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Soliton and Periodic Wave Solutions of the Nonlinear Loaded (3+1)-Dimensional Version of the Benjamin-Ono Equation by Functional Variable Method*

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Abstract In this article, we establish new travelling wave solutions for the nonlinear loaded (3+1)-dimensional version of the Benjamin-Ono equation by the functional variable method. The performance of this method is reliable and effective and the method provides the exact solitary wave solutions and periodic wave solutions. The solution procedure is very simple and the traveling wave solutions are expressed by hyperbolic functions and trigonometric functions. After visualizing the graphs of the soliton solutions and the periodic wave solutions, the use of distinct values of random parameters is demonstrated to better understand their physical features. It has been shown that the method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics.

Keywords Nonlinear loaded Benjamin-Ono equation, solitary wave solutions, functional variable method, nonlinear evolution equations, periodic wave solutions, trigonometric function, hyperbolic function

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1. Introduction

Nonlinear partial differential equations are important equations used in modeling many phenomena in science and engineering applications. One of the most important nonlinear evolution equations is the Benjamin-Ono(BO) equation

$$u_{tt} + \alpha \left(u^2\right)_{xx} + \beta u_{xxxx} = 0, \tag{1.1}$$

where the constant coefficient α controls the nonlinearity and the characteristic speed of the long waves, the other constant β is the fluid depth, α, β are non-zero parameters and $u(x,t)$ is the elevation of the free surface of the fluid. In 1967, Benjamin examined a general theoretical treatment of a new class of long stationary waves with finite amplitude. Ono extended Benjamin's theory to obtain a class of

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nonlinear evolution equations in 1975. The BO equation describes internal waves between two stratified homogenous fluids with different densities, where one of the layers is infinitely deep [1–4]. The literature is rich in different studies aimed at discovering special solutions to the nonlinear BO equation, such as the existence of multi-soliton solution by [1, 5] and certain discrete solutions by Tutiya [6]. Many powerful and direct methods have been developed to find special solutions to nonlinear BO equations such as analytically method [7], exp-function method [8], numerical method [9, 10], Hirota bilinear method [11], extended truncated expansion method [12], generalization of the homoclinic breather method [13], tanh expansion method [14] and constructive method [15], which are used for searching the exact solutions.

In [16], (2+1)-dimensional extension of the BO equation is given as:

$$u_{tt} + \alpha \left(u^2\right)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \tag{1.2}$$

where α, β and γ are non zero constants and $u(x, y, t)$ is the elevation of the free surface of the fluid. The multiple soliton solutions and multiple complex solitons solutions to (1.2) were taken using the simplified form of the Hirota's method [16]. The exact solutions have great importance in revealing the internal mechanism of the physical phenomena.

In this paper, we consider the following nonlinear loaded (3+1)-dimensional version of the BO equation with variable coefficient

$$u_{tt} - \alpha \left(u^2\right)_{xx} - \beta u_{xxxx} - \gamma u_{yyyy} - \lambda u_{zzzz} + \varphi(t)u(0, 0, 0, t)u_{xx} = 0, \tag{1.3}$$

where $\varphi(t)$ is the given real continuous function, $x \in R$, $y \in R$, $z \in R$, $t \geq 0$, α, β, γ and λ are arbitrary constants, and $u(x, y, z, t)$ is the elevation of the free surface of the fluid. Equation (1.3) is an extension form of (1.2), which contains an additional term with a variable coefficient. As in equation (1.2), equation (1.3) can be used to describe long internal gravity waves in deeply stratified fluids.

Nowadays due to intensive research on optimal management of the agroecosystem, there has been a significant increase in interest in loaded equations. For example, one area of focus is long-term forecasting and regulation of groundwater and soil moisture levels. Among the works devoted to loaded equations, one should especially note the works of A. Kneser [17], L. Lichtenstein [18], A. M. Nakhushev [19, 20], and others. It is known that the loaded differential equations contain some of the traces of an unknown function. In [19, 20], the term of loaded equation was used for the first time, the most general definitions of the loaded differential equation were given and also a detailed classification of the differential loaded equations as well as their numerous applications was presented. A complete description of solutions to the nonlinear loaded equations and their applications can be found in papers [21–29].

In this study we construct exact travelling wave solutions of the nonlinear loaded (3+1)-dimensional version of the BO equation (1.3). The functional variable method is used to obtain exact solutions for these equations, including solitary wave solutions and periodic wave solutions. The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions. The solution procedure is very simple and the traveling wave solutions are expressed by hyperbolic functions and trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features. It has been shown that the method provides a very effective and powerful mathematical tool for solving nonlinear equations in mathematical physics.

The article is organized as follows. In Section 2, we present some basic information about the description of the functional variable method. Section 3 is devoted to solutions of equation (1.3). In Section 4, we present the graphical representation to equation (1.3). Finally, conclusions are presented in Section 5.

2. Description of the functional variable method

We consider NPDE of the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{tt}, u_{yy}, u_{xy}, u_{xt}, u_{yt}, ...) = 0, \tag{2.1}$$

where P is a polynomial in $u = u(x, y, t)$ and its partial derivatives.

Step 1. The following transformation is used for a travelling wave solution of (2.1):

$$u(x, y, t) = u(\xi),\ \xi = px + qy - kt,\ p = const,\ q = const,\ k = const, \tag{2.2}$$

with

$$\frac{\partial u}{\partial x} = p\frac{du}{d\xi}, \frac{\partial u}{\partial y} = q\frac{du}{d\xi}, \frac{\partial u}{\partial t} = -k\frac{du}{d\xi}, ..., \tag{2.3}$$

where k is the speed of the traveling wave.

Substituting Eq.(2.2) and (2.3) into NPDE (2.1) we get the following ODE of the form

$$F(u,\, u', u'', u''', ...) = 0, \qquad u' = \frac{du}{d\xi}, \tag{2.4}$$

where F is a polynomial in $u(\xi), u'(\xi), u''(\xi), u'''(\xi), ...$.

Step 2. Let

$$u' = F(u). \tag{2.5}$$

It follows that

$$\int \frac{du}{F(u)} = \xi + \xi_0. \tag{2.6}$$

We suppose that $\xi_0 = 0$ for convenience. Now we can calculate higher order derivatives of u:

$$\begin{aligned} &u'' = \tfrac{dF(u)}{du}\tfrac{du}{d\xi} = \tfrac{dF(u)}{du}F(u) = \tfrac{1}{2}\tfrac{d\left(F^2(u)\right)}{du}, \\ &u''' = \tfrac{1}{2}\tfrac{d^2\left(F^2(u)\right)}{du^2}\sqrt{F^2(u)}, \\ &u^{(IV)} = \tfrac{1}{2}\left[\tfrac{d^3\left(F^2(u)\right)}{du^3}F^2(u) + \tfrac{d^2\left(F^2(u)\right)}{du^2}\tfrac{d\left(F^2(u)\right)}{du}\right], \\ &... \,. \end{aligned} \tag{2.7}$$

Step 3. Putting Eq.(2.7) into Eq.(2.4), we obtain

$$H(u, \frac{dF(u)}{du}, \frac{d^2F(u)}{du^2}, \frac{d^3F(u)}{du^3}, ...) = 0. \tag{2.8}$$

The key idea of this particular form Eq.(2.8) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq.(2.8) provides the expression of F and this, together with Eq.(2.5), gives solutions to the original problem.

3. Solutions of the nonlinear loaded Benjamin-Ono equation

We will find the exact solution to the equation (1.3) by the functional variable method. For doing this, in equation (1.3), let's use the following transformation.

$$u(x,y,z,t)=u(\xi),\ \xi=px+qy+mz-kt, \tag{3.1}$$

where $p=const, q=const$, $m=const$, and k is the speed of the traveling wave.

It is easy to show that after transformation (3.1), the nonlinear partial differential Eq.(1.3) can be transformed into an ordinary differential equation of the form

$$u''=\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}u-\frac{\alpha p^2}{\beta p^4+\gamma q^4+\lambda m^4}u^2. \tag{3.2}$$

According to (2.7) Eq.(3.2) can be written as follows

$$\frac{1}{2}\frac{d\left(F^2(u)\right)}{du}=\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}u-\frac{\alpha p^2}{\beta p^4+\gamma q^4+\lambda m^4}u^2. \tag{3.3}$$

Integrating Eq.(3.3) and by simplifying, we get

$$F(u)=u\sqrt{\mu(t)-\eta u}, \tag{3.4}$$

where $\mu(t)=\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}$ and $\eta=\frac{2\alpha p^2}{3(\beta p^4+\gamma q^4+\lambda m^4)}$.

From Eq.(2.5) and Eq.(3.4) we deduce that

$$\frac{du}{u\sqrt{\mu(t)-\eta u}}=d\xi, \tag{3.5}$$

$$\sqrt{\mu(t)-\eta u}=z,\quad dz=-\frac{\eta du}{2\sqrt{\mu(t)-\eta u}}=-\frac{\eta du}{2z}, \tag{3.6}$$

$$\frac{du}{u\sqrt{\mu(t)-\eta u}}=\frac{2zdz}{(z^2-\mu(t))z}=\frac{2dz}{z^2-\mu(t)}, \tag{3.7}$$

$$\frac{2dz}{z^2-\mu(t)}=d\xi. \tag{3.8}$$

After integrating Eq.(3.8), we have

$$\frac{1}{\sqrt{\mu(t)}}\ln\frac{z-\sqrt{\mu(t)}}{z+\sqrt{\mu(t)}}=\xi. \tag{3.9}$$

Finally, we get the following expression

$$u(x,y,z,t)=-\frac{6\left(k^2+\varphi(t)u(0,0,0,t)p^2\right)}{\alpha p^2}\frac{e^{\sqrt{\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}}(px+qy+mz-kt)}}{\left(1-e^{\sqrt{\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}}(px+qy+mz-kt)}\right)^2}. \tag{3.10}$$

The function $u(0,0,0,t)$ can be easily obtained based on expression (3.10).

We get two types of solutions to the equation (1.3) as follows:

1) When $\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4} > 0$, we get the solitary solution

$$u(x,y,z,t) = -\frac{6\left(k^2+\varphi(t)u(0,0,0,t)p^2\right)}{\alpha p^2}\left(cth^2\left(w(t)\right)-1\right), \tag{3.11}$$

where $w(t) = \sqrt{\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}}\frac{px+qy+mz-kt}{2}$, $cthz = \frac{e^z+e^{-z}}{e^z-e^{-z}}$.

2) When $\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4} < 0$, we get the periodic solution

$$u(x,y,z,t) = \frac{6\left(k^2+\varphi(t)u(0,0,0,t)p^2\right)}{\alpha p^2}\left(ctg^2\left(w(t)\right)+1\right), \tag{3.12}$$

where $w(t) = \sqrt{\frac{k^2+\varphi(t)u(0,0,0,t)p^2}{\beta p^4+\gamma q^4+\lambda m^4}}\frac{px+qy+mz-kt}{2}$, $ctgz = \frac{cosz}{sinz}$.

The graphs of solutions to the equation (1.3) by using distinct values of random parameter will be demonstrated.

If $k=-1,\ \alpha=6,\ \beta=1,\ \gamma=1,\ \varphi(t)=-t^2,\ p=1,\ q=1,\ \lambda=-1,$ and $m=1$, then we have

$$u(x,y,z,t) = -\frac{5}{t^2+1}\left(cth^2\left(\sqrt{\frac{5}{t^2+1}}\frac{x+y+z+t}{2}\right)-1\right). \tag{3.13}$$

If $k=-1,\ \alpha=6,\ \beta=-1,\ \gamma=1,\ \ \varphi(t)=-t^2,\ p=1,\ q=1,\ \lambda=-1,$ and $m=1$, then we have

$$u(x,y,z,t) = -\frac{3}{t^2+1}\left(ctg^2\left(\sqrt{\frac{3}{t^2+1}}\frac{x+y+z+t}{2}\right)+1\right). \tag{3.14}$$

4. Graphical representation of the nonlinear loaded Benjamin-Ono equation

We have presented some graphs of solitary and periodic waves of equation (1.3) constructed by taking suitable values of the unknown parameters involved, in order to visualize the underlying mechanism to the original physical phenomena. Graphical representation is an effective tool for communication and it exemplifies evidently the solutions of the problems. The graphical illustrations of the solutions are depicted in Figure 1 and Figure 2. Solitary and periodic wave solutions represent an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to possess a variety of solitary wave solutions. Solitons are solutions to a common class of nonlinearly partially differential equations with weak linearity describing physical systems. The existence of periodic travelling waves usually depends on the parameter values in a mathematical equation. The amplitude and velocities are controlled by parameters of various kind. The soliton is a self-reinforcing wave packet maintaining its shape while propagating at a constant velocity.

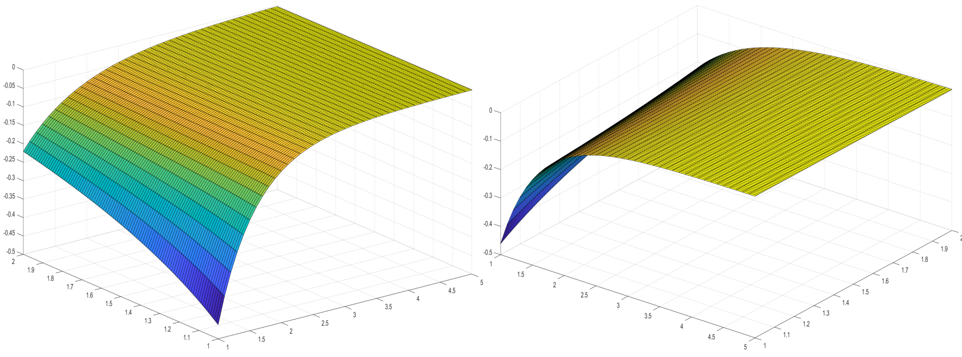

Figure 1. Solitary wave solution to the equation (1.3) for $k = -1,\ \alpha = 6,\ \beta = 1,\ \gamma = 1,\ \varphi(t) = -t^2$, $p = 1,\ q = 1,\ \lambda = -1,\ \ m = 1,\ x \in [1, 5],\ t \in [1, 2]$.

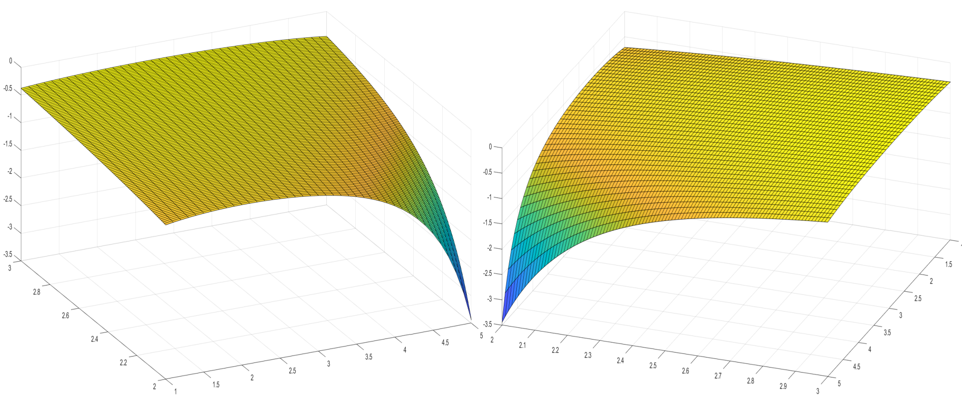

Figure 2. Periodic wave solution to the equation (1.3) for $k = -1,\ \alpha = 6,\ \beta = -1,\ \gamma = 1,\ \ \varphi(t) = -t^2$, $p = 1,\ q = 1,\ \lambda = -1,\ \ m = 1,\ x \in [1, 5],\ t \in [2, 3]$.

5. Conclusion

The functional variable method has been successfully used to obtain several traveling wave solutions of the nonlinear loaded (3+1)-dimensional version of the BO equation. We have shown that, this method can provide a useful way to efficiently find the exact structures of solutions to a variety of nonlinear wave equations. After visualizing the graphs of the soliton solutions and the periodic solutions, distinct values of random parameter are demonstrated to better understand their physical features. The advantage of the method is giving more solution functions such as periodic solutions and hyperbolic solutions than other popular analytical methods. We conclude that the exact solutions are of great importance in revealing the internal mechanism of the physical phenomena.

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