Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2011.m12si05

Vol. **5**, No. 1, pp. 85-98
February 2012

A Projected Algebraic Multigrid Method for Linear Complementarity Problems

Jari Toivanen[1,*] and Cornelis W. Oosterlee[2]

[1] *Institute for Computational and Mathematical Engineering, Durand Building, room 023A, Stanford University, Stanford, CA 94305, USA and Department of Mathematical Information Technology, Agora, FI-40014 University of Jyväskylä, Finland.*
[2] *Centrum voor Wiskunde en Informatica, Modelling, Analysis and Simulation (MAS2), Kruislaan 413, Amsterdam, The Netherlands and Delft University of Technology, Delft Institute of Applied Mathematics, Mekelweg 4, 2628 CD Delft, The Netherlands.*

Received 13 January 2011; Accepted (in revised version) 05 May 2011

Available 21 December 2011

Abstract. We present an algebraic version of an iterative multigrid method for obstacle problems, called projected algebraic multigrid (PAMG) here. We show that classical algebraic multigrid algorithms can easily be extended to deal with this kind of problem. This paves the way for efficient multigrid solution of obstacle problems with partial differential equations arising, for example, in financial engineering.

AMS subject classifications: 65M55, 65F99, 91G20

Key words: Linear complementarity problem, algebraic multigrid, American options, elasto-plastic torsion problem.

1. Introduction

In this paper we show that the algebraic multigrid (AMG) method, as it is commonly used to solve partial differential equations on unstructured grids in a robust and efficient way, can relatively easily be extended to dealing with obstacle problems. These problems are often encountered in engineering practice, ranging from classical engineering applications like elasto-plastic torsion applications to relatively recent applications occurring, for example, in computational finance.

One of the motivations to develop the AMG method for obstacle problems here is to transfer efficient iterative solution methods of black-box type to application fields like financial engineering. Multigrid has been used in that field, but mainly by academic researchers from universities.

*Corresponding author. *Email addresses:* toivanen@stanford.edu (J. Toivanen), c.w.oosterlee@cwi.nl (C.W. Oosterlee)

AMG was popularised by the overview article of Ruge and Stüben from 1987 [25], with the basic principles of AMG for so-called M-matrices. Moreover, the software related to the AMG solver described in that article was initially provided as open source software. AMG solvers from the eighties were particularly efficient for matrix problems originating from two-dimensional discrete partial differential equations. At that time, with limited computer resources, this was sufficient. In the early nineties of the previous century, however, computer capacity had increased and partial differential equation software for three-dimensional applications had been developed and a need to solve three-dimensional problems efficiently by black-box methods arose. The AMG solver from the eighties had to be upgraded in terms of reducing its coarse grid operator complexity for three-dimensional problems. A revival of AMG started at that time, for example, by [20,26], as well as in [6]. The resulting AMG solvers with reduced operator complexities due to so-called aggressive coarsening [26], or by other means [8] were parallelized as well for enhanced efficiency, for example in [10, 21], amongst several others. The development of AMG has been described in textbooks such as [4] or [27].

Independent of this development, geometric multigrid, which is explicitly based on grids and structures, has been applied to obstacle problems with partial differential equations (PDEs), also in the eighties of last century. Obstacle problems can be formulated as linear complementarity problems (LCPs), which have a long tradition regarding their efficient numerical solution, see, for example, [7,9]. The LCP formulation is beneficial for iterative solution, since the unknown boundary (as obstacle problems are governed by unknown *free* boundaries) does not appear explicitly and can be obtained in a post-processing step.

In the pioneering paper [3] from 1983, regarding the use of multigrid to this type of problem, Brandt and Cryer introduced the projected full approximation scheme (PFAS) multigrid method for LCPs. Unlike in the basic geometric multigrid correction scheme, which is based on transferring corrections to the numerical solution from coarse to fine grids, each level of PFAS approximates the complete solution of the fine LCP. It is thus based on the non-linear full approximation scheme (FAS) developed by Brandt in [2]. In 1987, Hoppe developed a solver for obstacle problems which employs a multigrid method to solve reduced linear algebraic systems in [13]. A later multigrid approach for obstacle problems was called the monotone multigrid method, developed by Kornhuber in [19].

Reisinger and Wittum proposed in [24] a projected multigrid method for LCPs. It resembles the standard multigrid method more closely than PFAS. The coarse grid right-hand side is the restriction of the defect (residual). In that strain of literature we here present the so-called projected algebraic multigrid (PAMG) method. We will base our algorithm on this latter idea, and use the original Ruge-Stüben AMG framework.

A wide range of financial engineering applications in which obstacle problems occur in the form of LCPs is presented in [14,18]. The basic obstacle problem considered in computational finance is the calculation of the value of an American-style option in a stochastic volatility setting. It leads to the solution of a two-dimensional (plus time) convection-diffusion type PDE with a free boundary. In [28], it has been shown that for the American-style options the theory of linear complementarity applies, so that it is possible to rewrite

the problem as an LCP. Numerical discretization schemes of finite element type, amongst others, are presented in [1].

A first application of multigrid to obstacle problems in computational finance is found in [5], with the concept of line-wise smoothers to deal with stretched non-equidistant computational grids. Semi-coarsening in multigrid as well as multigrid as a preconditioner for Krylov methods, tailored to the LCPs in computational finance, can be found in [22]. With the AMG solver developed here, stretched grids are efficiently handled, as algebraic coarsening based on matrix elements typically results in some form of semi-coarsening.

A detailed study of the performance of multigrid for linear complementarity problems arising from the stochastic volatility Heston model [11] can be found in the work by Toivanen and co-workers [15], [16], [17].

The outline of the paper is the following. The model linear complementarity problem is presented in Section 2. The PAMG method and details of its components are described in Section 3. Numerical experiments with elasto-plastic torsion problems and American option pricing under a stochastic volatility model demonstrating the efficiency of the proposed method are presented in Section 4. The conclusions are given at the end of the paper.

2. Linear Complementarity Problem

We consider a linear complementarity problem (LCP)

$$\begin{cases} \mathbf{Ax} \geq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{g}, \\ (\mathbf{Ax} - \mathbf{b})^T (\mathbf{x} - \mathbf{g}) = 0 \end{cases}$$

with the inequalities holding component-wise. We denote this problem by LCP($\mathbf{A}$,$\mathbf{x}$,$\mathbf{b}$,$\mathbf{g}$). Thus, for each i either

$$[\mathbf{Ax}]_i = \mathbf{b}_i \quad \text{or} \quad \mathbf{x}_i = \mathbf{g}_i$$

holds. The set of i-values for which the later equation holds is called the active set, which is given by $\{i \ : \ \mathbf{x}_i = \mathbf{g}_i\}$. The complementary set $\{i \ : \ \mathbf{x}_i > \mathbf{g}_i\}$ is called the inactive set.

We assume the matrix $\mathbf{A}$ to be an M-matrix or nearly so. For an M-matrix $\mathbf{A}$ the following properties hold:

- positive diagonal entries: $\mathbf{A}_{ii} > 0$,
- non-positive off-diagonal entries: $\mathbf{A}_{ij} \leq 0, \ \ i \neq j$,
- diagonally dominant: $\sum_{j \neq i} |\mathbf{A}_{ij}| \leq \mathbf{A}_{ii}$.

Often low-order finite difference or element discretizations of partial differential operators like the convection-diffusion-reaction operator, $-\Delta + \mathbf{v} \cdot \nabla + c$, lead to M-matrices.

The classical iterative solution method for LCPs is the projected successive overrelaxation (PSOR) method proposed by Cryer in [7]. One iteration of this method for LCP($\mathbf{A}$,$\mathbf{x}$,$\mathbf{b}$,$\mathbf{g}$) is given by:

Algorithm 2.1. *Algorithm PSOR(***A**,**x**,**b**,**g***)*

Do $i = 1, \dim \mathbf{A}$
$\mathbf{r}_i = \mathbf{b}_i - \sum_j \mathbf{A}_{ij}\mathbf{x}_j$
$\mathbf{x}_i = \mathbf{x}_i + \omega \mathbf{r}_i / \mathbf{A}_{ii}$
$\mathbf{x}_i = \max\{\mathbf{x}_i,\ \mathbf{g}_i\}$
End Do

In the case $\omega = 1$, the PSOR method reduces to the projected Gauss-Seidel (PGS) method.

A helpful observation here is that PSOR is based on the iterative solution process, by means of SOR, of the matrix equation $\mathbf{Ax} = \mathbf{b}$, where matrix $\mathbf{A}$ originates from a discretization of a PDE. The iterative solution of this matrix equation can be performed fully algebraically, for example, by AMG. This will form the basis of the PAMG algorithm proposed here.

3. Projected Algebraic Multigrid

We present a PAMG method which follows the idea of the projected multigrid method proposed by Reisinger and Wittum in [24]. The geometric components of this method are replaced by the algebraic counterparts described by Ruge and Stüben in [25].

To show the similarity and differences of AMG methods for LCPs and systems of linear equations, we describe the algorithms for one V-cycle of PAMG and AMG side-by-side in the following:

Algorithm 3.1. *Algorithm PAMG and Algorithm AMG*

*Algorithm PAMG(***A**,**x**,**b**,**g***)*	*Algorithm AMG(***A**,**x**,**b***)*
If coarsest level Then	*If coarsest level Then*
*solve LCP(***A**,**x**,**b**,**g***)*	$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
Else	*Else*
*PS(***A**,**x**,**b**,**g***)*	*S(***A**,**x**,**b***)*
$\mathbf{x}^c = \mathbf{0}$	$\mathbf{x}^c = \mathbf{0}$
$\mathbf{r}^c = \bar{\mathbf{R}}(\mathbf{b} - \mathbf{Ax})$	$\mathbf{r}^c = \mathbf{R}(\mathbf{b} - \mathbf{Ax})$
$\mathbf{g}^c = \widehat{\mathbf{R}}(\mathbf{g} - \mathbf{x})$	
*PAMG(***A**^{c},**x**^{c},**r**^{c},**g**^{c}*)*	*AMG(***A**^{c},**x**^{c},**r**^{c}*)*
$\mathbf{x} = \mathbf{x} + \bar{\mathbf{P}}\mathbf{x}^c$	$\mathbf{x} = \mathbf{x} + \mathbf{P}\mathbf{x}^c$
*PS(***A**,**x**,**b**,**g***)*	*S(***A**,**x**,**b***)*
End If	*End If*

DOI https://doi.org/10.4208/nmtma.2011.m12si05 | Generated on 2025-01-14 12:20:41

In the above algorithms, PS and S are smoothers for the LCPs and the systems of linear equations, respectively. The matrices $\mathbf{P}$ and $\mathbf{R}$ are the standard algebraic prolongation and restriction operators. The restriction operators for the solution of the LCP and its constraint are denoted by $\bar{\mathbf{R}}$ and $\widehat{\mathbf{R}}$, respectively. The prolongation for the LCPs is denoted by $\bar{\mathbf{P}}$. The coarse grid matrix is $\mathbf{A}^c$. We describe the choice of these operators in the following sections.

3.1. PAMG components

In this section we will give details about the different components the PAMG solver is based on.

Coarse grid selection. At a given multigrid level, we need to select a set of coarse grid points which define the next coarser level. These points are denoted by C-points. The rest of grid points, the fine grid points, are denoted by F-points. Thus, the intersection of C-points and F-points is empty and their union gives all grid points at the current level.

In the selection of C-points, we use the coarsening proposed by Ruge and Stüben in [25]. This is based on the graph defined by the non-zero entries of the matrix $\mathbf{A}$. A graph connection from i to j is called *strong* when

$$\mathbf{A}_{ij} \leq \alpha \min_{l \neq i} \mathbf{A}_{il} \qquad \text{and} \qquad \mathbf{A}_{ij} < 0.$$

Otherwise the connection is called weak. For the parameter α, we use the fairly standard value $\alpha = 0.25$. The coarse grid points (C-points) are chosen so that the following two conditions hold:

- C-points are only weakly connected to other C-points, and
- each fine point (F-point) with any strong connections has at least one strong connection to a C-point.

We use the following algorithm to choose the C-points. During the algorithm the points are divided into three sets: undecided points (U-points), a subset of C-points, and a subset of F-points. We denote the number of strong connections to the ith U-point from the other U-points by n_i.

Mark all points as U-points
Calculate n_is
Do while $\max_i n_i > 0$
    Find smallest k such that $n_k = \max_i n_i$
    Mark k to be C-point
    Mark all U-points having a strong connection to k to be F-points
    Update n_is
End Do
Mark all remaining U-points to be F-points

Prolongations. The interpolated value of a fine grid point k is given by the formula

$$\sum_{i \in C_k} w_{ik} \mathbf{x}_i,$$

where C_k is the set of coarse grid points strongly connected to the fine grid point k, w_{ik} are the interpolation weights, and $\mathbf{x}_i$ is the value at the ith coarse grid point. Following a proposal by Ruge and Stüben in [25], we choose the weights to be

$$w_{ik} = \frac{1}{\mathbf{A}_{ii}} \left(|\mathbf{A}_{ik}| + \sum_{j \in D_i} \frac{|\mathbf{A}_{jk}|}{\sum_{l \in C_i} |\mathbf{A}_{jl}|} |\mathbf{A}_{ij}| \right),$$

where D_i is the set of coarse grid points weakly connected to i. As long as the diagonal entries of $\mathbf{A}$ are positive, the interpolation weights w_{ik} are non-negative. We denote the matrix defining the above interpolation from C-points to all points by $\mathbf{P}$. We use this standard interpolation to form the coarse grid matrix.

The interpolation matrix for the correction from the coarse grid is denoted by $\bar{\mathbf{P}}$. This can be the standard interpolation matrix $\mathbf{P}$. Alternatively, Brandt and Cryer suggested in [3] the following one-sided prolongation

$$[\bar{\mathbf{P}}\mathbf{x}^c]_i = \begin{cases} [\mathbf{P}\mathbf{x}^c]_i & \text{if} \quad \mathbf{x}_i > \mathbf{g}_i, \\ 0 & \text{if} \quad \mathbf{x}_i = \mathbf{g}_i. \end{cases} \tag{3.1}$$

With this prolongation the coarse grid correction cannot change an active grid point to be inactive. This has to be done by the smoother. We will mainly use the standard prolongation in PAMG, that is, $\bar{\mathbf{P}} = \mathbf{P}$.

Restrictions. The standard restriction matrix $\mathbf{R}$ is given by the transpose of the prolongation matrix $\mathbf{P}$, that is,

$$\mathbf{R} = \mathbf{P}^T.$$

For the defect (residual), we use however the *one-sided restriction* operator proposed by Hoppe and Kornhuber in [12]. It is defined by

$$[\bar{\mathbf{R}}\mathbf{r}]_i = \begin{cases} [\mathbf{R}\mathbf{r}_{\mathscr{A}}]_i & \text{if} \quad \mathbf{x}_i = \mathbf{g}_i, \\ [\mathbf{R}\mathbf{r}_{\mathscr{I}}]_i & \text{if} \quad \mathbf{x}_i > \mathbf{g}_i, \end{cases}$$

where $\mathbf{R}$ is the standard restriction and

$$[\mathbf{r}_{\mathscr{A}}]_i = \begin{cases} \mathbf{r}_i & \text{if} \quad \mathbf{x}_i = \mathbf{g}_i, \\ 0 & \text{else}, \end{cases} \qquad \text{and} \qquad [\mathbf{r}_{\mathscr{I}}]_i = \begin{cases} \mathbf{r}_i & \text{if} \quad \mathbf{x}_i > \mathbf{g}_i, \\ 0 & \text{else}. \end{cases}$$

With the one-sided restriction the active set residual does not contribute to the coarse inactive set residual. Notice that we only need to have explicit information about the vector $\mathbf{g}$, next to the matrix elements $\mathbf{A}_{ij}$ to define this restriction operator algebraically.

Reisinger and Wittum also use in [24] this one-sided restriction $\bar{\mathbf{R}}$ in their projected multigrid method. They further state for the restriction operators the minimum requirements

$$\begin{cases} \bar{\mathbf{R}}(\mathbf{A}\mathbf{x}-\mathbf{b}) \geq \mathbf{0}, \\ \widehat{\mathbf{R}}(\mathbf{x}-\mathbf{g}) \geq \mathbf{0}, \\ \left[\bar{\mathbf{R}}(\mathbf{A}\mathbf{x}-\mathbf{b})\right]^T \widehat{\mathbf{R}}(\mathbf{x}-\mathbf{g}) = 0, \end{cases}$$

which guarantee the solution $\mathbf{x}$ to be also a fixed point of projected multigrid iterations. In general, using the standard restriction $\mathbf{R}$ as $\bar{\mathbf{R}}$ does not satisfy these conditions. In practice, one can observe that geometric and algebraic multigrid methods often fail to converge with the standard restriction, so that we adopt the one-sided restriction operator for $\bar{\mathbf{R}}$.

In order to satisfy the constraint $\mathbf{x} \geq \mathbf{g}$ at each coarse grid point, the restriction $\widehat{\mathbf{R}}$ for the constraint has to be the *injection operator*. This restriction operator copies the C-point values to the coarse grid and does not add any contribution from F-point values. We also choose for $\widehat{\mathbf{R}}$ the injection operator.

Coarse grid matrix. We use the Galerkin coarsening, that is, the coarse grid matrix $\mathbf{A}^c$ is chosen to be

$$\mathbf{A}^c = \mathbf{R}\mathbf{A}\mathbf{P} = \mathbf{P}^T\mathbf{A}\mathbf{P}.$$

We use the standard prolongation and restriction operators, $\mathbf{P}$ and $\mathbf{R}$, in the definition of the Galerkin coarse grid matrices. The modified restriction operators, $\bar{\mathbf{R}}$ and $\widehat{\mathbf{R}}$, are solely used for the restriction of the residuals and the constraints for the LCP, respectively.

Smoother for LCPs. Our smoother for LCPs is based on the PGS method which is a point-wise smoothing method. In the case of grid anisotropies, or any other anisotropies in the engineering problems, the AMG coarsening strategy should automatically, based on the rules regarding the strong connections, detect strong coupling and some form of semi-coarsening should automatically take place within AMG. This is one of the strong points we aim for here, as this will give rise to efficient algebraic coarsening strategies on non-uniform stretched grids that are often encountered in (financial) engineering applications.

4. Numerical experiments

In this section we give a couple of motivating examples to show the quality of PAMG for problems on regular finite difference and finite element grids. The third example is from computational finance and deals with a stretched grid. The performance of PAMG on these examples gives some insight in our choice of the PAMG components.

4.1. Elasto-plastic torsion problem

For an elastic rod with a crosscut Ω under a twist, a stress function u satisfies the LCP:

$$\begin{cases} -\Delta u \geq -2C & \text{in } \Omega, \\ u \geq -d(\mathbf{x}, \partial\Omega) & \text{in } \Omega, \\ (\Delta u - 2C)(u + d(\mathbf{x}, \partial\Omega)) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $d(\mathbf{x}, \partial\Omega)$ is the distance from the boundary $\partial\Omega$ at $\mathbf{x}$. In the region where $-\Delta u = -2C$ is satisfied, the rod is elastic while in the region where $-u = d(\mathbf{x}, \partial\Omega)$ the rod behaves plastically. In the following, we consider this problem with the coefficient $C = 10$. For the iterations, we use the stopping criterion

$$\|\bar{\mathbf{r}}^{(k)}\|_\infty \leq 10^{-5}\|\mathbf{b}\|_\infty,$$

where $\bar{\mathbf{r}}^{(k)}$ is the reduced residual defined by:

$$\bar{\mathbf{r}} = \begin{cases} [\mathbf{A}\mathbf{x} - \mathbf{b}]_i & \text{if} \quad \mathbf{x}_i > \mathbf{g}_i, \\ 0 & \text{if} \quad \mathbf{x}_i = \mathbf{g}_i. \end{cases} \tag{4.2}$$

We report the convergence rates for the iterations defined by

$$\exp\left(\frac{1}{k}\log\frac{\|\mathbf{r}^{(k)}\|_\infty}{\|\mathbf{r}^{(0)}\|_\infty}\right).$$

Two tests have been performed for this equation, in a square with finite differences and in an ellipse with finite elements.

Square. Following [22], we have chosen the first example the crosscut Ω to be the square $[0,1]^2$. For the discretization of the Laplace operator, we use the standard five-point finite difference stencil. The stress function u and the plastic region are shown in Fig. 1. We study the convergence of the iteration with multigrid V- and F-cycles. Furthermore, we compare the use of one and two pre-smoothing and post-smoothing sweeps. Mainly, we performed the smoothing by the PGS relaxation method, but we also studied the convergence with an overrelaxation parameter in one test. In the latter case, the smoother is the PSOR method.

Convergence results for the standard prolongation and for the one-sided prolongation in (3.1), as suggested by Brandt and Cryer, are reported in Tables 1 and 2, respectively. With the V(1,1)-cycle the PAMG method failed to converge in a few hundreds of iterations on the finest grid 513×513 with the standard prolongation. The number of iterations grows with the V-cycle whereas with the F-cycle PAMG scales well, with both choices for the
prolongation operator. We therefore choose the standard prolongation in the experiments to follow.

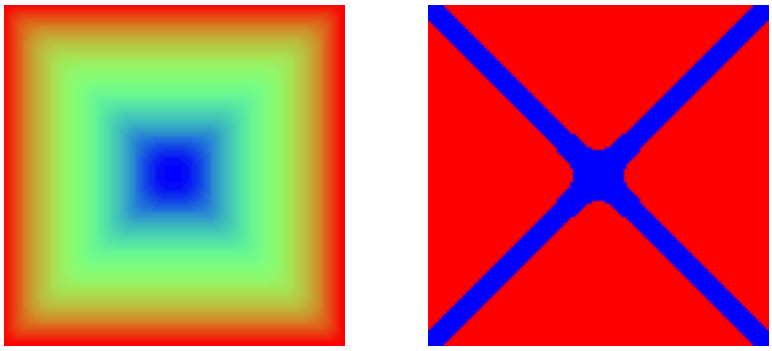

Figure 1: The solution u in the square (left) and the corresponding plastic region, that is, the active set (red, right).

Table 1: The number of iterations and the convergence rate with the standard prolongation for the elasto-plastic torsion problem in the square.

	V(1,1)		V(2,2)		F(1,1)		F(2,2)	
grid	iter	rate	iter	rate	iter	rate	iter	rate
129×129	164	0.93	15	0.46	7	0.19	6	0.10
257×257	76	0.86	20	0.55	15	0.45	9	0.26
513×513		1.00	29	0.67	11	0.32	7	0.16

Table 2: The number of iterations and the convergence rate with the one-sided prolongation by Brandt and Cryer for the elasto-plastic torsion problem in the square.

	V(1,1)		V(2,2)		F(1,1)		F(2,2)	
grid	iter	rate	iter	rate	iter	rate	iter	rate
129×129	35	0.72	13	0.40	7	0.17	5	0.08
257×257	37	0.73	17	0.51	11	0.35	8	0.22
513×513	146	0.92	24	0.61	8	0.24	6	0.13

Table 3: The number of iterations and the convergence rate with the one-sided prolongation by Brandt and Cryer and overrelaxation for the elasto-plastic torsion problem in the square.

	V(1,1) $\omega = 1.2$		V(2,2) $\omega = 1.3$		F(1,1) $\omega = 1.2$		F(2,2) $\omega = 1.3$	
grid	iter	rate	iter	rate	iter	rate	iter	rate
129×129	18	0.52	9	0.26	10	0.29	7	0.14
257×257	25	0.62	12	0.37	10	0.30	7	0.18
513×513	42	0.76	14	0.43	10	0.28	6	0.14

Table 3 gives the results for the V- and F-cycles with PSOR used as smoother. The overrelaxation parameter ω is chosen in an optimal way from the set $\{1, 1.1, 1.2, \cdots, 2\}$. Overrelaxation is especially beneficial for the V-cycle results, as expected. V-cycles may have an unsatisfactory low-frequency error treatment for LCPs and PSOR may help because overrelaxation can reduce low-frequency error components. The maximum number of

Table 4: The number of iterations and the convergence rate with the standard prolongation for the elasto-plastic torsion problem in the ellipse.

	PGS	V(1,1)		V(2,2)		F(1,1)		F(2,2)	
mesh	iter	iter	rate	iter	rate	iter	rate	iter	rate
861	430	15	0.46	9	0.24	8	0.21	6	0.14
3277	1751	16	0.48	9	0.25	10	0.30	6	0.14
12803	6990	17	0.50	10	0.28	8	0.23	6	0.12
50576	28070	31	0.68	12	0.38	10	0.30	8	0.19
200979		32	0.70	14	0.41	10	0.31	7	0.19

iterations reduces only by one for both F(1,1)- and F(2,2)-cycles.

Based on these convergence results, we prefer to stay with F-cycles and PGS ($\omega = 1$) in the experiments to follow. The improved performance of F-cycles compared to V-cycles can be understood because the injection operator $\widehat{\mathbf{R}}$ forms a basic ingredient of PAMG. The coarse grid correction should therefore be made more robust, for example by using F-cycles.

Ellipse. As a second example, we consider the elasto-plastic torsion problem in an ellipse

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^2 \ : \ \frac{1}{4}x_1^2 + x_2^2 < 1 \right\}.$$

For discretization, we use linear finite elements on quasi-uniform triangulations. We show here that PAMG also performs well on triangular meshes. These discretizations also lead to M-matrices. The stress function u and the plastic region are shown in Fig. 2. Fig. 3 shows an example of coarser grids constructed by the AMG coarsening. Convergence results for the standard prolongation and PGS smoother are reported in Table 4. Also, the number of iterations with the PGS method is given in that table.

4.2. Pricing American options

We consider pricing American put options under Heston's stochastic volatility model [11]. The values of options are given by the point values of the solution u of the time-dependent LCP

$$\begin{cases} \mathscr{L}u \geq 0 & \text{in } \Omega \\ u \geq g & \text{in } \Omega \\ (\mathscr{L}u)(u-g) = 0 & \text{in } \Omega, \end{cases}$$

where $\Omega = (0, x_1^{\max}) \times (0, x_2^{\max}) \times (0, T]$. The parabolic partial differential operator $\mathscr{L}$ is defined by

$$\begin{aligned} \mathscr{L}u = u_\tau + \frac{1}{2}x_2 x_1^2 u_{x_1x_1} + \rho\gamma x_2 x_1 u_{x_1x_2} + \frac{1}{2}\gamma^2 x_2 u_{x_2x_2} \\ + r x_1 u_{x_1} + \alpha(\beta - x_2)u_{x_2} - ru, \end{aligned}$$

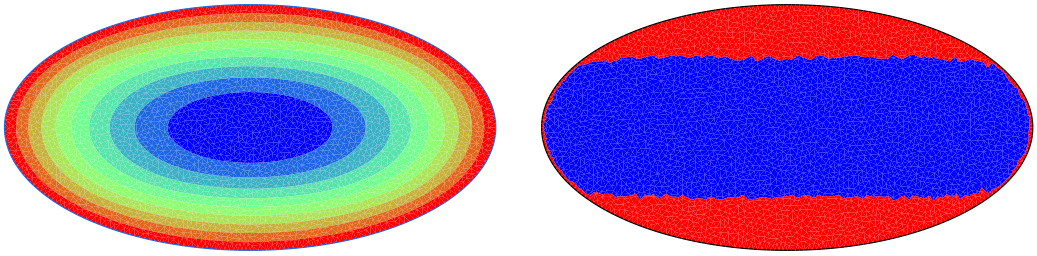

Figure 2: The solution u in the ellipse on the mesh with 3277 nodes (left) and the corresponding plastic region, that is, the active set (red, right).

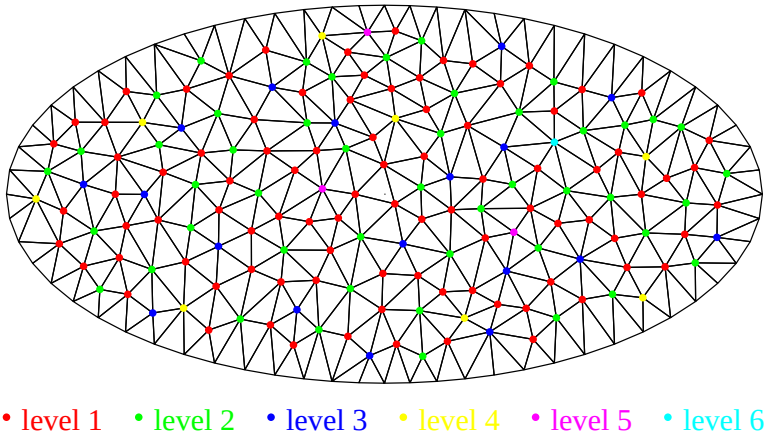

Figure 3: The different levels constructed by the AMG coarsening on a mesh with 231 nodes. The kth finest grid includes the levels from k to 6.

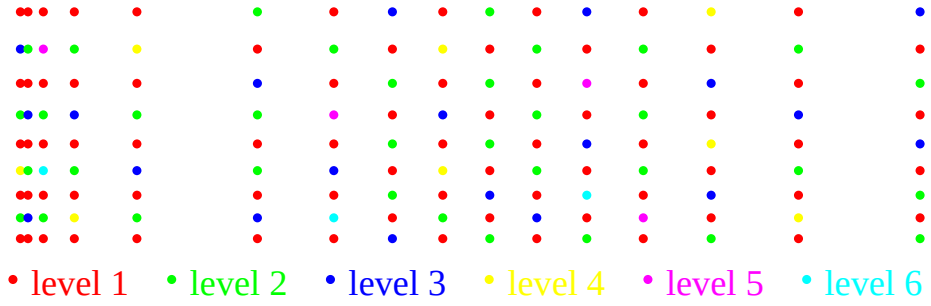

Figure 4: The different levels constructed by the AMG coarsening on a 17×9 grid. The kth finest grid includes the levels from k to 6.

Table 5: The average number of iterations, the errors at the 10 reference points, the ratios of consecutive errors, and the CPU times in seconds.

method	space time grid	iter.	error	ratio	CPU
PAMG	$64 \times 32 \times 34$	1.2	0.002361		0.06
	$128 \times 64 \times 66$	1.6	0.000747	3.16	0.55
	$256 \times 128 \times 130$	2.0	0.000428	1.74	5.65
	$512 \times 256 \times 258$	2.0	0.000112	3.83	45.59
PMG	$64 \times 32 \times 34$	1.9	0.002443		0.06
	$128 \times 64 \times 66$	2.0	0.000900	2.71	0.56
	$256 \times 128 \times 130$	2.5	0.000426	2.11	5.78
	$512 \times 256 \times 258$	3.0	0.000108	3.96	58.06

where x_1 is the value of the underlying asset, x_2 is the variance, and τ is the time to expiry T. The volatility of the variance process is γ and the variance will drift back to a mean value $\beta > 0$ at a rate $\alpha > 0$. The correlation between the value and variance processes is ρ. The risk-free interest rate is r. For a put option with a strike price K, the payoff function reads

$$g = \max\{K - x_1, 0\}.$$

We price options under the parameter values

$$K = 10, \;\; T = 0.25, \;\; r = 0.1, \;\; \alpha = 5, \;\; \beta = 0.16, \;\; \gamma = 0.9, \;\; \text{and} \;\; \rho = 0.1$$

which have been used also in [5, 15–17, 22, 29].

The boundary conditions are given by

$$\begin{aligned} u(0, x_2) &= K, && 0 < x_2 < x_2^{\max}, \\ u_{x_1}(x_1^{\max}, x_2) &= 0, && 0 < x_2 < x_2^{\max}, \\ u_{x_2}(x_1, x_2^{\max}) &= 0, && 0 < x_1 < x_1^{\max}. \end{aligned}$$

With the parameter values chosen, it is not necessary to pose a boundary condition on the $(0, x_1^{\max}) \times \{0\}$ boundary as $\mathscr{L}$ reduced to a first-order operator with the characteristic curve pointing outward.

Similarly to [15] and [22], we truncate domain so that $x_1^{\max} = 20$ and $x_2^{\max} = 1$. We use the finite difference discretizations with non-uniform space steps described in [15] and the Rannacher time-stepping [23]. The time steps are uniform except the first four time steps taken, using the implicit Euler method with the time step being half of that of the other steps.

Table 5 reports results for the PAMG method and also for the geometric projected multigrid (PMG) method, described in [15]. The triplet (m, n, l) represents the number of steps in the x_1, x_2, and τ directions, respectively. The reported error is the l_2-norm of the error at the reference points $\{8, 9, 10, 11, 12\} \times \{0.0625, 0.25\}$. The ratio in the table is the ratio of consecutive errors.

Here we use the PAMG method with the multigrid V-cycle. When moving downwards and upwards the smoother is one PGS iteration over all points, followed by one PGS iteration over F-points only. The stopping criterion for multigrid methods is

$$\|\bar{\mathbf{r}}^{(k)}\|_2 \leq \frac{0.02}{m\,n}\|\mathbf{b}\|_2, \tag{4.3}$$

where $\mathbf{b}$ is the right-hand side vector and $\bar{\mathbf{r}}$ is the reduced residual vector, defined in (4.2). The number of iterations presented is the average number of iterations per time step. The main reason for the increase in the number of iterations when the grid is refined is stopping criterion (4.3) which becomes more strict then. The runs have been performed on a PC with 3.8 GHz Xeon processor, using Fortran implementations. Fig. 4 shows coarser grids constructed by the AMG coarsening.

5. Conclusions

We presented a PAMG method for solving iteratively linear complementarity problems based on the AMG coarsening technique described by Ruge and Stüben in [25]. This is an easy-to-use robust and efficient black-box solver which only requires the matrix and vectors defining the problem. An underlying assumption is that the matrix is an M-matrix or nearly so.

Numerical experiments demonstrated the PAMG method to be robust and efficient for partial differential operators discretized using structured and unstructured grids/meshes. In experiments with the elasto-plastic torsion problems the method with F-cycle appears to be scalable, that is, the number of iterations seem to be bounded when discretizations became finer. Experiments pricing American options under a stochastic volatility model showed the PAMG method to be faster than a geometric multigrid tailored for the problems. This demonstrates that PAMG is easy-to-use and efficient for pricing American-style options under multi-factor financial models.

References

[1] Y. Achdou and O. Pironneau, *Computational Methods for Option Pricing*, vol. 30 of Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 2005.

[2] A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp., 31 (1977), pp. 333–390.

[3] A. Brandt and C. W. Cryer, *Multigrid algorithms for the solution of linear complementarity problems arising from free boundary problems*, SIAM J. Sci. Statist. Comput., 4 (1983), pp. 655–684.

[4] W. L. Briggs, V. E. Henson, and S. F. McCormick, *A multigrid tutorial*, SIAM, Philadelphia, PA, second ed., 2000.

[5] N. Clarke and K. Parrott, *Multigrid for American option pricing with stochastic volatility*, Appl. Math. Finance, 6 (1999), pp. 177–195.

[6] A. J. Cleary, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick, G. N. Miranda, and J. W. Ruge, *Robustness and scalability of algebraic multigrid*, SIAM J. Sci. Comput., 21 (2000), pp. 1886–1908.

[7] C. W. Cryer, *The solution of a quadratic programming problem using systematic overrelaxation*, SIAM J. Control, 9 (1971), pp. 385–392.
[8] H. De Sterck, U. Meier Yang, and J. J. Heys, *Reducing complexity in parallel algebraic multigrid preconditioners*, SIAM J. Matrix Anal. Appl., 27 (2006), pp. 1019–1039.
[9] C. M. Elliott and J. R. Ockendon, *Weak and variational methods for moving boundary problems*, vol. 59 of Research Notes in Mathematics, Pitman, Boston, MA, 1982.
[10] V. E. Henson and U. Meier Yang, *BoomerAMG: a parallel algebraic multigrid solver and preconditioner*, Appl. Numer. Math., 41 (2002), pp. 155–177.
[11] S. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Rev. Financial Stud., 6 (1993), pp. 327–343.
[12] R. Hoppe and R. Kornhuber, *Multi-grid methods for two phase Stefan problem*, Tech. Report 171, TU Berlin, 1987.
[13] R. H. W. Hoppe, *Multigrid algorithms for variational inequalities*, SIAM J. Numer. Anal., 24 (1987), pp. 1046–1065.
[14] J. Huang and J.-S. Pang, *Option pricing and linear complementarity*, J. Comput. Finance, 2 (1998), pp. 31–60.
[15] S. Ikonen and J. Toivanen, *Efficient numerical methods for pricing American options under stochastic volatility*, Numer. Methods Partial Differential Equations, 24 (2008), pp. 104–126.
[16] ———, *Operator splitting methods for pricing American options under stochastic volatility*, Numer. Math., 113 (2009), pp. 299–324.
[17] K. Ito and J. Toivanen, *Lagrange multiplier approach with optimized finite difference stencils for pricing American options under stochastic volatility*, SIAM J. Sci. Comput., 31 (2009), pp. 2646–2664.
[18] P. Jaillet, D. Lamberton, and B. Lapeyre, *Variational inequalities and the pricing of American options*, Acta Appl. Math., 21 (1990), pp. 263–289.
[19] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities. I*, Numer. Math., 69 (1994), pp. 167–184.
[20] A. Krechel and K. Stüben, *Operator dependent interpolation in algebraic multigrid*, in Multigrid methods V, vol. 3 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 1998, pp. 189–211.
[21] ———, *Parallel algebraic multigrid based on subdomain blocking*, Parallel Comput., 27 (2001), pp. 1009–1031.
[22] C. W. Oosterlee, *On multigrid for linear complementarity problems with application to American-style options*, Electron. Trans. Numer. Anal., 15 (2003), pp. 165–185.
[23] R. Rannacher, *Finite element solution of diffusion problems with irregular data*, Numer. Math., 43 (1984), pp. 309–327.
[24] C. Reisinger and G. Wittum, *On multigrid for anisotropic equations and variational inequalities: Pricing multi-dimensional European and American options*, Comput. Vis. Sci., 7 (2004), pp. 189–197.
[25] J. W. Ruge and K. Stüben, *Algebraic multigrid*, in Multigrid methods, vol. 3 of Frontiers Appl. Math., SIAM, Philadelphia, PA, 1987, pp. 73–130.
[26] K. Stüben, *Algebraic multigrid: An introduction with applications*, in Multigrid, Academic Press Inc., San Diego, CA, 2001.
[27] U. Trottenberg, C. W. Oosterlee, and A. Schüller, *Multigrid*, Academic Press Inc., San Diego, CA, 2001.
[28] P. Wilmott, J. Dewynne, and S. Howison, *Option pricing: mathematical models and computation*, Oxford Financial Press, Oxford, 1993.
[29] R. Zvan, P. A. Forsyth, and K. R. Vetzal, *Penalty methods for American options with stochastic volatility*, J. Comput. Appl. Math., 91 (1998), pp. 199–218.