, \\ v_t = \mathcal{L}_2 v + v[r_2(x) - cu - v], & x \in \Omega,\ t > 0, \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0, & x \in \Omega, \\ v(x,0) = v_0(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^N$ with $1 \leqslant N \in \mathbb{Z}$, $u(x,t)$ and $v(x,t)$ represent the population density of two competing species at location $x \in \Omega$

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and time $t > 0$, respectively, $r_i(x)$ $(i = 1,2)$ is a bounded and positive function accounting for the intrinsic growth rate, and the positive constants b and c measure the inter-specific competition intensities (note that the intra-specific competition coefficients have been normalized by 1). The linear differential operator $\mathcal{L}_i$ is defined by

$$\mathcal{L}_i w := \operatorname{div}\Big(d_i \nabla w - \alpha_i w \nabla P_i(x)\Big), \quad i = 1,2, \tag{1.2}$$

with $d_i > 0$ denoting the rate of random diffusion, $P_i(x) \in C^1(\overline{\Omega})$ specifying the advective direction and $\alpha_i \in \mathbb{R}$ measuring the advection speed. The boundary operator $\mathcal{B}_i$ is defined by

$$\mathcal{B}_i w = d_i \frac{\partial w}{\partial \nu} - \alpha_i w \frac{\partial P_i}{\partial \nu} = 0, \quad i = 1,2, \tag{1.3}$$

where ν denotes the outward unit normal vector on the boundary $\partial\Omega$. The no-flux boundary conditions imposed in (1.3) mean that no individuals can cross over the boundary of the habitat, i.e., the environment is closed.

Recently, system (1.1)-(1.3) has been systematically investigated by Zhou et al., [37,38], where the competition coefficients b and c are chosen as bifurcation/variable parameters, and the global dynamics is determined in a certain range of b and c on the b-c plane. Specifically, the authors first gave a clear picture of the local stability around the two semi-trivial steady states in terms of critical competition coefficients by the principal eigenvalue theory, then established an important estimate on the linear stability of any positive steady state via an analytic argument (see also a similar result by Guo, He and Ni [9]), and finally obtained the global dynamics in a certain range of b and c by appealing to the theory of monotone dynamical systems [10,11,14]. The main result suggests that either one of the two semi-trivial steady state is globally asymptotically stable (competitive exclusion) or there is a unique positive steady state which is globally asymptotically stable (coexistence), depending on the competition intensities (see details in [37, Theorems 4 and 5]).

To some extent, the works [37, 38] can be seen as a study on the parameter region of b and c where the global dynamics of system (1.1)-(1.3) can be completely determined. In the current paper, as a further development of [37, 38], we pursue further to understand the complicated dynamics of system (1.1)-(1.3), especially the structure of positive steady states. To this end, we primarily employ the bifurcation approach to present a global result on the structure of positive steady states.

Here, we mention several bifurcation results by considering some special cases or variants of system (1.1)-(1.3). For example, system (1.1)-(1.3) with $d_1 = d_2 = 1$, $\alpha_1 = \alpha_2 = 0$, $r_1 = r_2 = a > 0$ and $b, c > 1$ (the strong competition case) and with zero Dirichlet boundary conditions (as well as Neumann and Robin boundary conditions) have been investigated by Gui and Lou [8], where the existence and multiplicity of positive steady states are carefully examined by both bifurcation approach and degree method. A spatially one-dimensional case of system (1.1)-(1.3) together with Danckwerts boundary conditions, modeling the competition between two aquatic species in a free-flow environment, was studied by Wang, Tian, and Nie [32], where among other things, a picture on the structure of positive steady states is given by bifurcation approach. Moreover, Cantrell et al., [3] investigated the following slightly different model from (1.1)-(1.3)

$$\begin{cases} u_t = \nabla \cdot \Big(\alpha(x)\nabla \frac{u}{m}\Big) + u\Big[m(x) - u - bv\Big], & x \in \Omega,\ t > 0, \\ v_t = \nabla \cdot \Big(\beta(x)\nabla v\Big) + v\Big[m(x) - cu - v\Big], & x \in \Omega,\ t > 0, \\ \nabla \frac{u}{m} \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.4}$$

where one species (i.e., u) uses an ideal free dispersal strategy with a positive coefficient $\alpha(x)$, the other one (i.e., v) adopts a fickian-type diffusion with a positive rate $\beta(x)$, and the two species are supposed to compete for the same resource as measured by a positive function $m(x)$. Again, by employing the bifurcation theory, the authors obtained a global bifurcation result for the positive steady states of system (1.4).

Motivated by the above mentioned works, in the sequel, we shall consider the general competitive system (1.1)-(1.3) and utilize the global bifurcation theory for a C^1 Fredholm mapping developed by Pejsachowicz and Rabier [22] to present a global bifurcation result (see Theorem 4.1 in Section 4). Also, some results on the phenomenon of multiple positive steady states are included in Section 5.

The remainder of this paper is organized as follows. In Section 2, we give some fundamental results that are useful in later analysis. Section 3 is devoted to the analysis of boundary behaviors of positive steady states. Based on this, we then present the main global bifurcation result in Section 4, and finally discuss the multiple solutions phenomenon in Section 5.

2. Preliminaries

In this section, we mainly give some fundamental results which will be used later.

2.1. Bifurcation theory

First, we recall some standard bifurcation theories.

Let X_1 and Y be two Banach spaces, U be an open connected subset of $\mathbb{R} \times X_1$ with $(\varrho_0, x_0) \in U$, and let F be a continuously differentiable mapping from U into Y, i.e., $F \in C^1(U, Y)$. Assume that

(H_1) $F\big(\varrho, x_0\big) = 0$ for $\big(\varrho, x_0\big) \in U$;

(H_2) $D_x F$, $D_\varrho F$, $D_{\varrho x} F$ exist and are continuous in U;

(H_3) $D_x F\big(\varrho_0, x_0\big)$ is a Fredholm operator with index 0, and for some $\theta_0 \in X_1$,

$$\mathrm{Ker}\Big(D_x F\big(\varrho_0, x_0\big)\Big) = \mathrm{span}\{\theta_0\};$$

(H_4) $D_{\varrho x} F\big(\varrho_0, x_0\big)[\theta_0] \notin \mathrm{Range}\Big(D_x F\big(\varrho_0, x_0\big)\Big)$.

The following local bifurcation result follows from [4].

Theorem 2.1. *Let Z be any complement of Ker$\Big(D_x F\big(\varrho_0, x_0\big)\Big)$ in X_1 and assume that (H_1)-(H_4) hold. Then, there exists an open interval $I = (-\epsilon, \epsilon)$ and continuous functions*

$$\varrho : I \to \mathbb{R} \qquad \textit{and} \qquad f : I \to Z$$

such that $\varrho(0) = \varrho_0$, $f(0) = 0$, *and if* $x(s) = x_0 + s\theta_0 + sf(s)$ *for* $s \in I$, *then* $F\big(\varrho(s), x(s)\big) = 0$. *Moreover,* $F^{-1}\big(\{0\}\big)$ *near* $\big(\varrho_0, x_0\big)$ *consists precisely of the curves*

$$\Big\{\big(\varrho, x\big) \,:\, x = x_0\Big\} \quad \textit{and} \quad \Gamma := \Big\{\big(\varrho(s), x(s)\big) : s \in I\Big\}.$$

If, in addition, $D_xF\big(\varrho, x\big)$ *is a Fredholm operator for all* $\big(\varrho, x\big) \in U$, *then the curve* Γ *is contained in* $\mathcal{C}$, *which is a connected component of* $\bar{\mathcal{E}}$, *where*

$$\mathcal{E} := \Big\{\big(\varrho, x\big) \in U : F\big(\varrho, x\big) = 0, x \neq x_0\Big\},$$

and either $\mathcal{C}$ *is not compact in* U, *or* $\mathcal{C}$ *contains a point* $\big(\varrho_1, x_0\big)$ *with* $\varrho_1 \neq \varrho_0$.

In many nonlinear biological systems, people are particularly interested in positive solutions. Therefore, we further present an unilateral version of the above theorem (see below). The proof can be found in [26].

Theorem 2.2. *In addition to* (H_1)*-*(H_4)*, we further assume that*

(H_5) *the norm function* $x \longmapsto \|x\|$ *in* X_1 *is continuously differentiable for any* $x \neq 0$;

(H_6) *for* $\kappa \in (0,1)$, *if both* $\big(\varrho, x_0\big)$ *and* $\big(\varrho, x\big)$ *are in* U, *then* $(1-\kappa)D_xF\big(\varrho, x_0\big) + \kappa D_xF\big(\varrho, x\big)$ *is a Fredholm operator.*

Define

$$\Gamma^+ := \Big\{\big(\varrho(s), x(s)\big) : s \in (0, \epsilon)\Big\} \quad \textit{and} \quad \Gamma^- := \Big\{\big(\varrho(s), x(s)\big) : s \in (-\epsilon, 0)\Big\}.$$

Let $\mathcal{C}$ *be defined as in Theorem 2.1, and* $\mathcal{C}^+$ *(resp.* $\mathcal{C}^-$*) be the connected component of* $\mathcal{C} \setminus \Gamma^-$ *(resp.* $\mathcal{C} \setminus \Gamma^+$*) which includes* Γ^+ *(resp.* Γ^-*). Let* Z *be any complement of* $\operatorname{Ker}(D_xF\big(\varrho_0, x_0\big))$ *in* X_1, *then* $\mathcal{C}^+$ *or* $\mathcal{C}^-$ *satisfies one of the following alternatives:*

(i) *it is not compact;*

(ii) *it contains a point* $\big(\varrho_1, x_0\big)$ *with* $\varrho_1 \neq \varrho_0$;

(iii) *it contains a point* $\big(\varrho, x_0 + z\big)$ *with* $z \in Z \setminus \{0\}$.

2.2. Some notations and definitions

Concerning system (1.1)-(1.3), here and hereafter, we fix the operators $\mathcal{L}_i, \mathcal{B}_i$ ($i = 1, 2$), the domain Ω, the functions $r_1(x), r_2(x)$ and the parameter $c > 0$, and choose $b > 0$ as a bifurcation parameter to study the bifurcation diagram.

Let us set

$$X_1 = \Big\{\big(m, n\big) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) : \mathcal{B}_1 m = \mathcal{B}_2 n = 0 \text{ on } \partial\Omega\Big\},$$

$$Y = L^p(\Omega) \times L^p(\Omega), \quad p > N,$$

and

$$U = \mathbb{R}^+ \times \Big\{\big(u, v\big) \in X_1 \,:\, u > 0,\, v > -\delta \text{ in } \Omega\Big\}, \tag{2.1}$$

for some small $\delta > 0$.

Define the map $\mathcal{F}: U \longrightarrow Y$ by

$$\mathcal{F}: \ \Big(b,(u,v)\Big) \mapsto \begin{pmatrix} \mathcal{L}_1 u + u\Big[r_1(x) - u - bv\Big] \\ \mathcal{L}_2 v + v\Big[r_2(x) - cu - v\Big] \end{pmatrix}.$$

Clearly, for fixed $c > 0$, $\Big(b,(u,v)\Big)$ is a stationary solution of system (1.1)-(1.3) if and only if $\mathcal{F}\Big(b,(u,v)\Big) = 0$. Note that $\mathcal{F}\Big(b,(0,\tilde{v})\Big) = 0$ for all $b > 0$.

Given any $c > 0$, let $\Big(b,(u,v)\Big)$ be a steady state of system (1.1)-(1.3). Then,

$$\begin{cases} 0 = \mathcal{L}_1 u + u\Big[r_1(x) - u - bv\Big], & x \in \Omega, \\ 0 = \mathcal{L}_2 v + v\Big[r_2(x) - cu - v\Big], & x \in \Omega, \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Let us denote all possible steady states of system (1.1)-(1.3) in U by $\mathcal{S}_c$, namely,

$$\mathcal{S}_c := \Big\{\Big(b,(u,v)\Big) \in U : \ \mathcal{F}\Big(b,(u,v)\Big) = 0\Big\},$$

and the positive ones by

$$\mathcal{S}_c^+ := \Big\{\Big(b,(u,v)\Big) \in U : \ u > 0, \ v > 0 \text{ for } x \in \overline{\Omega}, \text{ and } \mathcal{F}\Big(b,(u,v)\Big) = 0\Big\}. \tag{2.3}$$

Note that the set $\mathcal{S}_c^+$ remains unchanged if the strict inequalities are relaxed to $u \geqslant, \not\equiv 0$ and $v \geqslant, \not\equiv 0$ in $\overline{\Omega}$ due to the strong maximum principle [6].

Denote by $\big(\tilde{u},0\big)$ and $\big(0,\tilde{v}\big)$ respectively, the two semi-trivial steady states of system (1.1)-(1.3), where $\tilde{u}$ and $\tilde{v}$ are positive solutions of

$$\begin{cases} 0 = \mathcal{L}_1 u + u\Big[r_1(x) - u\Big], & x \in \Omega, \\ \mathcal{B}_1 u = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} 0 = \mathcal{L}_2 v + v\Big[r_2(x) - v\Big], & x \in \Omega, \\ \mathcal{B}_2 v = 0, & x \in \partial\Omega, \end{cases}$$

respectively. The existence and uniqueness of $\tilde{u}$ and $\tilde{v}$ are standard [2, 36], and clearly they are independent of the competition coefficients b and c.

Linearizing system (1.1)-(1.3) at $\big(\tilde{u},0\big)$ and $\big(0,\tilde{v}\big)$ respectively, one sees

$$\begin{cases} 0 = \mathcal{L}_2 \varphi + \Big[r_2(x) - c\tilde{u}\Big]\varphi + \lambda\varphi, & x \in \Omega, \\ \mathcal{B}_2 \varphi = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

and

$$\begin{cases} 0 = \mathcal{L}_1 \psi + \Big[r_1(x) - b\tilde{v}\Big]\psi + \mu\psi, & x \in \Omega, \\ \mathcal{B}_1 \psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

It is well-known (see, e.g., [12,27]) that problem (2.4) (resp. (2.5)) admits a principal eigenvalue, denoted in the sequel by λ_1 (resp. μ_1), which is simple and real, and its corresponding eigenfunction φ_1 (resp. ψ_1) can be chosen strictly in $\overline{\Omega}$. Moreover, similar to [15, Corollary 2.10], one can show that the linear stability of $(\tilde{u},0)$, and $(0,\tilde{v})$ can be determined respectively by the sign of λ_1 and μ_1. Taking $(0,\tilde{v})$ as an example, it is linearly stable, neutrally stable and linearly unstable provided $\mu_1>0$, $\mu_1=0$ and $\mu_1<0$, respectively.

Regarding λ_1 and μ_1, we recall a result from [37, Proposition 16] as below.

Proposition 2.1. *Regard λ_1 as a function of c and μ_1 as a function of b. Then,*

(*i*) *there exists a critical value $c^*>0$ such that*

$$\mathit{sign}(\lambda_1)=\mathit{sign}\ (c-c^*);$$

(*ii*) *there exists a critical value $b^*>0$ such that*

$$\mathit{sign}(\mu_1)=\mathit{sign}\ (b-b^*).$$

Based on the above result, we can define $\bar{\varphi}$ as the principal eigenfunction corresponding to $\lambda_1|_{c=c^*}=0$, namely,

$$\begin{cases} 0=\mathcal{L}_2\bar{\varphi}+\Big[r_2(x)-c^*\tilde{u}\Big]\bar{\varphi}, & x\in\Omega,\\ \mathcal{B}_2\bar{\varphi}=0, & x\in\partial\Omega,\\ \bar{\varphi}>0, \quad \|\bar{\varphi}\|_{L^\infty}=1. \end{cases} \tag{2.6}$$

Furthermore, we define $\bar{\theta}$ as the unique positive solution of

$$\begin{cases} \mathcal{L}_1\bar{\theta}+\Big[r_1(x)-2\tilde{u}\Big]\bar{\theta}=-\bar{\varphi}\tilde{u}, & x\in\Omega,\\ \mathcal{B}_1\bar{\theta}=0, & x\in\partial\Omega, \end{cases} \tag{2.7}$$

where the positivity of $\bar{\theta}$ in $\overline{\Omega}$ is due to the strong maximum principle [6]. Note that $\overline{\theta}$ is well defined since $-\mathcal{L}_1-[r_1-2\tilde{u}]I$ is invertible (the spectrum lies in $\{z\in\mathbb{C}:\ \mathrm{Re}\, z>0\}$).

In a similar way, we define $\bar{\psi}$ to be the principal eigenfunction corresponding to $\mu_1|_{b=b^*}=0$, i.e.,

$$\begin{cases} 0=\mathcal{L}_1\bar{\psi}+\Big[r_1(x)-b^*\tilde{v}\Big]\bar{\psi}, & x\in\Omega,\\ \mathcal{B}_1\bar{\psi}=0, & x\in\partial\Omega,\\ \bar{\psi}>0, \quad \|\bar{\psi}\|_{L^\infty}=1, \end{cases} \tag{2.8}$$

and $\tilde{\psi}$ to be the unique negative solution of

$$\begin{cases} \mathcal{L}_2\tilde{\psi}+\Big[r_2(x)-2\tilde{v}\Big]\tilde{\psi}=c\tilde{v}\bar{\psi}, & x\in\Omega,\\ \mathcal{B}_2\tilde{\psi}=0, & x\in\partial\Omega, \end{cases} \tag{2.9}$$

where, again, the maximum principle [6] is used.

Finally, we define a critical value as follows

$$\bar{L}:=\frac{\int_\Omega \bar{\varphi}^3\cdot e^{-\frac{\alpha_2}{d_2}P_2(x)}\mathrm{d}x}{\int_\Omega \bar{\varphi}^2\cdot\bar{\theta}\cdot e^{-\frac{\alpha_2}{d_2}P_2(x)}\mathrm{d}x}>0. \tag{2.10}$$

2.3. Local bifurcation

For system (1.1)-(1.3), by Theorem 2.1, we have the following local bifurcation result at $\Big(b^*, \big(0, \tilde{v}\big)\Big)$.

Proposition 2.2. *For any $M > 1$, there are $\delta, \eta > 0$ such that for each $c \in [0, M]$, there exist smooth functions $\hat{b}(s), \hat{u}(s)$ and $\hat{v}(s)$ defined in $s \in (-\delta, \delta)$ such that*

(i) $\Big(\hat{b}(s), \big(\hat{u}(s), \hat{v}(s)\big)\Big)\Big|_{s=0} = \Big(b^*, \big(0, \tilde{v}\big)\Big)$;

(ii) *in the neighborhood*

$$\mathcal{N}_\eta := \Big\{\Big(b, \big(u, v\big)\Big) \in U : |b - b^*| + \|\big(u, v\big) - \big(0, \tilde{v}\big)\| < \eta\Big\},$$

the set $\mathcal{S}_c = \mathcal{F}^{-1}(0)$ is exactly determined by the trivial solution curve

$$\mathcal{C}_0 = \Big\{\Big(b, \big(0, \tilde{v}\big)\Big) : b \in \mathbb{R}^+\Big\},$$

and the nontrivial one

$$\mathcal{C}_1 = \Big\{\Big(\hat{b}(s), \big(\hat{u}(s), \hat{v}(s)\big)\Big) : s \in (-\delta, \delta)\Big\},$$

and that is,

$$\mathcal{S}_c \cap \mathcal{N}_\eta = (\mathcal{C}_0 \cup \mathcal{C}_1) \cap \mathcal{N}_\eta;$$

(iii) *in the neighborhood $\mathcal{N}_\eta$, the curve of positive solutions is determined by*

$$\mathcal{S}_c^+ \cap \mathcal{N}_\eta = \mathcal{C}_1^+ := \Big\{\Big(b, \big(u, v\big)\Big) \in \mathcal{C}_1 \,:\, u > 0 \ \mathit{in}\ \overline{\Omega}\Big\};$$

(iv) *the curve $\mathcal{C}_1$ is contained in $\mathcal{C}_*$ which is a connected component of $\overline{\mathcal{E}_1}$ with*

$$\mathcal{E}_1 := \Big\{\Big(b, \big(u, v\big)\Big) \in U : \mathcal{F}\Big(b, \big(u, v\big)\Big) = 0, \big(u, v\big) \neq \big(0, \tilde{v}\big)\Big\}.$$

Remark 2.1. Set $\theta_0 = (\bar{\psi}, \tilde{\psi})$, where $\bar{\psi}$ and $\tilde{\psi}$ are defined in (2.8) and (2.9), respectively. Then,

$$\operatorname{Ker}\Big(D_{(u,v)}\mathcal{F}\Big(b^*, \big(0, \tilde{v}\big)\Big)\Big) = \operatorname{span}\Big\{\theta_0\Big\}. \tag{2.11}$$

The complement of $\operatorname{Ker}\Big(D_{(u,v)}\mathcal{F}\Big(b^*, \big(0, \tilde{v}\big)\Big)\Big)$ can be conveniently chosen as

$$Z := \Big\{\big(u', v'\big) \in X_1 : \int_\Omega \Big[u'\bar{\psi} + v'\tilde{\psi}\Big]\mathrm{d}x = 0\Big\}, \tag{2.12}$$

so that $X_1 = \operatorname{span}\Big\{\theta_0\Big\} \oplus Z$.

Similar to the above $\mathcal{C}_1^+$, we can define

$$\mathcal{C}_1^- := \Big\{\Big(b, \big(u, v\big)\Big) \in \mathcal{C}_1 \,:\, u < 0 \ \text{ in } \overline{\Omega}\Big\}.$$

Let $\mathcal{C}_*^+$ be the connected component of $\mathcal{C}_* \backslash \mathcal{C}_1^-$ which includes $\mathcal{C}_1^+$. Clearly, Theorem 2.2 is applicable to $\mathcal{C}_*^+$. Therefore, we have the following result.

Proposition 2.3. *$\mathcal{C}_*^+$ must satisfy one of the following alternatives:*

(i) it is not compact;

(ii) it contains a point $\Big(b_1, \big(0, \tilde{v}\big)\Big)$ with $b_1 \neq b^*$;

(iii) it contains a point $\Big(b, \big(0, \tilde{v}\big) + z\Big)$ with $z = \big(u', v'\big) \in Z \setminus \{0\}$.

Proof. It suffices to verify that the following two cases hold:

(H_5) the norm function $\big(u, v\big) \longmapsto \|\big(u, v\big)\|$ in X_1 is continuously differentiable for any $\big(u, v\big) \neq \big(0, 0\big)$;

(H_6) for $\kappa \in (0, 1)$, if both $\Big(b, \big(0, \tilde{v}\big)\Big)$ and $\Big(b, \big(u, v\big)\Big)$ are in U, then

$$(1-\kappa)D_xF\Big(b, \big(0, \tilde{v}\big)\Big) + \kappa D_xF\Big(b, \big(u, v\big)\Big)$$

is a Fredholm operator.

Clearly, (H_5) is satisfied for $\|\cdot\|$ in X_1, because $(u, v) \mapsto \int_\Omega |(u, v)|^p \, dx$ is differentiable in $L^p(\Omega)$ for all $p \in (1, \infty)$ (see also [25]). For (H_6), one should notice that for any $\Big(b, \big(0, \tilde{v}\big)\Big)$ and $\Big(b, \big(u, v\big)\Big)$ in U, $(1-\kappa)D_{(u,v)}\mathcal{F}\Big(b, \big(0, \tilde{v}\big)\Big) + \kappa D_{(u,v)}\mathcal{F}\Big(b, \big(u, v\big)\Big)$ can be written as the sum of an isomorphism (the linear elliptic operator minus identity) and a multiplication operator (which is a compact operator from $W^{2,p}$ to L^p). Hence, it is Fredholm of index zero [22]. $\square$

3. Boundary behaviors

To better understand the geometric structure of the positive solution set $\overline{\mathcal{S}_c^+}$, we need to do some qualitative analysis on the boundary behaviors.

First, we discuss the possibility of positive steady states converging to $(\tilde{u}, 0)$.

Proposition 3.1. *Assume that there is a sequence of positive solutions of system (1.1)-(1.3) denoted by $\Big\{\big(b_n, c_n, u_n, v_n\big)\Big\}_{n=1}^{\infty}$ with $b_n, c_n > 0$ and $u_n, v_n > 0$ in $\overline{\Omega}$ for each n. Then the following statements are valid:*

(i) if $\limsup\limits_{n\to\infty} \frac{c_n - c^*}{\|v_n\|_{L^\infty}} \le 0$, $u_n \rightharpoonup \tilde{u}$ in H^1, and $v_n \to 0$ in L^∞, then $\limsup\limits_{n\to\infty} b_n c_n \le \bar{L}$;

(ii) if $\liminf\limits_{n\to\infty} \frac{c_n - c^*}{\|v_n\|_{L^\infty}} \ge 0$, $u_n \rightharpoonup \tilde{u}$ in H^1, and $v_n \to 0$ in L^∞, then $\liminf\limits_{n\to\infty} b_n c_n \ge \bar{L}$,

where c^* is determined in Proposition 2.1, and $\bar{L}$ is defined in (2.10).

Proof. Since $\big(u_n, v_n\big) \to \big(\tilde{u}, 0\big)$ as $n \to \infty$, $\big(\tilde{u}, 0\big)$ must be neutrally stable. Therefore, $c_n \to c^*$ as $n \to \infty$ (see Proposition 2.1).

Let $w_n = \frac{v_n}{\|v_n\|_{L^\infty}}$. Then for each $n > 0$, we have

$$\begin{cases} 0 = \mathcal{L}_2 w_n + w_n\Big[r_2(x) - c_n u_n - v_n\Big], & x \in \Omega, \\ \mathcal{B}_2 w_n = 0, & x \in \partial\Omega, \\ \|w_n\|_{L^\infty} = 1, \quad w_n > 0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

Using the elliptic regularity and imbedding theorem [7], we may assume, passing to a subsequence if necessary, that

$$w_n \longrightarrow w^* \quad \text{in} \quad C^1(\Omega) \quad \text{as} \quad n \to \infty,$$

where w^* satisfies

$$\begin{cases} 0 = \mathcal{L}_2 w^* + w^* \Big[r_2(x) - c^* \tilde{u} \Big], & x \in \Omega, \\ \mathcal{B}_2 w^* = 0, & x \in \partial\Omega, \\ \|w^*\|_{L^\infty} = 1, \quad w^* > 0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.2}$$

Hence, $w^* = \bar{\varphi}$ (recall $\bar{\varphi}$ is the principal eigenfunction of (2.6)).

Set $z_n = u_n - \tilde{u}$. Obviously, $z_n \leqslant 0$ for all n, $z_n \to 0$ as $n \to \infty$, and z_n satisfies

$$\begin{aligned} \mathcal{L}_1 z_n + \Big[r_1(x) - 2\tilde{u} \Big] z_n &= u_n \Big[u_n + b_n \|v_n\|_{L^\infty} w_n - 2\tilde{u} \Big] + \tilde{u}^2 \\ &= \Big[u_n^2 - 2\tilde{u} u_n + \tilde{u}^2 \Big] + b_n \|v_n\|_{L^\infty} u_n w_n \\ &= z_n^2 + b_n \|v_n\|_{L^\infty} w_n z_n + b_n \|v_n\|_{L^\infty} w_n \tilde{u}. \end{aligned} \tag{3.3}$$

By the standard L^p theory for elliptic equations [7], there exists a constant $C > 0$, which is independent of n such that

$$\|z_n\|_{W^{2,p}} \leqslant C \cdot \Big[\|z_n^2\|_{L^p} + b_n \|v_n\|_{L^\infty} \|w_n z_n\|_{L^p} + b_n \|v_n\|_{L^\infty} \|w_n \tilde{u}\|_{L^p} \Big].$$

Set $\theta_n = \frac{z_n}{b_n \|v_n\|_{L^\infty}}$. Then,

$$\|\theta_n\|_{W^{2,p}} \leqslant C \cdot \Big[\|z_n \theta_n\|_{L^p} + \|w_n z_n\|_{L^p} + \|w_n \tilde{u}\|_{L^p} \Big]. \tag{3.4}$$

Due to the boundedness of w_n and $\tilde{u}$ as well as the fact that $\|z_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$, one sees from (3.4) that θ_n is actually uniformly bounded in $W^{2,p}$ (and thus in C^1). Passing to a sequence, we may assume that θ_n converges weakly in $W^{2,p}(\Omega)$, and satisfies

$$\begin{cases} \mathcal{L}_1 \theta_n + \Big[r_1(x) - 2\tilde{u} \Big] \theta_n = z_n \theta_n + w_n z_n + w_n \tilde{u}, & x \in \Omega, \\ \mathcal{B}_1 \theta_n = 0, & x \in \partial\Omega, \end{cases}$$

which, together with $z_n \to 0$ and $w_n \to w^* = \bar{\varphi}$, further implies that $\theta_n \rightharpoonup -\bar{\theta}$, where $\bar{\theta} > 0$ is uniquely determined by (2.7).

Now, rewrite (3.1) as

$$\begin{aligned} \mathcal{L}_2 w_n + \Big[r_2(x) - c^* \tilde{u} \Big] w_n &= \Big[c_n u_n + v_n - c^* \tilde{u} \Big] w_n \\ &= \Big[(c_n - c^*) \tilde{u} + c_n b_n \|v_n\|_{L^\infty} \theta_n + \|v_n\|_{L^\infty} w_n \Big] w_n, \end{aligned} \tag{3.5}$$

and recall that for each n,

$$\mathcal{B}_2 w_n = 0 \quad \text{for } x \in \partial\Omega, \quad \|w_n\|_{L^\infty} = 1, \quad \text{and} \quad w_n > 0 \quad \text{in} \quad \Omega.$$

Multiplying (3.2) by $e^{-\frac{\alpha_2}{d_2}P_2(x)}w_n$ and (3.5) by $e^{-\frac{\alpha_2}{d_2}P_2(x)}w^*$, subtracting the resulting equations and integrating over Ω, one then observes that for each n,

$$\begin{aligned}\|v_n\|_{L^\infty}\int_\Omega \Big[b_n c_n w_n \theta_n \bar{\varphi} + w_n^2\bar{\varphi}\Big]e^{-\frac{\alpha_2}{d_2}P_2(x)}\mathrm{d}x \\ + \int_\Omega \Big[c_n - c^*\Big] w_n \bar{\varphi}\tilde{u} e^{-\frac{\alpha_2}{d_2}P_2(x)}\mathrm{d}x = 0.\end{aligned} \tag{3.6}$$

Dividing both sides of (3.6) by $\|v_n\|_{L^\infty}$ and taking $n \to \infty$, one then finds that statements (i) and (ii) hold. □

Now, we talk about several other cases.

Proposition 3.2. *Assume that there is a sequence of positive steady states of system* (1.1)-(1.3) *denoted by* $\Big\{\big(b_n, c_n, u_n, v_n\big)\Big\}_{n=1}^\infty$ *with* $b_n, c_n > 0$ *and* $u_n, v_n > 0$ *in* $\overline{\Omega}$ *for each* n. *Then the following statements hold true:*

(i) if $\big(u_n, v_n\big) \to \big(0, \tilde{v}\big)$ *in* $[L^\infty(\Omega)]^2$ *as* $n \to \infty$ *and* $\sup_n b_n < \infty$, *then* $b_n \to b^*$ *as* $n \to \infty$;

(ii) if $u_n \rightharpoonup \tilde{u}$ *in* $H^1(\Omega)$, $v_n \to 0$ *in* $L^\infty(\Omega)$ *as* $n \to \infty$ *and* $c_n \equiv c^*$ *for all* $n > 0$, *then* $b_n \to \frac{\bar{L}}{c^*}$ *as* $n \to \infty$;

(iii) if $b_n \to \infty$ *as* $n \to \infty$, *then* $c_n > c^*$ *for* $n \gg 1$;

(iv) if $b_n \to 0$, $c_n \to c \in [0,\infty]$ *as* $n \to \infty$ *and* $c_n \not\equiv c^*$, *then* $c \in [0, c^*]$. *Moreover, if* $c \in [0, c^*)$, *then* $\big(u_n, v_n\big) \to \big(\tilde{u}, v_c\big)$ *as* $n \to \infty$, *where* v_c *is the unique positive solution of*

$$\begin{cases}\mathcal{L}_2 v + \Big[r_2(x) - c\tilde{u} - v\Big]v = 0, & x \in \Omega,\\ \mathcal{B}_2 v = 0, & x \in \partial\Omega,\end{cases} \tag{3.7}$$

and if $c = c^*$, *then* $\big(u_n, v_n\big) \to \big(\tilde{u}, 0\big)$ *as* $n \to \infty$.

Proof. First, we prove statement (i). If the sequence of $\{b_n\}_{n=1}^\infty$ is uniformly bounded and does not converge to b^*, then (after passing to a subsequence if necessary) there exists a small number $\epsilon_0 > 0$ such that $b_n \geqslant b^* + \epsilon_0$ or $b_n \leqslant b^* - \epsilon_0$ for all large $n > 0$. This in view of Proposition 2.1 implies that for all large $n > 0$, $\big(0, \tilde{v}\big)$ is either linearly stable or linearly unstable, contradicting $\big(u_n, v_n\big) \to \big(0, \tilde{v}\big)$.

For statement (ii), we apply Proposition 3.1 to conclude

$$\bar{L} \le \liminf_{n\to\infty} b_n c_n \le \limsup_{n\to\infty} b_n c_n \le \bar{L}, \quad \text{i.e.,} \quad b_n c_n \to \bar{L}.$$

Since $c_n \to c^* > 0$, we have $b_n \to \bar{L}/c^*$.

Statement (iii) is a direct consequence of Lemma 3.1 (given later).

Finally, we prove statement (iv). Note that $\big(b_n, c_n, u_n, v_n\big)$ satisfies

$$\begin{cases}0 = \mathcal{L}_1 u_n + u_n\Big[r_1(x) - u_n - b_n v_n\Big], & x \in \Omega,\\ 0 = \mathcal{L}_2 v_n + v_n\Big[r_2(x) - c_n u_n - v_n\Big], & x \in \Omega,\\ \mathcal{B}_1 u_n = \mathcal{B}_2 v_n = 0, & x \in \partial\Omega.\end{cases}$$

Since $b_n \to 0$ as $n \to \infty$, it follows from [37, Theorems 4, 5] that $c_n \in (0, c^*)$ for all large n. Hence, $c = \lim_{n\to\infty} c_n \in [0, c^*]$. Also, it is easy to see that $\|u_n\|_{W^{2,p}(\Omega)}$ is uniformly bounded, and thus u_n should converge to some function u^* in C^1 which satisfies (weakly)

$$\begin{cases} 0 = \mathcal{L}_1 u + u\Big[r_1(x) - u\Big], & x \in \Omega, \\ \mathcal{B}_1 u = 0, & x \in \partial\Omega. \end{cases}$$

Hence, $u^* = \tilde{u}$ or $u^* = 0$. If $u^* = 0$, one can integrate the equation of u_n over Ω to deduce a contradiction for large n. Therefore, $u^* = \tilde{u}$, i.e., $u_n \to \tilde{u}$ as $n \to \infty$. This fact, together with $c = \lim_{n\to\infty} c_n \in [0, c^*]$, further implies that v_n also converges in C^1 to some function v^* satisfying (3.7) in the weak sense.

If $c = c^*$, then it is easy to see $\big(u_n, v_n\big) \to \big(\tilde{u}, 0\big)$ as $n \to \infty$, as it is the only nonnegative solution of (3.7). If $c \in [0, c^*)$, we claim that $v_n \to v_c$. Indeed, passing to a convergent subsequence, we may assume either $v_n \to 0$ or $v_n \to v_c$ in $L^\infty(\Omega)$. Suppose to the contrary that $v_n \to 0$. Multiplying the equation of v_n by $e^{-\frac{\alpha_2}{d_2}P_2(x)}\bar{\varphi}$ and (2.6) by $e^{-\frac{\alpha_2}{d_2}P_2(x)}v_n$, subtracting the resulting equations and integrating over Ω, one then sees

$$\int_\Omega \Big[c^*\tilde{u} - c_n u_n - v_n\Big] e^{-\frac{\alpha_2}{d_2}P_2(x)} \bar{\varphi} v_n \mathrm{d}x = 0 \quad \text{for any} \quad n > 0,$$

which is impossible for large n since $c_n u_n + v_n \to c\tilde{u} < c^*\tilde{u}$. □

Remark 3.1. Proposition 3.2 *(iv)* excludes the possibility of positive steady states (u_n, v_n) when $b_n \to 0$ and $c_n \searrow c^*$ as $n \to \infty$. Indeed, it also includes the special situation when $c_n \equiv c^*$ for each $n > 0$ and $b_n \to 0$ as $n \to \infty$.

Next, we exhibit the boundedness of $\overline{\mathcal{S}_c^+}$.

Lemma 3.1. *$\overline{\mathcal{S}_c^+}$ is uniformly bounded in $c \in [0, c^*]$ with respect to the topology in $(0, \infty) \times X_1$.*

Proof. This result follows directly from [37, Proposition 19]. □

Finally, we give a description on how $\overline{\mathcal{S}_c^+}$ is connected to

$$\mathcal{S}_c^0 := \Big\{\Big(0, \big(u, v\big)\Big) \in U : \ u > 0, \ v > 0 \text{ for } x \in \overline{\Omega} \text{ and } \mathcal{F}\Big(0, \big(u, v\big)\Big) = 0\Big\}.$$

Lemma 3.2. *Let $c > 0$. The following statements on $\overline{\mathcal{S}_c^+}$ and $\mathcal{S}_c^0$ are true:*

(i) $\overline{\mathcal{S}_c^+} \cap \mathcal{S}_c^0 \neq \emptyset$, *if and only if* $c \in (0, c^*)$;

(ii) *for each* $c \in (0, c^*)$, $\overline{\mathcal{S}_c^+} \cap \mathcal{S}_c^0 = \Big\{\Big(0, \big(\tilde{u}, v_c\big)\Big)\Big\}$, *where v_c is the unique positive solution of problem* (3.7);

(iii) *for each* $c \in (0, c^*)$, *there exist* $\delta, \eta > 0$ *and* C^1 *functions* $\Big(b_1(s), \big(u_1(s), v_1(s)\big)\Big)$ *such that* $\Big(b_1(0), \big(u_1(0), v_1(0)\big)\Big) = \Big(0, \big(\tilde{u}, v_c\big)\Big)$ *and*

$$\overline{\mathcal{S}_c^+} \cap \mathcal{N}'_\eta = \Big\{\Big(b_1(s), \big(u_1(s), v_1(s)\big)\Big) : s \in [0, \delta)\Big\}, \tag{3.8}$$

where

$$\mathcal{N}'_\eta := \Big\{\Big(b, \big(u, v\big)\Big) \in U : |b| + \|\big(u, v\big) - \big(\tilde{u}, v_c\big)\| < \eta\Big\}.$$

Proof. By the Proposition 3.2 (iv), one sees that $\mathcal{S}_c^0 \neq \emptyset$, if and only if problem (3.7) admits a unique positive solution v_c. This fact, together with $c > 0$, further yields $c \in (0, c^*)$. Hence, statements (i) and (ii) are proved.

For statement (iii), linearizing system (1.1)-(1.3) at $\big(u(x), v(x)\big)$, one obtains the following linear eigenvalue problem

$$\begin{cases} \mathcal{L}_1\varphi + \Big[r_1(x) - u - bv\Big]\varphi - u\Big[\varphi + b\psi\Big] + \lambda\varphi = 0, & x \in \Omega, \\ \mathcal{L}_2\psi + \Big[r_2(x) - cu - v\Big]\psi - v\Big[c\varphi + \psi\Big] + \lambda\psi = 0, & x \in \Omega, \\ \mathcal{B}_1\varphi = \mathcal{B}_2\psi = 0. & x \in \partial\Omega. \end{cases} \tag{3.9}$$

Restricting problem (3.9) at $\Big(b, \big(u, v\big)\Big) = \Big(0, \big(\tilde{u}, v_c\big)\Big)$, one easily sees that $\lambda > 0$. That is, the principal eigenvalue of the linearized operator $D_{(u,v)}\mathcal{F}\Big(0, \big(\tilde{u}, v_c\big)\Big)$ is positive, which implies that the operator $D_{(u,v)}\mathcal{F}\Big(0, \big(\tilde{u}, v_c\big)\Big)$ is invertible. Then by employing the implicit function theorem, one can obtain the solution curve $\Big(b_1(s), \big(u_1(s), v_1(s)\big)\Big)$ satisfying (3.8). $\square$

Remark 3.2. Lemma 3.2 (iii) shows that nearby the point $\Big(0, \big(\tilde{u}, v_c\big)\Big)$, and $\overline{\mathcal{S}_c^+}$ is a simple curve as given in (3.8). Indeed, if we assume $\frac{\alpha_2}{d_2}P_2(x) \equiv \frac{\alpha_1}{d_1}P_1(x)$ in Ω, then $\overline{\mathcal{S}_c^+}$ $(0 < c < c^*)$ is a simple curve in the following sense

$$\overline{\mathcal{S}_c^+}\Big|_{b \in \left[0,\ \min\{b^*, \frac{1}{c}\}\right)} = \Big\{\Big(b_1(s), \big(u_1(s), v_1(s)\big)\Big) : s \in [0, \delta)\Big\}, \text{ for some } \delta > 0. \tag{3.10}$$

To see (3.10), one needs to combine the existence of a connected component of $\overline{\mathcal{S}_c^+}$ that connects $\Big(b^*, \big(0, \tilde{v}\big)\Big)$ and $\Big(0, \big(\tilde{u}, v_c\big)\Big)$ (see Theorem 4.1 (i) given later) with the linear stability (non-degeneracy) of any positive steady state (see [38, Theorem 1.1]) together. In particular, if $0 < c < c^*$ and $b^* \leqslant \frac{1}{c}$, then $\overline{\mathcal{S}_c^+}\Big|_{b \in \left[0, b^*\right)}$ is a simple curve connecting $\Big(b^*, \big(0, \tilde{v}\big)\Big)$ and $\Big(0, \big(\tilde{u}, v_c\big)\Big)$, and if $0 < c \ll 1$, then $\overline{\mathcal{S}_c^+}$ is totally a simple curve (this is because $\overline{\mathcal{S}_c^+}$ is a bounded set in b (see Lemma 3.1). We also note here that if, instead, one uses the weaker condition $\frac{\alpha_2}{d_2}P_2(x) \leqslant \frac{\alpha_1}{d_1}P_1(x)$ in Ω, similar results to the above can be obtained but one needs to shrink the range of parameter b (see [37]).

4. Global bifurcation

Let us define

$$\mathcal{U}_c := \Big\{\Big(b, \big(u, v\big)\Big) \in U : \ u > 0, v > 0 \quad \text{for} \quad x \in \overline{\Omega}\Big\}, \tag{4.1}$$

where U is defined in (2.1).

Now, we are in a position to present the main result of this paper, which provides a complete understanding on the global structure of $\overline{\mathcal{S}_c^+}$.

Theorem 4.1. *The following statements on $\overline{\mathcal{S}_c^+}$ are valid:*

(*i*) *for* $0<c<c^*$, $\overline{\mathcal{S}_c^+}$ *remains bounded in* U,

$$\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c=\Big\{\Big(b^*,(0,\tilde v)\Big),\Big(0,(\tilde u,v_c)\Big)\Big\},\tag{4.2}$$

and $\overline{\mathcal{S}_c^+}$ *has a connected component containing* $\Big(b^*,(0,\tilde v)\Big)$ *and* $\Big(0,(\tilde u,v_c)\Big)$;

(*ii*) *for* $c=c^*$, $\overline{\mathcal{S}_c^+}$ *remains bounded in* U,

$$\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c=\Big\{\Big(b^*,(0,\tilde v)\Big),\Big(\frac{\bar L}{c^*},(\tilde u,0)\Big)\Big\},\tag{4.3}$$

and $\overline{\mathcal{S}_c^+}$ *has a connected component containing* $\Big(b^*,(0,\tilde v)\Big)$ *and* $\Big(\frac{\bar L}{c^*},(\tilde u,0)\Big)$;

(*iii*) *for* $c>c^*$, $\overline{\mathcal{S}_c^+}$ *is unbounded in* U,

$$\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c=\Big\{\Big(b^*,(0,\tilde v)\Big)\Big\},\tag{4.4}$$

and $\overline{\mathcal{S}_c^+}$ *has an unbounded connected component emanating from* $\Big(b^*,(0,\tilde v)\Big)$.

Proof. We prove statement $\underline{(i)}$ by the following three steps $(i.1)$-$(i.3)$.

$(i.1)$: The boundedness of $\overline{\mathcal{S}_c^+}$ in U for $c\in(0,c^*)$ is guaranteed by Lemma 3.1.

$(i.2)$: Validity of (4.2). Let $\Big(b',(u',v')\Big)$ be any element of $\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c$. Then, $\Big(b',(u',v')\Big)$ is a solution of system (2.2), and clearly, $b'\geqslant 0$ and $u',v'\geqslant 0$ in $\overline{\Omega}$. If $b'=0$, then $u'=\tilde u$ or $u'=0$. We claim that $u'=0$ is impossible, because $(0,0)$ is always linearly unstable, and $(0,\tilde v)$ is linearly unstable for $b'>0$ small. Therefore, $u'=\tilde u>0$. Furthermore, $v'=0$ is also impossible, as $(\tilde u,0)$ is linearly unstable for $c\in\big(0,c^*\big)$. Hence, if $b'=0$, then $u'=\tilde u$, and $v'>0$ for $x\in\overline{\Omega}$ in view of the maximum principle [23], i.e., $\big(u',v'\big)=\big(\tilde u,v_c\big)$ due to Lemma 3.2 (*ii*). If $b'>0$, then $\big(u',v'\big)$ could be one of $\big(0,0\big)$, $\big(\tilde u,0\big)$ and $\big(0,\tilde v\big)$. Using the linear instability of $\big(0,0\big)$ and $\big(\tilde u,0\big)$ again, we see $\big(u',v'\big)=\big(0,\tilde v\big)$ and $b'=b^*$, since $b=b^*$ is the unique point where there is a branch of positive steady states emanating from $\big(0,\tilde v\big)$. Such an analysis shows that

$$\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c\subset\Big\{\Big(b^*,(0,\tilde v)\Big),\Big(0,(\tilde u,v_c)\Big)\Big\}.$$

On the other hand, by Proposition 2.2,

$$\Big(b^*,(0,\tilde v)\Big)\in\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c,$$

and by Lemma 3.2 (*iii*),

$$\Big(0,(\tilde u,v_c)\Big)\in\overline{\mathcal{S}_c^+}\cap\partial\mathcal{U}_c.$$

This establishes (4.2).

$(i.3)$: Now, we verify that $\mathcal{C}_*\cap\mathcal{U}_c$ is a connected component of $\overline{\mathcal{S}_c^+}$ whose closure contains $\Big(b^*,(0,\tilde v)\Big)$ and $\Big(0,(\tilde u,v_c)\Big)$ (recall that $\mathcal{C}_*$ and $\mathcal{C}_*^+$ are defined in subsection 3.2). Clearly, $\Big(b^*,(0,\tilde v)\Big)\in\overline{\mathcal{C}_*\cap\mathcal{U}_c}$. It remains to verify $\Big(0,(\tilde u,v_c)\Big)\in\overline{\mathcal{C}_*\cap\mathcal{U}_c}$. By definition, $\big(\mathcal{C}_* \cap \mathcal{U}_c\big) \subset \mathcal{C}_*^+$. It follows from Proposition 2.3 that $\mathcal{C}_*^+$ must satisfy one of those three alternatives given in Theorem 2.2. We claim that alternative (i) holds. Otherwise,

$$\overline{\mathcal{C}_*^+ \cap \mathcal{U}_c} \subseteq \Big\{\Big(b, \big(u,v\big)\Big) \,:\, 0<b<M\Big\},$$

for some large $M>0$, and at least one of the alternatives (ii) or (iii) holds. However, the boundedness of $\mathcal{C}_*^+$ and the non-degeneracy of $\big(0,0\big)$, $\big(\tilde{u},0\big)$ and $\big(0,\tilde{v}\big)$ for $b \neq b^*$ imply that

$$\overline{\mathcal{C}_*^+} \cap \partial\mathcal{U}_c = \Big\{\Big(b^*, \big(0,\tilde{v}\big)\Big)\Big\}.$$

Therefore, $\mathcal{C}_*^+$ intersects the trivial branch at a single point, i.e., alternative (ii) is impossible. In fact, alternative (iii) can also be ruled out, since u remains nonnegative as $\Big(b, \big(u,v\big)\Big)$ varies in $\overline{\mathcal{C}_*^+ \cap \mathcal{U}_c}$. Hence, $\mathcal{C}_*^+$ satisfies alternative (i), which is equivalent to stating that either the closure of $\mathcal{C}_*^+$ intersects $\partial\mathcal{U}_c$ or $\mathcal{C}_*^+$ is unbounded in the norm of $U=(0,\infty)\times X_1$. By the uniform boundedness of $\overline{\mathcal{S}_c^+}$ in $c\in[0,c^*]$, $\mathcal{C}_*^+$ cannot be unbounded in the norm of $(0,\infty)\times X_1$. Therefore, the closure of $\mathcal{C}_*^+$ contains a point $\Big(0, \big(\bar{u},\bar{v}\big)\Big) \in \partial\mathcal{U}_c$. Since $\big(0,0\big)$ and $\big(0,\tilde{v}\big)$ are linearly unstable for small $b'>0$ and since $\big(\tilde{u},0\big)$ is linearly unstable for given $c\in(0,c^*)$, $\big(\bar{u},\bar{v}\big)$ is a nonnegative steady state. By Lemma 3.2 (ii), $\Big(0,\big(\bar{u},\bar{v}\big)\Big) = \Big(0,\big(\tilde{u},v_c\big)\Big)$, and one can further use Lemma 3.2 (iii) to conclude $\Big(0,\big(\tilde{u},v_c\big)\Big) \in \overline{\mathcal{C}_*^+ \cap \mathcal{U}_c}$. This finishes $(i.3)$.

Statement (ii) is verified by the following three aspects $(ii.1)$-$(ii.3)$.

$(ii.1)$: The boundedness of $\overline{\mathcal{S}_c^+}$ for $c=c^*$, again, is due to Lemma 3.1.

$(ii.2)$: Validity of (4.3). For any $\Big(b^1, \big(u^1,v^1\big)\Big) \in \overline{\mathcal{S}_c^+} \cap \partial\mathcal{U}_c$, by continuity, it is a solution of system (2.2) with $b^1 \geqslant 0$ and $u^1, v^1 \geqslant 0$ in $\overline{\Omega}$. We claim that $b^1=0$ is impossible. If $b^1=0$, then $u^1=0$ is impossible in view of the above $(i.2)$, and so $u^1=\tilde{u}>0$. Now, since $c=c^*$, we have $v^1=0$. However, by Proposition 3.2(ii), for $c=c^*$, there are no positive steady states converging to $\big(\tilde{u},0\big)$ as $b\to 0$. Hence, $b^1=0$ is impossible. Next, we consider $b^1>0$. Then, $\big(u^1,v^1\big)$ could be one of $\big(0,0\big)$, $\big(\tilde{u},0\big)$ and $\big(0,\tilde{v}\big)$. Clearly, $\big(0,0\big)$ is impossible due to its linear instability. If $\big(u^1,v^1\big)=\big(0,\tilde{v}\big)$, then $b^1=b^*$. If $\big(u^1,v^1\big)=\big(\tilde{u},0\big)$, by Proposition 3.2 (ii), we see $b^1=\frac{\bar{L}}{c^*}$. This indicates

$$\overline{\mathcal{S}_c^+} \cap \partial\mathcal{U}_c \subset \Big\{\Big(b^*, \big(0,\tilde{v}\big)\Big), \Big(\frac{\bar{L}}{c^*}, \big(\tilde{u},0\big)\Big)\Big\}.$$

Moreover, it is easy to see $\Big(b^*, \big(0,\tilde{v}\big)\Big) \in \overline{\mathcal{S}_c^+} \cap \partial\mathcal{U}_c$. The relation $\Big(\frac{\bar{L}}{c^*}, \big(\tilde{u},0\big)\Big) \in \overline{\mathcal{S}_c^+} \cap \partial\mathcal{U}_c$ is guaranteed by the next step. Hence, (4.3) is finished.

$(ii.3)$: Similar to $(i.3)$, now we verify that $\mathcal{C}_* \cap \mathcal{U}_c$ is a connected component of $\overline{\mathcal{S}_c^+}$ whose closure contains $\Big(b^*, \big(0,\tilde{v}\big)\Big)$ and $\Big(\frac{\bar{L}}{c^*}, \big(\tilde{u},0\big)\Big)$. By construction, we only need to show $\Big(\frac{\bar{L}}{c^*}, \big(\tilde{u},0\big)\Big) \in \overline{\mathcal{C}_* \cap \mathcal{U}_c}$ $\big(\subset \mathcal{C}_*^+\big)$. By the above reasoning,

$$\overline{\mathcal{C}_*^+ \cap \mathcal{U}_c} \subseteq \Big\{\Big(b, \big(u,v\big)\Big) \,:\, 0<b<M\Big\},$$

for some $M > 0$. Therefore, we may repeat the argument in $(i.3)$ to prove that $\mathcal{C}_*^+$ should satisfy alternative (i) in Theorem 2.2, and due to the boundedness of $\overline{\mathcal{S}_c^+}$ for $c = c^*$, the closure of $\mathcal{C}_*^+$ contains a point $\big(b', (u', v')\big)$ other than $\big(b^*, (0, \tilde{v})\big)$. By the maximum principle [23], either $u' = 0$ or $v' = 0$. Since $(0, 0)$ is linearly unstable and since $b = b^*$ is the unique bifurcation point at $(0, \tilde{v})$, we see $u' \neq 0$, and thereby $(u', v') = (\tilde{u}, 0)$. By Proposition 3.2 (ii), we further have $b' = \frac{\bar{L}}{c^*}$. Hence, $\big(\frac{\bar{L}}{c^*}, (\tilde{u}, 0)\big) \in \overline{\mathcal{C}_* \cap \mathcal{U}_c}$, as desired.

Finally, we finish statement (iii) by the following $(iii.1)$-$(iii.3)$.

$(iii.1)$: Unboundedness of $\overline{\mathcal{S}_c^+}$. It suffices to show that $\mathcal{C}_*^+$ is unbounded. Suppose to the contrary that it is bounded. If $\mathcal{C}_*^+$ satisfies alternative (i) in Theorem 2.2, then the closure of $\mathcal{C}_*^+$ should contain a point $\big(b_1, (u_1, v_1)\big) \in \partial\mathcal{U}_c$ other than $\big(b^*, (0, \tilde{v})\big)$. We claim that $b_1 > 0$, since $(0, 0)$ and $(0, \tilde{v})$ are linearly unstable for small $b_1 \geq 0$, and since $(\tilde{u}, 0)$ is linearly stable for $c > c^*$. Moreover, (u_1, v_1) cannot be a positive steady state due to Lemma 3.2 (i). Hence, there is a point $\big(b_1, (u_1, v_1)\big) \in \overline{\mathcal{C}_* \cap \mathcal{U}_c} \cap \partial\mathcal{U}_c$ different from $\big(b^*, (0, \tilde{v})\big)$ such that $\big(b_1, (u_1, v_1)\big)$ is a non-negative solution of system (2.2) with $b_1 > 0$, and either $u_1 = 0$ or $v_1 = 0$. Using the linear instability of $(0, 0)$, the linear stability of $(\tilde{u}, 0)$ and the fact that $b_1 = b^*$ is the unique bifurcation point at $(0, \tilde{v})$, one derives a contradiction. Having excluded alternative (i) and noting that alternative (ii) is also impossible, $\mathcal{C}_*^+$ must satisfy alternative (iii). Then, one can use the same idea as in step $(i.3)$ to derive a contradiction. Hence, for $c > c^*$, $\overline{\mathcal{S}_c^+}$ is unbounded.

$(iii.2)$: Validity of (4.4). The proof is very similar to the above $(i.2)$ or $(ii.2)$, and thus is omitted.

$(iii.3)$: We prove that $\mathcal{C}_* \cap \mathcal{U}_c$ is an unbounded connected component of $\overline{\mathcal{S}_c^+}$ whose closure contains $\big(b^*, (0, \tilde{v})\big)$. By construction, $\big(b^*, (0, \tilde{v})\big) \in \overline{\mathcal{C}_* \cap \mathcal{U}_c}$. It remains to show the unboundedness of the connected component $\mathcal{C}_* \cap \mathcal{U}_c$. By careful reading, step $(iii.1)$ indeed shows that $\mathcal{C}_*^+$ is unbounded in $U = (0, \infty) \times X_1$. Therefore, to finish the proof, we only need to verify $\mathcal{C}_* \cap \mathcal{U}_c = \mathcal{C}_*^+$. To this end, we turn to illustrate that there are no positive solutions in $\mathcal{C}_*^+$ converging to a point $\big(b_2, (u_2, v_2)\big) \in \partial\mathcal{U}_c$ with $u_2 = 0$ or $v_2 = 0$ and $\big(b_2, (u_2, v_2)\big) \neq \big(b^*, (0, \tilde{v})\big)$. Clearly, such $\big(b_2, (u_2, v_2)\big)$ should be one of

$$\big(b_2, (0, 0)\big), \quad \big(b_2, (\tilde{u}, 0)\big) \quad \text{and} \quad \big(b_2, (0, \tilde{v})\big),$$

for which, there are clearly no positive solutions converging to them, because $(0, 0)$ is linearly unstable, $(\tilde{u}, 0)$ is linearly stable and $(0, \tilde{v})$ is linearly unstable for small $b_2 > 0$. Therefore, the desired result follows. □

Remark 4.1. We make some discussion on the connection between Theorem 4.1 and the main results in [37], which may help readers get a better understanding. Theorem 4.1 (i), in particular, implies that positive steady states do exist for both small b and c, which biologically means that coexistence of two populations occurs when the competition is weak. Such a result is consistent with the observation in [37, Figure 2, R_1 and R_8] or [37, Figure 3, I]. Theorem 4.1 (ii), compared with (i), indicates that $c = c^*$ is a critical value such that the branch $\overline{\mathcal{S}_c^+}$ cannot be close to $b = 0$ anymore, which is in line with [37, Theorems 4(3) and 5(3)] (see also [37, Figures 2 and 3]). Theorem 4.1 (iii) shows that there are always positive steady states for the strong competition case. A tough issue is to determine further the bifurcation direction nearby the critical point (b^*, c^*), so that one may achieve a more clear understanding on the dynamics near the degenerate point (b^*, c^*). We leave it for future investigation.

5. Multiple solutions

In this subsection, we aim to investigate the multiple solutions phenomenon (abbreviated by "$\mathcal{MSP}$" in the sequel) for system (1.1)-(1.3).

First, we give a definition of "$\mathcal{MSP}$" as follows.

> Fix all parameters in system (1.1)-(1.3) except the competition coefficient b. We say "$\mathcal{MSP}$" occurs, if there are $b_0 > 0$ and $\big(u_i, v_i\big) \in X_1$ with $u_i, v_i > 0$ in $\overline{\Omega}$ $(i = 1, 2)$ and $\big(u_1, v_1\big) \neq \big(u_2, v_2\big)$ such that both $\Big(b_0, \big(u_1, v_1\big)\Big)$ and $\Big(b_0, \big(u_2, v_2\big)\Big)$ satisfy the elliptic system (2.2).

Recall the connected component $\mathcal{C}_* \cap \mathcal{U}_c$ of $\overline{\mathcal{S}_c^+}$ in the proof of Theorem 4.1. For notation brevity, we define

$$\Gamma_c := \mathcal{C}_* \cap \mathcal{U}_c.$$

Note that Γ_c makes sense for all $c \in (0, \infty)$, and is quite different for $c \in (0, c^*)$, $c = c^*$ and $c \in (c^*, \infty)$ in view of Theorem 4.1.

Now, we state the main result on "$\mathcal{MSP}$" as below.

Theorem 5.1. *Fix all parameters in system* (1.1)-(1.3) *except the competition coefficient b. Then, "$\mathcal{MSP}$" always happens in a small neighborhood of the point* (b^*, c^*).

Proof. Note that Γ_{c^*} is a connected component of $\overline{\mathcal{S}_c^+}$ that connects $\Big(b^*, \big(0, \tilde{v}\big)\Big)$ and $\Big(\frac{\bar{L}}{c^*}, \big(\tilde{u}, 0\big)\Big)$ (see Theorem 4.1 (ii)), where b^* and c^* are determined in Proposition 2.1, and $\bar{L}$ is defined in (2.10).

We proceed with this proof by considering the following three cases

$$(1)\ \frac{\bar{L}}{c^*} = b^*; \qquad (2)\ \frac{\bar{L}}{c^*} > b^*; \qquad (3)\ \frac{\bar{L}}{c^*} < b^*.$$

For case (1), if $\Gamma_{c^*} \subset \Big\{\Big(b, \big(u, v\big)\Big) \in U : b \equiv b^*\Big\}$, then fixing the parameter $b = b^*$, there is a branch of positive steady states connecting $\big(0, \tilde{v}\big)$ and $\big(\tilde{u}, 0\big)$, and thus "$\mathcal{MSP}$" occurs. If Γ_{c^*} contains a point $\Big(b^1, \big(u^1, v^1\big)\Big)$ with $b^1 > b^*$, due to the connectedness of Γ_{c^*} and $\frac{\bar{L}}{c^*} = b^*$, "$\mathcal{MSP}$" must occur for all $b \in (b^*, b^1)$. The same conclusion holds provided Γ_{c^*} contains a point $\Big(b, \big(u, v\big)\Big)$ with $b < b^*$. This finishes case (1).

For case (2), taking $c_n \nearrow c^*$, then using the property of Γ_{c^*}, one finds that Γ_{c_n} connects $\Big(b^*, \big(0, \tilde{v}\big)\Big)$ to $\Big(b^1, \big(u^1, v^1\big)\Big)$ $\Big(\approx \Big(\frac{\bar{L}}{c^*}, \big(\tilde{u}, 0\big)\Big)\Big)$, and then to $\Big(0, \big(\tilde{u}, v_c\big)\Big)$.

Since $b^* < b^1$ and $b^1 > 0$, we deduce that "$\mathcal{MSP}$" must occur for $b \in (b^*, b^1)$. This finishes case (2).

Case (3) can be dealt with in a similar way to case (2). Here, we only note that in a certain situation, one needs to consider Γ_{c_n} with $c_n \searrow c^*$ instead of the one in case (2) and also needs to use the property of Γ_{c_n} described in Theorem 4.1 (*iii*). $\square$

As a consequence of the above theorem, we have the following dynamics of system (1.1)-(1.3) nearby the degenerate point $\big(b, c\big) = \big(b^*, c^*\big)$ (where both semi-trivial steady states are neutrally stable).

Corollary 5.1. *Fix all parameters in system* (1.1)-(1.3) *except the competition coefficients b and c. Then, there exists a positive sequence $\Big\{\big(b_n, c_n\big)\Big\}_{n=1}^{\infty}$ with $\big(b_n, c_n\big) \to \big(b^*, c^*\big)$ such that for each $n > 0$, system* (1.1)-(1.3) *with $\big(b, c\big) = \big(b_n, c_n\big)$ has at least two different positive steady states. (note that it could happen that $\big(b_n, c_n\big) \equiv \big(b^*, c^*\big)$ for any $n > 0$.)*

Proof. By slightly modifying the above definition of "$\mathcal{MSP}$", we may define "$\mathcal{MSP}$" at a particular value of $\big(b, c\big)$ (two variables). In this sense, this corollary is equivalent to saying that "$\mathcal{MSP}$" always happens in a small neighborhood of $\big(b^*, c^*\big)$.

We prove this result by discussing the three cases in the proof of Theorem 5.1.

For case (1), if $\Gamma_{c^*} \subset \Big\{\Big(b, \big(u, v\big)\Big) \in U : b \equiv b^*\Big\}$, then "$\mathcal{MSP}$" occurs at a sequence of $\Big\{\big(b_n, c_n\big)\Big\}_{n=1}^{\infty}$ with $\big(b_n, c_n\big) \equiv \big(b^*, c^*\big)$, for any $n > 0$. If Γ_{c^*} contains a point $\Big(b, \big(u, v\big)\Big)$ with $b > b^*$, thanks to the connectedness of Γ_{c^*} and $\frac{\bar{L}}{c^*} = b^*$, Γ_{c^*} can decrease (in the value of b) continuously to two different points $\Big(b_1, \big(u_1, v_1\big)\Big)$ and $\Big(b_2, \big(u_2, v_2\big)\Big)$ with $b_1 = b_2 = b^*$ (note that u_i or v_i may be zero, $i = 1, 2$). This fact allows us to choose a sequence of $\Big\{\big(b_n, c_n\big)\Big\}_{n=1}^{\infty}$ with $b_n \searrow b^*$ as $n \to \infty$ and $c_n \equiv c^*$ for any $n > 0$ such that "$\mathcal{MSP}$" occurs at each $\big(b_n, c_n\big)$. If Γ_{c^*} contains a point $\Big(b, \big(u, v\big)\Big)$ with $b < b^*$, then we can choose a sequence of $\Big\{\big(b_n, c_n\big)\Big\}_{n=1}^{\infty}$ with $b_n \nearrow b^*$ as $n \to \infty$ and $c_n \equiv c^*$ for any $n > 0$ such that "$\mathcal{MSP}$" occurs at each $\big(b_n, c_n\big)$.

For cases (2) and (3), following the same idea as the above, one can demonstrate that "$\mathcal{MSP}$" always occurs along a sequence of $\Big\{\big(b_n, c_n\big)\Big\}_{n=1}^{\infty}$ with $b_n \to b^*$ and $c_n \to c^*$ as $n \to \infty$. The key point is the connectedness of Γ_{c^*} or Γ_{c_n} (when necessary). We leave the proof for the interested readers. $\square$

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