Numer. Math. Theor. Meth. Appl. Vol. **10**, No. 2, pp. 222-242
doi: 10.4208/nmtma.2017.s02 May 2017

# Deferred Correction Methods for Forward Backward Stochastic Differential Equations

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Received 5 November 2016; Accepted (in revised version) 10 January 2017

Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

**Abstract.** The deferred correction (DC) method is a classical method for solving ordinary differential equations; one of its key features is to iteratively use lower order numerical methods so that high-order numerical scheme can be obtained. The main advantage of the DC approach is its simplicity and robustness. In this paper, the DC idea will be adopted to solve forward backward stochastic differential equations (FBSDEs) which have practical importance in many applications. Noted that it is difficult to design high-order and relatively "clean" numerical schemes for FBSDEs due to the involvement of randomness and the coupling of the FSDEs and BSDEs. This paper will describe how to use the simplest Euler method in each DC step–leading to simple computational complexity–to achieve high order rate of convergence.

**AMS subject classifications**: 60H35, 65C20, 60H10
**Key words**: Deferred correction method, forward backward stochastic differential equations, Euler method, high-order scheme.

## 1. Introduction

This work is concerned with the forward-backward stochastic differential equations (FBSDEs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s, X_s, Y_s, Z_s)\mathrm{d}t + \int_0^t \sigma(s, X_s, Y_s, Z_s)\mathrm{d}W_s, & \text{(FSDE)} \\ Y_t = \xi + \displaystyle\int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, & \text{(BSDE)} \end{cases} \tag{1.1}$$

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where $t \in [0,T]$ with $T > 0$ being the deterministic terminal time; $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered complete probability space with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ being the natural filtration of the standard $m$-dimensional Brownian motion $W = (W_t)_{0\le t\le T}$; $X_0 \in \mathscr{F}_0$ is the initial condition for the forward stochastic differential equation (FSDE); $\xi \in \mathscr{F}_T$ is the terminal condition for the backward stochastic differential equation (BSDE); $b : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m} \to \mathbb{R}^d$ and $\sigma : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m} \to \mathbb{R}^{d\times m}$ are referred to the drift and diffusion coefficients, respectively; $f : \Omega \times [0.T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times m} \to \mathbb{R}^q$ is called the generator of BSDE, and $(X_t, Y_t, Z_t) : [0,T] \times \Omega \to \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times m}$ is the unknown.

We point out that $b(\cdot, x, y, z)$, $\sigma(\cdot, x, y, z)$, and $f(\cdot, x, y, z)$ are all $\mathcal{F}_t$-adapted for any fixed numbers $x$, $y$, and $z$, and that the two stochastic integrals with respect to $W_s$ are of the Itô type. A triple $(X_t, Y_t, Z_t)$ is called an $L^2$-adapted solution of the FBSDEs (1.1) if it is $\mathscr{F}_t$-adapted, square integrable, and satisfies (1.1). The FBSDEs (1.1) are called *decoupled* if $b$ and $\sigma$ are independent of $Y_t$ and $Z_t$.

Our interest is to design numerical schemes which can effectively find numerical solutions of the FBSDEs (1.1). Great efforts have been made to the numerical solutions of BSDEs, see, e.g., [1, 3, 8, 17, 19, 20]. However, for the fully coupled FBSDEs (1.1), there exist only few numerical studies and satisfactory results [6, 11, 18]. In fact, it is very difficult to design high-order and relatively "clean" numerical schemes for FBSDEs due to the fully coupling of the FSDEs and BSDEs. We mention the work in [18], where a class of multi-step type schemes are proposed, which turns out to be effective in obtaining relatively high accurate solutions for (1.1).

In this paper, we will approximate the solutions of the FBSDEs (1.1) based on the classical deferred correction (DC) method. The DC approach was first introduced in [16] to solve ordinary differential equations (ODEs). Its main idea is to use some low-order and simple schemes iteratively to achieve a high-order scheme. The terminology of *deffered correction* was formally appeared in [15], while its convergence theory for ODEs was established by Hairer [9]. In the past few decades, DC methods have been successfully applied to solve ODEs, see, e.g., [5, 10, 14] as well as partial differential equations (PDEs), see, e.g., [4, 7]. Our main task in this work is to design highly accurate numerical methods for the fully coupled FBSDEs based on the DC approach. More precisely, we will describe how to use the simplest Euler method in each iteration step–leading to lower overall computational complexity–to end up with high-order of convergence. Moreover, numerical experiments will demonstrate that the resulting DC-based scheme is highly accurate and stable.

The rest of the paper is organized as follows. Section 2 provides some relevant preliminaries, while Section 3 presents the general framework of the deferred correction methods for FBSDEs. More detailed construction of the DC-based algorithms for decoupled and coupled FBSDEs are discussed in Sections 4. In Section 5, several numerical experiments are presented to demonstrate the effectiveness of the proposed scheme. Some concluding remarks will be given in Section 6.

Some notation to be used:

- $|\cdot|$: the Euclidean norm in the Euclidean space $\mathbb{R}$, $\mathbb{R}^d$, and $\mathbb{R}^{d\times m}$;
- $A^\top$: the transpose of vector or matrix $A$;
- $\mathcal{F}_s^{t,x}$: the $\sigma$-algebra generated by the diffusion process $\{X_r, t \leq r \leq s, X_t = x\}$;
- $\mathbb{E}_s^{t,x}[\eta]$: the conditional expectation of the random variable $\eta$ under $\mathcal{F}_s^{t,x}$, i.e., $\mathbb{E}_s^{t,x}[\eta] = \mathbb{E}\left[\eta | \mathcal{F}_s^{t,x}\right]$, and we use $\mathbb{E}_t^x[\eta]$ to denote $\mathbb{E}_t^{t,x}[\eta]$ for simplicity;
- $C_b^k$: the set of functions $\phi(x)$ with uniformly bounded derivatives up to the order $k$;
- $C^{k_1,k_2}$: the set of functions $f(t,x)$ with continuous partial derivatives up to $k_1$ with respect to $t$, and up to $k_2$ with respect to $x$.

## 2. Preliminaries

This section is devoted to some preliminaries. We will begin by introducing the diffusion process generator and the Feynman-Kac formula. Moreover, a useful lemma in designing our DC-based schemes will be presented.

### 2.1. The diffusion process generator

A stochastic process $X_s$ is called a diffusion process starting at $(t,x)$ if it satisfies the SDE

$$X_s = x + \int_t^s b_r \mathrm{d}r + \int_t^s \sigma_r \mathrm{d}W_r, \quad s \in [t,T], \tag{2.1}$$

where $b_r = b(r,X_r)$ and $\sigma_r = \sigma(r,X_r)$ are measurable functions satisfying the linear growth and Lipschits continuous conditions, i.e.,

$$\begin{aligned} &|b(r,x)| + |\sigma(r,x)| \leq C(1+|x|), && x \in \mathbb{R}^d, \quad r \in [t_0,T], \\ &|b(r,x) - b(r,y)| + |\sigma(r,x) - \sigma(r,y)| \leq L\,|x-y|\,, && x,y \in \mathbb{R}^d, \quad r \in [t_0,T]. \end{aligned}$$

It is well known that under the above conditions the SDE (2.1) has a unique solution. Moreover, it follows from the Markov property of the diffusion process that

$$\mathbb{E}_t^x[X_s] = \mathbb{E}[X_s | X_t = x], \quad \forall t \leq s. \tag{2.2}$$

**Definition 2.1.** *Let $X_t$ be a diffusion process in $\mathbb{R}^d$ satisfying* (2.1). *The generator $A_t^x$ of $X_t$ on $g : [0,T] \times \mathbb{R}^d$ is defined by*

$$A_t^x g(t,x) = \lim_{s \downarrow t} \frac{\mathbb{E}_t^x[g(s,X_s)] - g(t,x)}{s-t}, \quad x \in \mathbb{R}^d, \tag{2.3}$$

*if the limit exists.*

We further define the second-order differential operator $L^0_{t,x}$, by

$$L^0_{t,x} = \frac{\partial}{\partial t} + \sum_i b_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}\left(\sigma\sigma^\top\right)_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.4}$$

Then, if $g \in C^{1,2}([0,T]\times\mathbb{R}^d)$, we have $A^x_t g(t,x) = L^0_{t,x}g(t,x)$.

By Definition 2.1, together with the Itô formula and the tower rule of conditional expectations, we have the following theorem [18]:

**Theorem 2.1.** *Let $t \le s$ be a fixed time, and $x_0 \in \mathbb{R}^d$ be a fixed space point. If*

$$f \in C^{1,2}([0,T]\times\mathbb{R}^d) \quad \text{and} \quad \mathbb{E}^x_t\left[\left|L^0_{t,X_t}f(t,X_t)\right|\right] < +\infty,$$

*then we have*

$$\frac{\mathrm{d}\mathbb{E}^x_t\left[f(s,X_s)\right]}{\mathrm{d}s} = \mathbb{E}^x_t\left[A^{X_s}_s f(s,X_s)\right], \quad s \ge t. \tag{2.5}$$

*Moreover, the following identity holds*

$$\left.\frac{\mathrm{d}\mathbb{E}^x_t\left[f(s,X_s)\right]}{\mathrm{d}s}\right|_{s=t} = \left.\frac{\mathrm{d}\mathbb{E}^x_t\left[f(s,\bar{X}_s)\right]}{\mathrm{d}s}\right|_{s=t}, \tag{2.6}$$

*where $\bar{X}_s$ is a diffusion process satisfying*

$$\bar{X}_s = x + \int_t^s \bar{b}_r \mathrm{d}r + \int_t^s \bar{\sigma}_r \mathrm{d}W_r, \tag{2.7}$$

*and $\bar{b}_r = \bar{b}(r,\bar{X}_r)$, $\bar{\sigma}_r = \bar{\sigma}(r,\bar{X}_r)$ are smooth functions of $(r,\bar{X}_r)$ that satisfy*

$$\bar{b}(t,\bar{X}_t) = b(t,x), \quad \bar{\sigma}(t,\bar{X}_t) = \sigma(t,x).$$

Note that by choosing different $\bar{b}_r$ and $\bar{\sigma}_r$, the identity (2.6) yields different ways for approximating $\frac{\mathrm{d}\mathbb{E}^x_t[f(s,X_s)]}{\mathrm{d}s}\big|_{s=t}$. The computational complexity can be significantly reduced if appropriate $\bar{b}_s$ and $\bar{\sigma}_s$ are chosen. For example, one can simply choose $\bar{b}(r,\bar{X}_r) = b(t,x)$ and $\bar{\sigma}(r,X_r) = \sigma(t,x)$ for all $r \in [t,s]$.

## 2.2. The non-linear Feynman-Kac formula and a useful lemma

Consider the following decoupled FBSDEs,

$$\begin{cases} X^{t,x}_s = x + \int_t^s b(r,X^{t,x}_r)\mathrm{d}r + \int_t^s \sigma(t,X^{t,x}_r)\mathrm{d}W_r, \\ Y^{t,x}_s = \varphi(X^{t,x}_T) + \int_s^T f(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)\mathrm{d}r - \int_s^T Z^{t,x}_r \mathrm{d}W_r, \end{cases} \quad \forall s \in [t,T]. \tag{2.8}$$

Here the superscript ${}^{t,x}$ indicates that the forward SDE starts from $(t,x)$, which will be omitted if no ambiguity arises. The existence and uniqueness of the solution to (2.8) was first addressed by Pardoux and Peng [12]. Moreover, it is shown [13] that

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad Z_s^{t,x} = \nabla u(s, X_s^{t,x})\sigma(s, X_s^{t,x}), \quad \forall s \in [t,T), \tag{2.9}$$

with $u(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^d)$ being the classical solution of the following Cauchy problem:

$$L_{t,x}^0 u(t,x) + f(t,x,u(t,x), \qquad \nabla u(t,x)\sigma(t,x)) = 0, \tag{2.10a}$$
$$u(T,x) = \varphi(x), \qquad t \in [0,T]. \tag{2.10b}$$

The representation in (2.9) is the so-called nonlinear Feynman-Kac formula.

Note that (2.9) provides a precise link between the solutions of FBSDEs and those of semi-linear PDEs. Motivated by the solution structure of decoupled FBSDEs, we introduce the following lemma which will play an important role in designing our DC schemes.

**Lemma 2.1.** *Suppose that the function* $g(t,X_t) \in C^{1,2}([0,T] \times \mathbb{R}^d)$, *where* $X_t$ *is the solution of the diffusion process in* (2.8). *Then under the conditions of Theorem 2.1, it holds that*

$$\frac{\mathrm{d}\mathbb{E}_t^x[g(s,X_s)]}{\mathrm{d}s}\bigg|_{s=t} = L_{t,x}^0 g(t,x), \tag{2.11a}$$
$$\frac{\mathrm{d}\mathbb{E}_t^x\left[g(s,X_s)(\Delta W_{t,s})^\top\right]}{\mathrm{d}s}\bigg|_{s=t} = \nabla g(t,x)\sigma(t,x), \tag{2.11b}$$

*where the increment* $\Delta W_{t,s}$ *is defined as* $\Delta W_{t,s} := W_s - W_t$.

*Proof.* It follows from Theorem 2.1 that

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}_t^x[g(s,X_s)]}{\mathrm{d}s} =& \mathbb{E}_t^x\left[A_s^{X_s} g(s,X_s)\right] = \mathbb{E}_t^x\left[L_{s,X_s}^0 g(s,X_s)\right] \\ =& \mathbb{E}_t^x\left[\frac{\partial}{\partial s} g(s,X_s) + \sum_i^d b_i \frac{\partial}{\partial x_i} g(s,X_s) + \frac{1}{2}\sum_{i,j}^d \left(\sigma\sigma^\top\right)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} g(s,X_s)\right].\end{aligned}$$

Then, under the condition of the lemma, (2.11a) follows by letting $s \to t$. We choose the functions $\bar{b}(s,\bar{X}_s)$ and $\bar{\sigma}(s,\bar{X}_s)$ in (2.7) as $\bar{b}(s,\bar{X}_s) = b(t,x)$ and $\bar{\sigma}(s,\bar{X}_s) = \sigma(t,x)$. Consequently, the diffusion process $\bar{X}_s$ can be presented as

$$\bar{X}_s = x + b(t,x)\Delta t_{t,s} + \sigma(t,x)\Delta W_{t,s},$$

where $\Delta t_{t,s} = s - t$ and $\Delta W_{t,s} = W_s - W_t$ for $t \le s$. By Theorem 2.1, we have

$$\frac{\mathrm{d}\mathbb{E}_t^x\left[g(s,X_s)(\Delta W_{t,s})^\top\right]}{\mathrm{d}s}\bigg|_{s=t} = \frac{\mathrm{d}\mathbb{E}_t^x\left[g(s,\bar{X}_s)(\Delta W_{t,s})^\top\right]}{\mathrm{d}s}\bigg|_{s=t}.$$

Let $\bar{g} = g(t, \bar{X})$, i.e.,

$$\bar{g} = g(s, x + b(t,x)\Delta t_{t,s} + \sigma(t,x)\Delta W_{t,s}).$$

Such a function $\bar{g}$ can be viewed as a function of $s$ and $\Delta W_{t,s}$, and $g(s, \bar{X}_s)(\Delta W_{t,s})^\top$ too. We denote $\bar{g} = g(s, x + b(t,x)\Delta t_{t,s} + \sigma(t,x)\Delta W_{t,s})$ by $G(s, \Delta W_{t,s})$. By Theorem 2.1, we deduce

$$\begin{aligned}
&\frac{\mathrm{d}\mathbb{E}_t^x\left[G(s,\Delta W_{t,s})\right]}{\mathrm{d}s} = \mathbb{E}_t^x\left[L^0_{s,\Delta W_{t,s}} G(s,\Delta W_{t,s})\right] \\
=&\mathbb{E}_t^x\left[\frac{\partial G}{\partial s}(s,\Delta W_{t,s}) + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(s,\Delta W_{t,s})\right] \\
=&\mathbb{E}_t^x\left[\frac{\partial \bar{g}}{\partial s}\Delta W_{t,s} + \frac{1}{2}\left(\sum_{i,j}^{d}(\sigma\sigma^\top)_{i,j}(t,x)\frac{\partial^2 \bar{g}}{\partial x_i \partial x_j}\Delta W_{t,s} + 2\nabla\bar{g}(t,x)\sigma(t,x)\right)\right],
\end{aligned}$$

which gives

$$\frac{\mathrm{d}\mathbb{E}_t^x[G(s,\Delta W_{t,s})]}{\mathrm{d}s}\bigg|_{s=t} = \nabla\bar{g}(t,x)\sigma(t,x) = \nabla g(t,x)\sigma(t,x).$$

This completes the proof of the lemma. □

## 3. The framework of DC method

### 3.1. The DC framework for ODEs

The idea of deferred correction (DC) method was first proposed in [15] for solving the following ODE problems

$$\begin{cases} y'(t) = f(t, y(t)), & t \in (0, T], \\ y(0) = y_0. \end{cases} \tag{3.1}$$

It aims to create high-order methods from low-order schemes. More precisely, the DC methods begin with a low-order scheme (such as the Euler scheme) and then promote it to a higher-order one by iteratively corrected numerical solutions of residual equations.

We simply give the DC procedure for solving ODEs as follows. First, introduce a regular time partition for $[0, T]$ as

$$0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T, \tag{3.2}$$

and a finer uniform partition $\mathbb{G}_K^n$ on the time sub-interval $I_n = [t_n, t_{n+1}]$

$$\mathbb{G}_K^n = \{t_{n,k} | t_n = t_{n,0} < t_{n,1} < \cdots < t_{n,k} < \cdots < t_{n,K} = t_{n+1}\} \tag{3.3}$$

with the time sub-step $\delta t = (t_{n+1} - t_n)/K$, where $K$ is a given positive integer. Let $I_{n,k} = [t_{n,k}, t_{n,k+1}]$.

Second, let $\{u^{n,k}\}_{k=0}^{K}$ be the approximated values of the solution $y(t)$ of (3.1) at time points $\{t_{n,k}\}_{k=0}^{K} \in \mathbb{G}_K^n$, which are obtained by using a low-order numerical scheme; based on the discrete values $\{u^{n,k}\}_{k=0}^{K}$, construct a continuous interpolation function $Iu(t)$; solve the *residual equation*

$$\delta'(t) = f(t, \delta(t) + Iu(t)) - \frac{\mathrm{d}}{\mathrm{d}t} Iu(t) \tag{3.4}$$

with $\delta(0) = 0$, where $\delta(t) = y(t) - Iu(t)$ is the error function. Note that this residual equation is of the same form as (3.1), so the same numerical scheme for (3.1) can be used to solve (3.4). This will yield approximated values $\{\delta^k\}_{i=0}^{K}$. Third, correct the approximation solution $u^{n,k}$ by $u^{n,k,\text{new}} = u^{n,k} + \delta^k$, $k = 0, 1, \cdots, K$. The above procedure can be repeated for $J$ times, where $J$ is a positive integer, and the rate of convergence is given by (see [9])

$$\mathcal{O}\left((\delta t)^{\min(J,K)+1}\right). \tag{3.5}$$

To summarize, we write the DC method for ODEs in Algorithm 3.1 below.

**Algorithm 3.1** (DC for ODEs).

1 Let $u^n = y_0$, for $n = 0$.

2 For $n = 1, 2, \cdots, N-1$, do (1)-(3).

(1) Let $u^{n,0} = u^{n-1}$.

(2) For $j = 1, 2, \cdots, J$, do (i)-(iii).

(i) For $k = 1, 2, \cdots, K$, solve $u^{n,k,[j]}$ by a lower-order numerical method at time points $t_{n,k} \in \mathbb{G}_K^n$.

(ii) Let $\delta^{0,[j]} = 0$. For $k = 1, 2, \cdots, K$, solve $\delta^{k,[j]}$ by the same lower-order method at time points $t_{n,k} \in \mathbb{G}_K^n$.

(iii) Update the numerical solutions $u^{n,k,[j]}$, $k = 1, 2, \cdots, K$, by

$$u^{n,k,[j+1]} = u^{n,k,[j]} + \delta^{k,[j]}.$$

(3) Let $u^n = u^{n,K,[J]}$.

### 3.2. The DC framework for FBSDEs

In our DC method for FBSDEs, we need to introduce a space partition $D_h$ of $\mathbb{R}^d$, which is a set of discrete grid points in $\mathbb{R}^d$, i.e., $D_h = \{x_i | x_i \in \mathbb{R}^d, i \in \mathbb{Z}\}$ with $\mathbb{Z}$ the set of integers. We define the density of the grids in $D_h$ by

$$h = \max_{x\in\mathbb{R}^d} \min_{y\in D_h} |x - y| = \max_{x\in\mathbb{R}^d} \text{dist}(x, D_h), \tag{3.6}$$

where $\text{dist}(x, D_h)$ is the distance from $x$ to $D_h$. For each $x \in \mathbb{R}^d$, we define a local subset $D_{h,x}$ of $D_h$ satisfying

- $\text{dist}(x, D_{h,x}) < \text{dist}(x, D_h \backslash D_{h,x})$;
- the number of elements in $D_{h,x}$ is finite and uniformly bounded, i.e., there exists a positive integer $\mathcal{N}_e$ such that $\# D_{h,x} \leq \mathcal{N}_e$.

We call $D_{h,x}$ the neighbor grid set in $D_h$ at $x$.

On the time partitions (3.2) and (3.3) and the space partition $D_h$, we will give a framework of our DC method for solving FBSDEs. Consider the backward stochastic differential equation on $I_n = [t_n, t_{n+1}] \subset [0, T]$

$$Y_t = Y_{t_{n+1}} + \int_t^{t_{n+1}} f(s, X_s, Y_s, Z_s) \text{d}s - \int_t^{t_{n+1}} Z_s \text{d}W_s, \quad t \in I_n. \tag{3.7}$$

Suppose that we have obtained a numerical approximation $(Y_i^{n,k}, Z_i^{n,k})$ of the solution $(Y_t, Z_t)$ of the BSDE (3.7) at time-space grid points $(t_{n,k}, X_{t_{n,k}} = x_i)$, $i \in \mathbb{Z}$, by a low-order numerical method (denoted as $M_l$ method). Based on these values $(Y_i^{n,k}, Z_i^{n,k})$, we can construct an interpolation approximate $(I_h Y_t, I_h Z_t)$ of $(Y_t, Z_t)$ for $t \in I_n$. Define the error terms $\delta Y_t$ and $\delta Z_t$ as

$$\delta Y_t = Y_t - I_h Y_t, \qquad \delta Z_t = Z_t - I_h Z_t. \tag{3.8}$$

It follows (3.7) and (3.8) that the processes $\delta Y_t$ and $\delta Z_t$ solve the following BSDE:

$$\delta Y_t = \delta Y_{t_{n+1}} + \int_t^{t_{n+1}} F(s, X_s, \delta Y_s, \delta Z_s) \text{d}s - \int_t^{t_{n+1}} \delta Z_s \text{d}W_s + \mathcal{E}(t), \tag{3.9}$$

where

$$\begin{aligned} F(s, X_s, \delta Y_s, \delta Z_s) &= f(s, X_s, \delta Y_s + I_h Y_s, \delta Z_s + I_h Z_s), \\ \mathcal{E}(t) &= I_h Y_{t_{n+1}} - \int_t^{t_{n+1}} I_h Z_s \text{d}W_s - I_h Y_t. \end{aligned}$$

After getting the approximated values $(\delta Y_i^{n,k}, \delta Z_i^{n,k})$ of $(\delta Y_t, \delta Z_t)$ at the grid points $(t_{n,k}, X_{t_{n,k}} = x_i)$, $i \in \mathbb{Z}$ by the method $M_l$ for (3.7), we update the approximated solutions $(Y_i^{n,k,\text{new}}, Z_i^{n,k,\text{new}})$ by

$$(Y_i^{n,k,\text{new}}, Z_i^{n,k,\text{new}}) = (Y_i^{n,k} + \delta Y_i^{n,k}, Z_i^{n,k} + \delta Z_i^{n,k}).$$

The corrected procedure may be repeated several times if needed.

We now summarize our DC method for FBSDEs in Algorithm 3.2 below.

The detailed construction for such algorithm can be highly non-trivial (as will be seen in the following sections). In Algorithm 3.2, we need to design some lower-order numerical method, which should be easy to be implemented, on the sub-interval $I_n$, $n = 0, 1, \cdots, N-1$.

For notation simplicity, in the sequel we will denote $t_{n,k}$ by $\tau_k$ for $k = 0, 1, \cdots, K$, unless otherwise specified.

**Algorithm 3.2** (DC for FBSDEs).

1. Give $Y_i^N$ and $Z_i^N$, $i \in \mathbb{Z}$.
2. For $n = N-1, \ldots, 1, 0$, $i \in \mathbb{Z}$, do (1)-(3).
   (1) Let $Y_i^{n,K} = Y_i^{n+1}$ and $Z_i^{n,K} = Z_i^{n+1}$.
   (2) For $j = 1, 2, \cdots, J$, do (i)-(iii).
      (i) For $k = K-1, \cdots, 1, 0$, solve $Y_i^{n,k,[j]}$ and $Z_i^{n,k,[j]}$ by a lower-order numerical method at grid points $(t_{n,k}, x_i) \in \mathbb{G}_K^n \times D_h$.
      (ii) Let $\delta Y_i^{K,[j]} = 0$ and $\delta Z_i^{K,[j]} = 0$. For $k = K-1, \cdots, 1, 0$, solve $\delta Y_i^{k,[j]}$ and $\delta Z_i^{k,[j]}$ by the same lower-order method at grid points $(t_{n,k}, x_i) \in \mathbb{G}_K^n \times D_h$.
      (iii) Update the numerical solution pairs $(Y_i^{n,k,[j]}, Z_i^{n,k,[j]})$, $k = 0, 1, \cdots, K-1$, by
$$Y_i^{n,k,[j+1]} = Y_i^{n,k,[j]} + \delta Y_i^{k,[j]}, \quad Z_i^{n,k,[j+1]} = Z_i^{n,k,[j]} + \delta Z_i^{k,[j]}.$$
   (3) Let $Y_i^n = Y_i^{n,0,[J]}$ and $Z_i^n = Z_i^{n,0,[J]}$.

## 4. The DC method for FBSDEs

In this section, we will discuss our DC method for decoupled and coupled FBSDEs in detail. The detailed derivation of the DC method for solving decoupled and coupled FBSDEs on $I_n$ is given in Subsections 4.1 and 4.2, respectively, and a conclusive algorithm of our DC method for solving FBSEDs is presented in Subsection 4.3.

### 4.1. DC schemes for decoupled FBSDEs

In this subsection, we shall focus on the DC methods for the following decoupled FBSDEs on the time sub-interval $\mathcal{I}_{n,k} = [\tau_k, t_{n+1}]$:

$$\begin{cases} X_t = X_{\tau_k} + \int_{\tau_k}^t b(s, X_s) \mathrm{d}s + \int_{\tau_k}^t \sigma(s, X_s) \mathrm{d}W_s, & t \in \mathcal{I}_{n,k}, \\ Y_t = Y_{t_{n+1}} + \int_t^{t_{n+1}} f(s, X_s, Y_s, Z_s) \mathrm{d}s - \int_t^{t_{n+1}} Z_s \mathrm{d}W_s, & t \in \mathcal{I}_{n,k}. \end{cases} \tag{4.1}$$

By taking the conditional expectation $\mathbb{E}_{\tau_k}^x[\cdot]$ on both sides of the BSDE in (4.1), we obtain

$$\mathbb{E}_{\tau_k}^x [Y_t] = \mathbb{E}_{\tau_k}^x \left[Y_{t_{n+1}}\right] + \int_t^{t_{n+1}} \mathbb{E}_{\tau_k}^x \left[f(s, X_s, Y_s, Z_s)\right] \mathrm{d}s, \quad t \in \mathcal{I}_{n,k}. \tag{4.2}$$

The integrand $\mathbb{E}^x_{\tau_k}[f(s,X_s,Y_s,Z_s)]$ is a continuous function of $s$. Then, by taking the derivative with respect to $t$ on both sides of (4.2), one obtains the following reference ODE:

$$\frac{\mathrm{d}\mathbb{E}^x_{\tau_k}[Y_t]}{\mathrm{d}t} = -\mathbb{E}^x_{\tau_k}[f(t,X_t,Y_t,Z_t)], \quad t\in\mathcal{I}_{n,k}. \tag{4.3}$$

Note that we also have

$$Y_{\tau_k} = Y_t + \int_{\tau_k}^t f(s,X_s,Y_s,Z_s)\mathrm{d}s - \int_{\tau_k}^t Z_s\mathrm{d}W_s, \quad t\in\mathcal{I}_{n,k}.$$

Let $\Delta W_{\tau_k,t} := W_t - W_{\tau_k}$ for $t\ge\tau_k$. By multiplying both sides of the above equation by $(\Delta W_{t_n,t})^\top$ and taking the conditional expectation $\mathbb{E}^x_{\tau_k}[\cdot]$ on both sides of the derived equation, we obtain for $t\in\mathcal{I}_{n,k}$:

$$\mathbb{E}^x_{\tau_k}\left[Y_t(\Delta W_{\tau_k,t})^\top\right] + \int_{\tau_k}^t \mathbb{E}^x_{\tau_k}\left[f(s,X_s,Y_s,Z_s)(\Delta W_{\tau_k,s})^\top\right]\mathrm{d}s - \int_{\tau_k}^t \mathbb{E}^x_{\tau_k}[Z_s]\,\mathrm{d}s = 0. \tag{4.4}$$

Again, the two integrands in (4.4) are continuous functions of $s$ [18]. Upon taking the derivative with respect to $t\in\mathcal{I}_{n,k}$ in (4.4), we get the following reference ODE:

$$\frac{\mathrm{d}\mathbb{E}^x_{\tau_k}\left[Y_t(\Delta W_{\tau_k,t})^\top\right]}{\mathrm{d}t} = -\mathbb{E}^x_{\tau_k}\left[f(t,X_t,Y_t,Z_t)(\Delta W_{\tau_k,t})^\top\right] + \mathbb{E}^x_{\tau_k}[Z_t]. \tag{4.5}$$

For (3.9), by using the same arguments in obtaining (4.3) and (4.5), we derive the following two reference ODEs for the error pair $(\delta Y_t,\delta Z_t)$ for $t\in\mathcal{I}_{n,k}$:

$$\frac{\mathrm{d}\mathbb{E}^x_{\tau_k}[\delta Y_t]}{\mathrm{d}t} = -\mathbb{E}^x_{\tau_k}\left[f\left(t,X_t,\delta Y_t+I_hY_t,\delta Z_t+I_hZ_t\right)\right] - \frac{\mathrm{d}\mathbb{E}^x_{\tau_k}[I_hY_t]}{\mathrm{d}t}, \tag{4.6a}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}^x_{\tau_k}\left[\delta Y_t(\Delta W_{\tau_k,t})^\top\right]}{\mathrm{d}t} =& -\mathbb{E}^x_{\tau_k}\left[f\left(t,X_t,\delta Y_t+I_hY_t,\delta Z_t+I_hZ_t\right)(\Delta W_{\tau_k,t})^\top\right]\\ &+\mathbb{E}^x_{\tau_k}[\delta Z_t+I_hZ_t] - \frac{\mathrm{d}\mathbb{E}^x_{\tau_k}\left[I_hY_t(\Delta W_{t_n,t})^\top\right]}{\mathrm{d}t}.\end{aligned} \tag{4.6b}$$

The Eqs. (4.3) and (4.5)-(4.6b) give us reference ODEs for solving the BSDE in (4.1), which will serve as fundamental tools in designing the DC-based numerical schemes. Specifically speaking, our DC schemes will be derived by approximating the conditional expectations and the derivatives in (4.3) and (4.5)-(4.6b).

#### 4.1.1. The semi-discrete scheme

We now propose the semi-discrete DC scheme for decoupled FBSDEs on $I_n$. We choose smooth functions $\bar{b}(t,x)$ and $\bar{\sigma}(t,x)$ for $t\in\mathcal{I}_{n,k}$ and $x\in\mathbb{R}^d$ with constraints $\bar{b}(\tau_k,x) = b(\tau_k,x)$ and $\bar{\sigma}(\tau_k,x)=\sigma(\tau_k,x)$. Define the diffusion process $\bar{X}^{\tau_k,x}_t$ by

$$\bar{X}^{\tau_k,x}_t = x + \int_{\tau_k}^t \bar{b}(s,\bar{X}^{\tau_k,x}_s)\mathrm{d}s + \int_{\tau_k}^t \bar{\sigma}(s,\bar{X}^{\tau_k,x}_s)\mathrm{d}W_s. \tag{4.7}$$

Let $(X_t^{\tau_k,x}, Y_t^{\tau_k,x}, Z_t^{\tau_k,x})$ be the solution of the decoupled FBSDEs (4.1), and $(\bar{Y}_t^{\bar{X}_t^{\tau_k,x}}, \bar{Z}_t^{\bar{X}_t^{\tau_k,x}})$ be the values of $(Y_t^{\tau_k,x}, Z_t^{\tau_k,x})$ at $(t, \bar{X}_t^{\tau_k,x})$. Then by Theorem 2.1 we have

$$\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[Y_t]}{\mathrm{d}t}\bigg|_{t=\tau_k} = \frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[\bar{Y}_t]}{\mathrm{d}t}\bigg|_{t=\tau_k} = \frac{\mathbb{E}_{\tau_k}^x[\bar{Y}_{\tau_{k+1}}] - Y_{\tau_k}}{\delta t} + \tilde{R}_y^k, \tag{4.8a}$$

$$\frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[Y_t(\Delta W_{\tau_k,t})^\top]}{\mathrm{d}t}\bigg|_{t=\tau_k} = \frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[\bar{Y}_t(\Delta W_{\tau_k,t})^\top]}{\mathrm{d}t}\bigg|_{t=\tau_k} = \frac{\mathbb{E}_{\tau_k}^x[\bar{Y}_{\tau_{k+1}}(\Delta W_{\tau_k,t})^\top]}{\delta t} + \tilde{R}_z^k, \tag{4.8b}$$

where $\tilde{R}_y^k$ and $\tilde{R}_z^k$ are truncation errors, defined by

$$\tilde{R}_y^k = \frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[Y_t]}{\mathrm{d}t}\bigg|_{t=\tau_k} - \frac{\mathbb{E}_{\tau_k}^x[\bar{Y}_{\tau_{k+1}}] - Y_{\tau_k}}{\delta t},$$

$$\tilde{R}_y^k = \frac{\mathrm{d}\mathbb{E}_{\tau_k}^x[Y_t(\Delta W_{\tau_k,t})^\top]}{\mathrm{d}t}\bigg|_{t=\tau_k} - \frac{\mathbb{E}_{\tau_k}^x[\bar{Y}_{\tau_{k+1}}(\Delta W_{\tau_k,t})^\top]}{\delta t}.$$

Inserting (4.8a) and (4.8b) into (4.3) and (4.5), respectively, we obtain the following reference equations for solving BSDE:

$$Y_{\tau_k} = \mathbb{E}_{\tau_k}^x[\bar{Y}_{\tau_{k+1}}] + \delta t \cdot f(\tau_k, X_{\tau_k}, Y_{\tau_k}, Z_{\tau_k}) + R_y^k, \tag{4.9a}$$

$$Z_{\tau_k} = \mathbb{E}_{\tau_k}^x[\bar{Y}_{\tau_{k+1}}(\Delta W_{\tau_k,t})^\top]/\delta t + R_z^k, \tag{4.9b}$$

where $R_y^k = -\tilde{R}_y^k$ and $R_z^k = \tilde{R}_z^k$. For the forward SDE, we choose the simplest form $\bar{b}(t, X_t^{\tau_k,x}) = b(\tau_k, x)$ and $\bar{\sigma}(t, X_t^{\tau_k,x}) = \sigma(\tau_k, x)$ for $t \in \mathcal{I}_{n,k}$, which results in the Euler scheme for the SDE, i.e.,

$$X^{k+1} = X^k + b(\tau_k, X^k)\delta t + \sigma(\tau_k, X^k)\Delta W_{\tau_k,\tau_{k+1}}.$$

Now let $Y^k$ and $Z^k$ be the numerical approximations for the solutions $Y_t$ and $Z_t$ of the BSDE in (2.8) at time $\tau_k$, respectively. By removing the truncation errors $R_y^k$ and $R_z^k$ from (4.9a) and (4.9b), respectively, we propose the time semi-discrete numerical scheme for solving $(Y_t, Z_t)$ of the BSDE (4.2)

**Scheme 4.1** (The Euler scheme). *Given $Y^K$ and $Z^K$ on $D_h$, for $k = K-1, \cdots, 1, 0$, solve $X^{k+1}$, $Y^k = Y^k(X^k)$ and $Z^k = Z^k(X^k)$ for all $X^k \in D_h$ by*

$$7X^{k+1} = X^k + b(\tau_k, X^k)\delta t + \sigma(\tau_k, X^k)\Delta W_{\tau_k,\tau_{k+1}}, \tag{4.10a}$$

$$Z^k = \mathbb{E}_{\tau_k}^{X^k}\left[\bar{Y}^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]/\delta t, \tag{4.10b}$$

$$Y^k = \mathbb{E}_{\tau_k}^{X^k}\left[\bar{Y}^{k+1}\right] + \delta t \cdot f(\tau_k, X^k, Y^k, Z^k), \tag{4.10c}$$

*where $\bar{Y}^{k+1}$ is the value of $Y^{k+1}$ at the space point $X^{k+1}$.*

The above scheme is of multi-step type as proposed in [18]. The main advantage of the use of the Euler scheme can dramatically reduce the total computational complexity [18]. However, unlike the multi-step schemes in [18], we can not expect an high-order convergence rate by only considering Scheme 4.1. To fix this, below we shall design high-order DC schemes based on the above Euler scheme.

Similarly, denote by $\delta Y^k$ and $\delta Z^k$ the approximated solution of $\delta Y_t$ and $\delta Z_t$ on $I_n$, respectively. We propose the Euler scheme to solve the solution $(\delta Y_t, \delta Z_t)$ of (4.6a) and (4.6b):

$$\begin{cases} \delta Z^k = \mathbb{E}_{\tau_k}^{X^k}[\delta \bar{Y}^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top]/\delta t - Z^k + \left.\dfrac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}[I_h Y_t(\Delta W_{\tau_k,\tau_{k+1}})^\top]}{\mathrm{d}t}\right|_{t=\tau_k}, \\ \delta Y^k = \mathbb{E}_{\tau_k}^{X^k}[\delta \bar{Y}^{k+1}] + \delta t \cdot \left( f(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k) + \left.\dfrac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}[I_h Y_t]}{\mathrm{d}t}\right|_{t=\tau_k} \right), \end{cases} \tag{4.11}$$

where $\delta \bar{Y}^{k+1}$ is the value of $\delta Y^{k+1}$ at the space point $X^{k+1}$. Notice that by Lemma 2.1, we have the two identities

$$\left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}[I_h Y_t]}{\mathrm{d}t}\right|_{t=\tau_k} = L^0_{\tau_k, X^k}(I_h Y_{\tau_k}), \tag{4.12a}$$

$$\left.\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{X^k}[I_h Y_t(\Delta W_{\tau_k,\tau_{k+1}})^\top]}{\mathrm{d}t}\right|_{t=\tau_k} = \nabla(I_h Y_{\tau_k})\sigma(\tau_k, X^k). \tag{4.12b}$$

Now combining Eqs. (4.11)-(4.12b), we propose our time semi-discrete for solving the error pair $(\delta Y_t, \delta Z_t)$ on $I_n$ as

**Scheme 4.2.** *Let $\delta Y^K = 0$ and $\delta Z^K = 0$ on $D_h$. Then for $k = K-1, \cdots, 1, 0$, solve $X^{n+1}$, $\delta Y^k = \delta Y^k(X^k)$ and $\delta Z^k = \delta Z^k(X^k)$ for all $X^k \in D_h$ by*

$$X^{k+1} = X^k + b(\tau_k, X^k)\delta t + \sigma(\tau_k, X^k)\Delta W_{\tau_k,\tau_{k+1}}, \tag{4.13a}$$

$$\delta Z^k = \mathbb{E}_{\tau_k}^{X^k}\left[\delta \bar{Y}^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]/\delta t - Z^k + \nabla(I_h Y_{\tau_k})\sigma(\tau_k, X^k), \tag{4.13b}$$

$$\delta Y^k = \mathbb{E}_{\tau_k}^{X^k}\left[\delta \bar{Y}^{k+1}\right] + \delta t\Big(f(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k) + L^0_{\tau_k, X^k}(I_h Y_{\tau_k})\Big), \tag{4.13c}$$

*where $\delta \bar{Y}^{k+1}$ is the value of $\delta Y^{k+1}$ at the space point $X^{k+1}$.*

Note that the above scheme involves the terms $\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{x}[I_h Y_t]}{\mathrm{d}t}$ and $\frac{\mathrm{d}\mathbb{E}_{\tau_k}^{x}[I_h Y_t(\Delta W_{\tau_k,t})^\top]}{\mathrm{d}t}$. By Lemma 1 and due to the definition of the operator $L^0$ in (2.4), we need to pay attention to the derivatives $\frac{\partial (I_h Y_t)}{\partial t}$, $\frac{\partial (I_h Y_t)}{\partial x}$, and $\frac{\partial^2 (I_h Y_t)}{\partial x^2}$ of $I_h Y_t$. Thus, high-order accuracy of the DC scheme relies heavily on the approximation quality of $\frac{\partial (I_h Y_t)}{\partial t}$, $\frac{\partial (I_h Y_t)}{\partial x}$, and $\frac{\partial^2 (I_h Y_t)}{\partial x^2}$.

### 4.1.2. The fully-discrete scheme

The main purpose here is to solve $Y^k$ and $Z^k$ at the grid points $x \in D_h$. Precisely, for each $x \in D_h$, $k = K-1, \cdots, 1, 0$, we seek to solve $Y^k = Y^k(x)$ and $Z^k = Z^k(x)$ by

$$\begin{cases} Z^k = \mathbb{E}^x_{\tau_k}\left[\bar{Y}^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]/\delta t, \\ Y^k = \mathbb{E}^x_{\tau_k}\left[\bar{Y}^{k+1}\right] + \delta t \cdot f(\tau_k, X^k, Y^k, Z^k), \end{cases} \tag{4.14}$$

where $\bar{Y}^{k+1}$ are the values of $Y^{k+1}$ at the space point $X^{k+1}$ defined by

$$X^{k+1} = X^k + b(\tau_k, X^k)\delta t + \sigma(\tau_k, X^k)\Delta W_{\tau_k,\tau_{k+1}}. \tag{4.15}$$

Generally, the $X^{k+1}$ does not belong to $D_h$ on the condition of $X^k = x \in D_h$. Thus, to solve $Y^k$ and $Z^k$, interpolation methods are needed to approximate the value of $Y^{k+1}$ at $X^{k+1}$ using the values of $Y^{k+1}$ on $D_h$. Here, we will adopt a local interpolation operator $I^k_{h,X}$ such that $I^k_{h,X}g$ is the interpolation value of the function $g$ at space point $X \in \mathbb{R}^d$ by using the values of $g$ only on $D^k_{h,X}$. Note that any interpolation methods can be used here, however, care should be made if one wants to guarantee the stability and accuracy.

In numerical simulations, the conditional expectations $\mathbb{E}^x_{\tau_k}\left[\bar{Y}^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]$ and $\mathbb{E}^x_{\tau_k}\left[\bar{Y}^{k+1}\right]$ in (4.14) should also be approximated. The approximation operator of $\mathbb{E}^x_{\tau_k}[\cdot]$ is denoted by $\mathbb{E}^{x,h}_{\tau_k}[\cdot]$, which can be any quadrature method such as the Monte Carlo method, the quasi-Monte Carlo method, and the Gaussian quadrature method, and so on.

Now by introducing the operators $I^k_{h,x}$ and $\mathbb{E}^{x,h}_{\tau_k}[\cdot]$, we rewrite (4.14) in the equivalent form

$$\begin{cases} Y_{\tau_k} = \mathbb{E}^{x,h}_{\tau_k}\left[I^{k+1}_{h,\bar{X}_{\tau_{k+1}}} Y_{\tau_{k+1}}\right] + \delta t \cdot f(\tau_k, x, Y_{\tau_k}, Z_{\tau_k}) + R^k_y + R^{k,\mathbb{E}}_y + R^{k,I_h}_y, \\ Z_{\tau_k} = \mathbb{E}^{x,h}_{\tau_k}\left[I^{k+1}_{h,\bar{X}_{\tau_{k+1}}} Y_{\tau_{k+1}}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]\Big/\delta t + R^k_z + R^{k,\mathbb{E}}_z + R^{k,I_h}_z, \end{cases} \tag{4.16}$$

where

$$\begin{aligned} R^{k,\mathbb{E}}_y &= \left(\mathbb{E}^x_{\tau_k} - \mathbb{E}^{x,h}_{\tau_k}\right)\left[\bar{Y}_{\tau_{k+1}}\right], \\ R^{k,\mathbb{E}}_z &= \left(\mathbb{E}^x_{\tau_k} - \mathbb{E}^{x,h}_{\tau_k}\right)\left[\bar{Y}_{\tau_{k+1}}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]\Big/\delta t, \\ R^{k,I_h}_y &= \mathbb{E}^{x,h}_{\tau_k}\left[\bar{Y}_{\tau_{k+1}} - I^{k+1}_{h,\bar{X}_{\tau_{k+1}}} Y_{\tau_{k+1}}\right], \\ R^{k,I_h}_z &= \mathbb{E}^{x,h}_{\tau_k}\left[\left(\bar{Y}_{\tau_{k+1}} - I^{k+1}_{h,\bar{X}_{\tau_{k+1}}} Y_{\tau_{k+1}}\right)(W_{\tau_k,\tau_{k+1}})^\top\right]\Big/\delta t. \end{aligned}$$

The two terms $R^{k,\mathbb{E}}_y$ and $R^{k,\mathbb{E}}_z$ are numerical errors introduced by approximating conditional expectations, and the other two terms $R^{k,I_h}_y$ and $R^{k,I_h}_z$ are numerical errors caused by numerical interpolations.

By removing the six error terms $R_y^k$, $R_y^{k,\mathbb{E}}$, $R_y^{k,I_h}$, $R_z^k$, $R_z^{k,\mathbb{E}}$, and $R_z^{k,I_h}$ from (4.16), we propose our fully discrete scheme for solving the solution $(X_t, Y_t, Z_t)$ of the decoupled FBSDEs (4.1) on $I_n$.

**Scheme 4.3.** *Given $Y^K$ and $Z^K$ on $D_h$, for $k = K-1, \cdots, 1, 0$, solve $X^{k+1}$, $Y^k = Y^k(X^k)$ and $Z^k = Z^k(X^k)$ for all $X^k \in D_h$ by*

$$\begin{aligned} X^{k+1} &= X^k + b(\tau_k, X^k)\delta t + \sigma(\tau_k, X^k)\Delta W_{\tau_k,\tau_{k+1}}, \\ Z^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right] \Big/ \delta t, \\ Y^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\right] + \delta t \cdot f(\tau_k, X^k, Y^k, Z^k). \end{aligned}$$

By using the same arguments, i.e., by approximating the two conditional expectations $\mathbb{E}_{\tau_k}^x[\delta \bar{Y}_{\tau_{k+1}}(\Delta W_{\tau_k,\tau_{k+1}})^\top]$ and $\mathbb{E}_{\tau_k}^x[\delta \bar{Y}_{\tau_{k+1}}]$, we propose our fully discrete Euler scheme for solving the error pair $(\delta Y_t, \delta Z_t)$ on $I_n$ as follows.

**Scheme 4.4.** *Let $\delta Y^K = 0$ and $\delta Z^K = 0$ on $D_h$, then for $k = K-1, \cdots, 1, 0$, solve $X^{k+1}$, $\delta Y^k = \delta Y^k(X^k)$ and $\delta Z^k = \delta Z^k(X^k)$ for all $X^k \in D_h$ by*

$$\begin{aligned} X^{k+1} &= X^k + b(\tau_k, X^k)\delta t + \sigma(\tau_k, X^k)\Delta W_{\tau_k,\tau_{k+1}}, \\ \delta Z^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} \delta Y^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right] \Big/ \delta t - Z^k + \nabla(I_h Y_{\tau_k})\sigma(\tau_k, X^k), \\ \delta Y^k &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} \delta Y^{k+1}\right] + \delta t\Big(f(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k) + L_{\tau_k,X^k}^0(I_h Y_{\tau_k})\Big). \end{aligned}$$

Consequently, Scheme 4.3 and Scheme 4.4 are the DC numerical schemes for solving $(Y^k, Z^k)$ and $(\delta Y^k, \delta Z^k)$ on $I_n$, respectively. They can be applied in Algorithm 3.2.

Note that the above schemes involve solving nonlinear equations with respect to $Y^k$ and $\delta Y^k$. Thus, some iteration methods are required. Suppose that the function $f(t, x, y, z)$ is Lipschitz continuous with respect to $y$, for small time partition $\delta t$, we propose the following iteration procedure to solve $Y^k$ :

$$Y^{k,[l+1]} = \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\right] + \delta t \cdot f(\tau_k, x, Y^{k,[l]}, Z^k), \tag{4.17}$$

until the iteration error $|Y^{k,[l+1]} - Y^{k,[l]}| \leq \epsilon_0$, where $\epsilon_0 > 0$ is a prescribed tolerance. Similar iteration procedures can be used for solving $\delta Y^k$.

The local truncation errors of the above Schemes 4.3-4.4 consist of six terms $R_y^k$, $R_y^{k,\mathbb{E}}$, $R_y^{k,I_h}$, $R_z^k$, $R_z^{k,\mathbb{E}}$, and $R_z^{k,I_h}$. The two terms $R_y^k$ and $R_z^k$ defined, respectively, in (4.9a) and (4.9b) come from the approximations of the derivatives, and the two terms $R_y^{k,I_h}$ and $R_z^{k,I_h}$ defined in (4.16) are the local interpolation errors. Under certain regular conditions on the data $b$, $\sigma$, $f$ and $\varphi$, by approximation theory, it holds that

$$R_y^k = \mathcal{O}\left((\Delta\tau_k)^2\right), \qquad R_z^k = \mathcal{O}\left((\Delta\tau_k)^2\right), \tag{4.18a}$$

$$R_z^{k,I_h} = \mathcal{O}\left(h^{l+1}\right), \qquad R_y^{k,I_h} = \mathcal{O}\left(h^{l+1}\right), \tag{4.18b}$$

where $l$ is the number of the grid points used in forming the interpolation polynomial $I^k_{h,X}$. The other two terms $R_y^{n,\mathbb{E}}$ and $R_z^{n,\mathbb{E}}$ are the local truncation errors resulted from the approximations of the conditional mathematical expectations in (4.16). It is noted that these conditional expectations are functions of Gaussian random variables, which can be represented as integrals with Gaussian kernels, and thus can be approximated by Gauss-Hermite quadrature with high accuracy.

## 4.2. DC schemes for coupled FBSDEs

In this subsection, we shall extend our DC schemes for solving fully coupled FBSDEs (1.1) on $I_n$. To this end, we make a trivial extension of Schemes 4.3-4.4 for decoupled cases to the following Schemes 4.5-4.6 for the coupled case on $I_n$.

**Scheme 4.5.** *Assume $Y^K$ and $Z^K$ defined on $D_h^K$ are known. For $k = K-1,\cdots,1,0$, solve $X^{k+1}$, $Y^k = Y^k(x)$ and $Z^k = Z^k(x)$ for all $x \in D_h$ by*

$$\begin{aligned} X^{k+1} &= X^k + b\left(\tau_k, X^k, Y^k, Z^k\right)\delta t + \sigma\left(\tau_k, X^k, Y^k, Z^k\right)\Delta W_{\tau_k,\tau_{k+1}},\\ Z^k &= \mathbb{E}^{x,h}_{\tau_k}\left[I^{k+1}_{h,X^{k+1}} Y^{k+1}(\Delta W_{\tau_k,t_{k=1}})^\top\right]\Big/\delta t,\\ Y^k &= \mathbb{E}^{x,h}_{\tau_k}\left[I^{k+1}_{h,X^{k+1}} Y^{k+1}\right] - \delta t\cdot f(\tau_k, x, Y^k, Z^k). \end{aligned}$$

**Scheme 4.6.** *Let $\delta Y^K = 0$ and $\delta Z^K = 0$ on $D_h$, then for $k = K-1,\cdots,1,0$, solve the errors $\delta Y^k$ and $\delta Z^k$ by*

$$\begin{aligned} X^{k+1} &= X^k + b(\tau_k, X^k, Y^k + \delta Y^k, Z^k + \delta Z^k)\delta t\\ &\quad + \sigma(\tau_k, X^k, Y^k + \delta Y^k, Z^k + \delta Z^k)\Delta W_{\tau_k,\tau_{k+1}},\\ \delta Z^k &= \mathbb{E}^{x,h}_{\tau_k}\left[I^{k+1}_{h,X^{k+1}}\delta Y^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right]\Big/\delta t - Z^k\\ &\quad + \nabla(I_h Y_{\tau_k})\sigma\left(\tau_k, X^k, Y^k + \delta Y^k, Z^k + \delta Z^k\right),\\ \delta Y^k &= \mathbb{E}^{x,h}_{\tau_k}\left[I^{k+1}_{h,X^{k+1}}\delta Y^{k+1}\right] + \delta t\Big(f(\tau_k, X^k, \delta Y^k + Y^k, \delta Z^k + Z^k) + L^0_{\tau_k,X^k}(I_h Y_{\tau_k})\Big). \end{aligned}$$

The main difference between Schemes 4.5-4.6 and Schemes 4.3-4.4 is that the equations in Schemes 4.5-4.6 are all coupled together, and thus, it requires to solve the relevant nonlinear equations. In the practical computation, we propose the following iterative Scheme 4.7 and Schem 4.8 to solve $(Y^k, Z^k)$ and $(\delta Y^k, \delta Z^k)$, respectively, on $I_n$.

**Scheme 4.7.** *Assume $Y^K$ and $Z^K$ defined on $D_h^K$ are known. For $k = K-1,\cdots,1,0$, and for $x \in D_h$, solve $Y^k = Y^k(x)$ and $Z^k = Z^k(x)$ by*

1. *Let $Y^{k,[0]} = Y^{k+1}(x)$ and $Z^{k,[0]} = Z^{k+1}(x)$, and let $l = 0$;*

2. *For* $l = 0, 1, \cdots$, *solve* $Y^{k,[l+1]} = Y^{k,[l+1]}(x)$ *and* $Z^{k,[l+1]} = Z^{k,[l+1]}(x)$ *by*

$$\begin{aligned} X^{k+1} &= X^k + b(\tau_k, X^k, Y^{k,[l]}, Z^{k,[l]})\delta t + \sigma(\tau_k, X^k, Y^{k,[l]}, Z^{k,[l]})\Delta W_{\tau_k,\tau_{k+1}}, \\ Z^{k,[l+1]} &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top\right] \Big/ \delta t, \\ Y^{k,[l+1]} &= \mathbb{E}_{\tau_k}^{x,h}\left[I_{h,X^{k+1}}^{k+1} Y^{k+1}\right] - \delta t \cdot f(\tau_k, x, Y^{k,[l+1]}, Z^{k,[l+1]}), \end{aligned}$$

*until* $\max(|Y^{k,[l+1]} - Y^{k,[l]}|, |Z^{k,[l+1]} - Z^{k,[l]}|) < \epsilon_0$;

3. *Let* $Y^k = Y^{k,[l+1]}$ *and* $Z^k = Z^{k,[l+1]}$.

**Scheme 4.8.** *For* $k = K-1, \cdots, 1, 0$, *solve the errors* $\delta Y^k$ *and* $\delta Z^k$ *by*

1. *Let* $\delta Y^{k,[0]} = \delta Y^{k+1}(x)$ *and* $\delta Z^{k,[0]} = \delta Z^{k+1}(x)$ *with* $\delta Y^K(x) = 0$ *and* $\delta Z^K(x) = 0$, *and let* $l = 0$;

2. *For* $l = 0, 1, \cdots$, *solve* $\delta Y^{k,[l+1]} = \delta Y^{k,[l+1]}(x)$ *and* $\delta Z^{k,[l+1]} = \delta Z^{k,[l+1]}(x)$ *by*

$$\begin{aligned} X^{k+1} &= X^k + b(\tau_k, X^k, Y^k + \delta Y^{k,[l]}, Z^k + \delta Z^{k,[l]})\delta t \\ &\qquad + \sigma(\tau_k, X^k, Y^k + \delta Y^{k,[l]}, Z^k + \delta Z^{k,[l]})\Delta W_{\tau_k,\tau_{k+1}}, \\ \delta Z^{k,[l+1]} &= \mathbb{E}_{\tau_k}^{x,h}[I_{h,X^{k+1}}^{k+1} \delta Y^{k+1}(\Delta W_{\tau_k,\tau_{k+1}})^\top]/\delta t - Z^k \\ &\qquad + \nabla(I_h Y_{\tau_k})\sigma(\tau_k, X^k, Y^k + \delta Y^{k,[l]}, Z^k + \delta Z^{k,[l]}), \\ \delta Y^{k,[l+1]} &= \mathbb{E}_{\tau_k}^{x,h}[I_{h,X^{k+1}}^{k+1} \delta Y^{k+1}] + \delta t\Big(f(\tau_k, X^k, \delta Y^{k,[l+1]} + Y^k, \delta Z^{k,[l+1]} + Z^k) \\ &\qquad + L_{\tau_k,X^k}^0(I_h Y_{\tau_k})\Big), \end{aligned}$$

*until* $\max(|\delta Y^{k,[l+1]} - Y^{k,[l]}|, |\delta Y^{k,[l+1]} - Y^{k,[l]}|) < \epsilon_1$;

3. *Let* $\delta Y^k = \delta Y^{k,[l+1]}$ *and* $\delta Z^k = \delta Z^{k,[l+1]}$.

**Remark 4.1.** If the drift coefficient $b$ and the diffusion coefficient $\sigma$ do not depend on $Y_.$ and $Z_.$, then Scheme 4.5 and Scheme 4.6 coincide with Scheme 4.3 and Scheme 4.4, respectively.

The mesh $D_h^k$ is essentially unbounded. In applications, one is often interested in obtaining certain values of $(Y_t, Z_t)$ at $(\tau_k, x)$ with $x$ in a bounded domain. For instance, in option pricing, people are only interested in the option values at the current option price. Thus, in practice, only a bounded submesh of $D_h$ is used on each time level. In our numerical experiments, we use the Gauss-Hermite integral rule to approximate the conditional expectations, in which only a small number of integral points are used.

## 4.3. DC algorithm for FBSDEs

Combining Algorithm 3.2 with the schemes on $I_n$ presented in Subsection 4.1 and 4.2, we are able to present our DC algorithm for solving decoupled and coupled FBSDEs on $[0, T]$ in Algorithm 4.1 below.

**Algorithm 4.1** (DC for decoupled & coupled FBSDEs).

1. Give $Y_i^N$ and $Z_i^N$, $i \in \mathbb{Z}$.
2. For $n = N-1, \cdots, 1, 0$, $i \in \mathbb{Z}$, do (1)-(3).
   (1) Let $Y_i^{n,K} = Y_i^{n+1}$ and $Z_i^{n,K} = Z_i^{n+1}$.
   (2) For $j = 1, 2, \cdots, J$, do (i)-(iii).
      (i) For $k = K-1, \cdots, 1, 0$, solve $Y_i^{n,k,[j]}$ and $Z_i^{n,k,[j]}$ by
      $$\begin{cases} \text{Schemes 4.3 for decoupled FBSDEs} \\ \text{Schemes 4.7 for coupled FBSDEs} \end{cases}$$
      at grid points $(t_{n,k}, x_i) \in \mathbb{G}_K^n \times D_h$.
      (ii) Let $\delta Y_i^{K,[j]} = 0$ and $\delta Z_i^{K,[j]} = 0$. For $k = K-1, \cdots, 1, 0$, solve $\delta Y_i^{k,[j]}$ and $\delta Z_i^{k,[j]}$ by
      $$\begin{cases} \text{Schemes 4.4 for decoupled FBSDEs} \\ \text{Schemes 4.8 for coupled FBSDEs} \end{cases}$$
      at grid points $(t_{n,k}, x_i) \in \mathbb{G}_K^n \times D_h$.
      (iii) Update the numerical solution pairs $(Y_i^{n,k,[j]}, Z_i^{n,k,[j]})$, $k = 0, 1, \cdots, K-1$, by
      $$Y_i^{n,k,[j+1]} = Y_i^{n,k,[j]} + \delta Y_i^{k,[j]}, \quad Z_i^{n,k,[j+1]} = Z_i^{n,k,[j]} + \delta Z_i^{k,[j]}.$$
   (3) Let $Y_i^n = Y_i^{n,0,[J]}$ and $Z_i^n = Z_i^{n,0,[J]}$.

## 5. Numerical experiments

In this section, we will provide several numerical examples to demonstrate the high accuracy and effectiveness of our DC schemes proposed in Section 4. For simplicity, we will use uniform partitions in both time and space. The time interval $[0, T]$ will be uniformly divided into $N$ parts with time step $\Delta t = \frac{T}{N}$ and time grids $t_n = n\Delta t$, $n = 0, 1, \cdots, N$, and each time sub-interval $I_n = [t_n, t_{n+1}]$ is divided into $K$ uniform parts with time substep $\delta t = \frac{T}{NK}$ and time grids $t_{n,k} = t_n + k\delta t$, $k = 0, 1, \cdots, K$. The space partition $D_h$ of one-dimensional real axis $\mathbb{R}$ is defined by

$$D_h = \{x_i : x_i = ih, \quad i = 0, \pm 1, \cdots\}$$

with $h$ the discretized spatial step. Let $D_{h,x} \subset D_h$ denote the subset of some neighbor grids near $x \in \mathbb{R}$. According to the error estimate (3.5), we set $J = K$, $I_{h,X^{k+1}}^{k+1}$ the local standard Lagrange interpolation operator on $D_{h,x}$, and $\mathbb{E}_{\tau_k}^{x,h}[\cdot]$ the Gaussian approximation operator of the conditional mathematical expectation $\mathbb{E}_{\tau_k}^{x}[\cdot]$. In all the examples, the terminal time $T$ is set to be $1.0$. And in all the tables, we use CR to denote the convergence rate. The numerical results, including numerical errors and convergence rates are obtained by using the Algorithm 4.1 to solve the following FBSDEs. The Algorithm 4.1 is coded in FORTRAN 95 with the intrinsic data type: REAL (KIND=16)

## 5.1. Test 1

In this subsection, we will test the stability, accuracy and effectiveness of our DC method for solving decoupled and coupled FBSDEs.

**The decoupled FBSDEs model:**

$$\begin{cases} \mathrm{d}X_t = \dfrac{1}{1+2\exp(t+X_t)}\mathrm{d}t + \dfrac{\exp(t+X_t)}{1+\exp(t+X_t)}\mathrm{d}W_t, \\ -\mathrm{d}Y_t = \left(-\dfrac{2Y_t}{1+2\exp(t+X_t)} - \dfrac{1}{2}\left(\dfrac{Y_tZ_t}{1+\exp(t+X_t)} - Y_t^2Z_t\right)\right)\mathrm{d}t - Z_t\mathrm{d}W_t, \end{cases} \tag{5.1}$$

with the initial condition $X_0 = x$ and terminal condition $Y_T = \frac{\exp(T+X_T)}{1+\exp(T+X_T)}$. The analytic solutions $Y_t$ and $Z_t$ of (5.1) are

$$Y_t = \frac{\exp(t+X_t)}{1+\exp(t+X_t)}, \quad Z_t = \frac{(\exp(t+X_t))^2}{(1+\exp(t+X_t))^3}. \tag{5.2}$$

**The coupled FBSDEs model:**

$$\begin{cases} \mathrm{d}X_t = \dfrac{1}{1+\exp(t+X_t)}\dfrac{1}{1+Y_t}\mathrm{d}t + Y_t\mathrm{d}W_t, \\ -\mathrm{d}Y_t = \left(-\dfrac{2Y_t}{1+2\exp(t+X_t)} - \dfrac{1}{2}\left(\dfrac{Y_tZ_t}{1+\exp(t+X_t)} - Y_t^2Z_t\right)\right)\mathrm{d}t - Z_t\mathrm{d}W_t, \end{cases} \tag{5.3}$$

with the initial condition $X_0 = x$ and the terminal condition $Y_T = \frac{\exp(T+X_T)}{1+\exp(T+X_T)}$. The analytic solutions $Y_t$ and $Z_t$ of (5.3) are

$$Y_t = \frac{\exp(t+X_t)}{1+\exp(t+X_t)}, \quad Z_t = \frac{(\exp(t+X_t))^2}{(1+\exp(t+X_t))^3}.$$

For decoupled problem (5.1) and coupled problem (5.3), we set $x = 1$ and $x = 0$, respectively. For $K = 1, \cdots, 4$, we solve (5.1) and (5.3) by Algorithm 4.1 for different time partitions. The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, and the convergence rates CR with respect to time step $\Delta t = \frac{1}{N}$ are listed in Table 1 and Table 2 for problem (5.1) and problem (5.3), respectively.

Numerical results listed in Tables 1 and 2 clearly show that our DC method proposed in this paper for solving FBSDEs is stable and effective, and is of $K$-th order method at least for $K = 1, \cdots, 4$. The results observed in these tests are consistent with those ones for DC methods to solve ODEs.

Table 1: Errors and convergence rates for (5.1).

| $K$ | | $N=4$ | $N=6$ | $N=8$ | $N=10$ | $N=12$ | CR |
|---|---|---|---|---|---|---|---|
| 1 | $\|Y^0-Y_0\|$ | 1.446E-02 | 9.670E-03 | 7.262E-03 | 5.813E-03 | 4.846E-03 | 0.995 |
| | $\|Z^0-Z_0\|$ | 1.923E-02 | 1.286E-02 | 9.665E-03 | 7.742E-03 | 6.456E-03 | 0.994 |
| 2 | $\|Y^0-Y_0\|$ | 2.617E-05 | 1.925E-05 | 1.475E-05 | 1.167E-05 | 9.456E-06 | 1.993 |
| | $\|Z^0-Z_0\|$ | 2.621E-05 | 1.933E-05 | 1.484E-05 | 1.175E-05 | 9.535E-06 | 1.980 |
| 3 | $\|Y^0-Y_0\|$ | 2.238E-06 | 6.237E-07 | 2.554E-07 | 1.285E-07 | 7.351E-08 | 3.109 |
| | $\|Z^0-Z_0\|$ | 1.536E-05 | 4.584E-06 | 1.945E-06 | 9.996E-07 | 5.801E-07 | 2.982 |
| 4 | $\|Y^0-Y_0\|$ | 2.346E-07 | 4.387E-08 | 1.352E-08 | 5.453E-09 | 2.603E-09 | 4.097 |
| | $\|Z^0-Z_0\|$ | 6.786E-07 | 1.318E-07 | 4.145E-08 | 1.694E-08 | 8.160E-09 | 4.024 |

Table 2: Errors and convergence rates for problem (5.3).

| $K$ | | $N=4$ | $N=6$ | $N=8$ | $N=10$ | $N=12$ | CR |
|---|---|---|---|---|---|---|---|
| 1 | $\|Y^0-Y_0\|$ | 1.013E-02 | 6.989E-03 | 5.334E-03 | 3.619E-03 | 4.312E-03 | 0.937 |
| | $\|Z^0-Z_0\|$ | 1.064E-02 | 7.054E-03 | 5.280E-03 | 4.220E-03 | 3.515E-03 | 1.008 |
| 2 | $\|Y^0-Y_0\|$ | 3.666E-04 | 1.618E-04 | 9.065E-05 | 5.785E-05 | 4.009E-05 | 2.014 |
| | $\|Z^0-Z_0\|$ | 6.196E-04 | 2.891E-04 | 1.669E-04 | 1.085E-04 | 7.618E-05 | 1.908 |
| 3 | $\|Y^0-Y_0\|$ | 1.312E-05 | 3.965E-06 | 1.681E-06 | 8.599E-07 | 4.960E-07 | 2.981 |
| | $\|Z^0-Z_0\|$ | 1.256E-05 | 3.798E-06 | 1.529E-06 | 7.261E-07 | 3.831E-07 | 3.171 |
| 4 | $\|Y^0-Y_0\|$ | 2.137E-07 | 3.930E-08 | 1.194E-08 | 4.756E-09 | 2.247E-09 | 4.146 |
| | $\|Z^0-Z_0\|$ | 4.800E-07 | 9.500E-08 | 2.995E-08 | 1.215E-08 | 5.776E-09 | 4.021 |

### 5.2. Test 2

The aim of this subsection is to show the performance of our DC method for solving FBSDEs in accuracy, effectiveness and stability. The chosen FBSDEs model is

$$\begin{cases} \mathrm{d}X_t = \sin(t+X_t)\mathrm{d}t + \dfrac{3}{10}\cos(t+X_t)\mathrm{d}W_t, \\ -\mathrm{d}Y_t = \left(\dfrac{3}{20}Y_tZ_t - \cos(t+X_t)(1+Y_t)\right)\mathrm{d}t - Z_t\mathrm{d}W_t, \end{cases} \tag{5.4}$$

with the initial condition $X_0 = x$ and the terminal condition $Y_T = \sin(T+X_T)$. The analytic solutions $Y_t$ and $Z_t$ of (5.4) are

$$Y_t = \sin(t+X_t), \quad Z_t = \frac{3}{10}\cos(t+X_t)^2. \tag{5.5}$$

In the tests, we set the initial condition $x = 0.5$. We use Algorithm 4.1 to solve the above decoupled FBSDEs. The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, and the convergence rates CR with respect to time step $\Delta t = \frac{1}{N}$ are listed in Table 3 for different time partitions. The results in Table 3 clearly show that our DC method is of $K$-th order method ($K = 1, 2, \ldots, 12$), stable and effective with very high convergence rate (up to 12). The results also show that our DC method is very efficient, which can be seen from the fact that the errors obtained with $N = 4$ and $K = 2$ are much smaller than those obtained with $N = 12$ and $K = 1$ (the Euler scheme).

Table 3: Errors and convergence rates for Example 5.3.

| $K$ | | $N=4$ | $N=6$ | $N=8$ | $N=10$ | $N=12$ | CR |
|---|---|---|---|---|---|---|---|
| 1 | $\|Y^0-Y_0\|$ | 2.512E-01 | 1.766E-01 | 1.360E-01 | 1.104E-01 | 9.286E-02 | 0.906 |
| | $\|Z^0-Z_0\|$ | 1.064E-01 | 7.021E-02 | 5.287E-02 | 4.235E-02 | 3.542E-02 | 1.001 |
| 2 | $\|Y^0-Y_0\|$ | 2.934E-04 | 2.260E-04 | 1.793E-04 | 1.456E-04 | 1.206E-04 | 2.193 |
| | $\|Z^0-Z_0\|$ | 5.994E-05 | 4.717E-05 | 3.808E-05 | 3.138E-05 | 2.630E-05 | 2.031 |
| 3 | $\|Y^0-Y_0\|$ | 1.400E-05 | 9.856E-06 | 7.197E-06 | 5.414E-06 | 4.175E-06 | 2.985 |
| | $\|Z^0-Z_0\|$ | 1.350E-06 | 9.692E-07 | 7.183E-07 | 5.468E-07 | 4.257E-07 | 2.848 |
| 4 | $\|Y^0-Y_0\|$ | 4.618E-08 | 2.819E-08 | 1.816E-08 | 1.222E-08 | 8.515E-09 | 4.170 |
| | $\|Z^0-Z_0\|$ | 6.305E-08 | 3.884E-08 | 2.520E-08 | 1.705E-08 | 1.194E-08 | 4.103 |
| 5 | $\|Y^0-Y_0\|$ | 8.858E-10 | 4.914E-10 | 2.901E-10 | 1.801E-10 | 1.165E-10 | 5.003 |
| | $\|Z^0-Z_0\|$ | 1.375E-09 | 7.669E-10 | 4.545E-10 | 2.833E-10 | 1.839E-10 | 4.962 |
| 6 | $\|Y^0-Y_0\|$ | 8.025E-09 | 7.979E-10 | 1.430E-10 | 3.702E-11 | 1.222E-11 | 5.911 |
| | $\|Z^0-Z_0\|$ | 7.891E-08 | 6.083E-09 | 9.958E-10 | 2.467E-10 | 7.932E-11 | 6.284 |
| 7 | $\|Y^0-Y_0\|$ | 4.346E-10 | 4.671E-11 | 7.536E-12 | 1.735E-12 | 5.117E-13 | 6.155 |
| | $\|Z^0-Z_0\|$ | 3.310E-09 | 1.973E-10 | 2.653E-11 | 5.589E-12 | 1.565E-12 | 6.970 |
| 8 | $\|Y^0-Y_0\|$ | 1.244E-10 | 4.416E-12 | 4.141E-13 | 6.637E-14 | 1.493E-14 | 8.219 |
| | $\|Z^0-Z_0\|$ | 1.061E-10 | 2.914E-12 | 2.357E-13 | 3.427E-14 | 7.210E-15 | 8.739 |
| 9 | $\|Y^0-Y_0\|$ | 3.227E-12 | 1.134E-13 | 9.450E-15 | 1.347E-15 | 2.702E-16 | 8.551 |
| | $\|Z^0-Z_0\|$ | 5.400E-12 | 1.101E-13 | 7.252E-15 | 8.975E-16 | 1.647E-16 | 9.465 |
| 10 | $\|Y^0-Y_0\|$ | 2.938E-13 | 4.422E-15 | 2.290E-16 | 2.325E-17 | 3.609E-18 | 10.293 |
| | $\|Z^0-Z_0\|$ | 1.449E-13 | 3.654E-15 | 2.241E-16 | 2.480E-17 | 4.057E-18 | 9.556 |
| 11 | $\|Y^0-Y_0\|$ | 7.235E-15 | 9.917E-17 | 4.448E-18 | 3.937E-19 | 5.396E-20 | 10.750 |
| | $\|Z^0-Z_0\|$ | 6.106E-15 | 7.201E-18 | 7.865E-19 | 1.202E-19 | 2.072E-20 | 11.083 |
| 12 | $\|Y^0-Y_0\|$ | 3.058E-16 | 1.663E-18 | 4.397E-20 | 2.702E-21 | 2.786E-22 | 12.658 |
| | $\|Z^0-Z_0\|$ | 9.577E-16 | 7.915E-18 | 2.465E-19 | 1.659E-20 | 1.763E-21 | 12.017 |

## 6. Conclusions

In this work, based on the theories of deferred correction methods, SDEs and FBSDEs, we proposed the deferred correction method (DC method) for solving FBSDEs. In this method, the solutions of FBSDEs are iteratively corrected by the Euler approximation solutions of FBSDEs and the associated residual FBSDEs. Our numerical experiments showed that the DC method is stable, effective, and admits high-order accuracy for solving FBSDEs. We believe that the DC methods proposed in this paper are promising in many practical applications, such as finance, stochastic control, risk measure, etc.

**Acknowledgments** The work is supported by the National Natural Science Foundation of China under grants Nos. 91630203, 91630312, 11571206 and 11571351.

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