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Review Article

A Review of Unified A Posteriori Finite Element Error Control

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Abstract. This paper aims at a general guideline to obtain a posteriori error estimates for the finite element error control in computational partial differential equations.
In the abstract setting of mixed formulations, a generalised formulation of the corresponding residuals is proposed which then allows for the unified estimation of the respective dual norms. Notably, this can be done with an approach which is applicable in the same way to conforming, nonconforming and mixed discretisations. Subsequently, the unified approach is applied to various model problems. In particular, we consider the Laplace, Stokes, Navier-Lamé, and the semi-discrete eddy current equations.

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1. Introduction

Numerical simulation in engineering and science involves all kinds of errors ranging from the modeling of the problem to round-off errors. We are concerned with problems that can be formulated as the linear equation

$$\mathscr{A}(p,u)=\ell$$

in function spaces Q and V with $\mathscr{A}(p,u)$ and ℓ in $(Q\times V)^*$. This paper is devoted to the control of the discretisation error that arises from the fact that the (unknown) exact solution $(p,u)\in Q\times V$ is approximated by a discrete solution (p_ℓ,u_ℓ) computed in a finite dimensional vector space $Q_\ell\times V_\ell$. The aim of a posteriori error control is the computation and justification of lower and upper error bounds for the unknown discretisation error $e:=(p,u)-(p_\ell,u_\ell)$. Apart from the standard case that the discrete spaces are subspaces of their infinite dimensional counterparts, we will also consider a violation of this inclusion in the case of non-conforming methods.

This paper is organized as follows. After some basic notations and definitions in this introductory section, a unifying formulation for different problem classes is introduced and applied to various examples in Section 2. Section 3 about the basic concepts of residual type error estimation is followed by a brief introduction to finite element spaces and interpolation operators in Section 4. The theoretical part is concluded by a compilation of crucial theorems in error estimation with proofs. In addition to the above statements, Sections 2 and 3 prepare applications discussed in detail in Sections 6 to 9.

Notation. In this paper, $a\lesssim b$ abbreviates $a\le Cb$ with some multiplicative mesh-size independent constant $C>0$ which only depends on the domain Ω and the shape (but not on the size) of finite element domains. Moreover, C is independent of crucial parameters of the partial differential equation (PDE) such as the Lamé parameter λ in the problem of linear elasticity below. Furthermore, $a\approx b$ abbreviates $a\lesssim b\lesssim a$.

Colon denotes the Euclidean scalar product of two matrices $A=(A_{jk}),B=(B_{jk})\in\mathbb{R}^{n\times n}$, that is, $A:B:=\sum_{j,k=1}^n A_{jk}B_{jk}$, the dyadic product of some vectors $a,b\in\mathbb{R}^n$ is denoted by $a\otimes b:=ab^T$ and the cross product of two vectors $a,b\in\mathbb{R}^3$ is written as $a\wedge b$. The space of symmetric matrices in $\mathbb{R}$ is defined by

$$\mathbb{R}^{n\times n}_{\text{sym}}:=\{A\in\mathbb{R}^{n\times n}:A=A^T\}.$$

There are different definitions of the differential operator curl which we use in the doc-

Table 1: Overview of the different definitions of the curl differential operator.

	$v:\Omega\subset\mathbb{R}^n\to\mathbb{R}$	$v:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$
$n=2$	$\operatorname{Curl} v=\begin{pmatrix}\partial_{x_2}v\\-\partial_{x_1}v\end{pmatrix}$,	$\operatorname{curl} v=\partial_{x_1}v_2-\partial_{x_2}v_1$
$n=3$	$\operatorname{Curl} v=\begin{pmatrix}\partial_{x_2}v-\partial_{x_3}v\\\partial_{x_3}v-\partial_{x_1}v\\\partial_{x_1}v-\partial_{x_2}v\end{pmatrix}$,	$\operatorname{curl} v=\begin{pmatrix}\partial_{x_2}v_3-\partial_{x_3}v_2\\\partial_{x_3}v_1-\partial_{x_1}v_3\\\partial_{x_1}v_2-\partial_{x_2}v_1\end{pmatrix}=\nabla\wedge v.$

ument. An overview is depicted in Table 1. Note that the curl of a function is always orthogonal to the gradient.

The Sobolev spaces of functions defined on a domain $\Omega\subset\mathbb{R}^n$ required for the formulation of the presented PDEs are as usual denoted by $H^k(\Omega)$ for the space of all functions in $L^2(\Omega)$ which allow weak derivation up to order k and

$$\begin{aligned}
H(\operatorname{div},\Omega) &:= \left\{v\in L^2(\Omega;\mathbb{R}^n)\;\middle|\;\operatorname{div} v=\nabla\cdot v\in L^2(\Omega)\right\},\\
H(\operatorname{curl},\Omega) &:= \left\{v\in L^2(\Omega;\mathbb{R}^3)\;\middle|\;\operatorname{curl} v=\nabla\wedge v\in L^2(\Omega;\mathbb{R}^3)\right\}.
\end{aligned}$$

Moreover, the spaces of functions with boundary conditions are written as

$$\begin{aligned}
L^2_0(\Omega) &:= \left\{q\in L^2(\Omega)\;\middle|\;\int_\Omega q\,dx=0\right\},\\
H^1_0(\Omega) &:= \left\{v\in H^1(\Omega)\;\middle|\;v|_{\partial\Omega}=0\right\},\\
H_0(\operatorname{div},\Omega) &:= \left\{v\in H(\operatorname{div},\Omega)\;\middle|\;v\cdot\nu|_{\partial\Omega}=0\right\},\\
H_0(\operatorname{curl},\Omega) &:= \left\{v\in H(\operatorname{curl},\Omega)\;\middle|\;(v\wedge\nu)|_{\partial\Omega}=0\right\}.
\end{aligned}$$

where ν is the unit exterior normal vector on the boundary $\partial\Omega$ of Ω.

Reliability and efficiency. The presented approach provides a general guideline in the derivation of reliable and efficient error estimators commonly denoted by η or μ. Any computable quantity in a numerical method is called *error estimator*. For the error e, we call an error estimator η *efficient*, if

$$\eta\lesssim\|e\|+\mathrm{hot}_{\mathrm{eff}},$$

and *reliable*, if

$$\|e\|\lesssim\eta+\mathrm{hot}_{\mathrm{rel}}.$$

Here "hot" denotes higher-order terms which usually are much smaller than η and the error e, and which tend to zero with decreasing mesh size much faster. However, in general this may depend on the (unknown) smoothness of the solution and the (known) smoothness of the data.

2. Well-posed continuous problems / unified notation of model problems

2.1. Generic mixed formulation

The problems in this text can be written in terms of linear algebra as follows. The sets Q and V are real vector spaces with norms $\|\cdot\|_Q$ and $\|\cdot\|_V$ and the operator $\mathscr{A}$ maps linearly from $Q \times V$ to its dual $(Q \times V)^*$. A linear operator $\mathscr{A}$ is continuous, if and only if it is bounded. It is bijective if, for any $\ell \in (Q \times V)^*$, the equation $\mathscr{A}(p,u) = \ell$ has a unique solution $(\bar{p},\bar{u})$. This defines an inverse mapping via $\mathscr{A}^{-1}\ell = (\bar{p},\bar{u})$. The linear problem to seek $(p,u) \in Q \times V$ with $\mathscr{A}(p,u) = \ell$ is called *well-posed* if it is uniquely solvable and if $\mathscr{A}$ and the inverse mapping $\mathscr{A}^{-1}$ are continuous.

In Section 3, we show in the context of error analysis by residual estimation, that well-posedness of the problem is also essential for error estimation, although the unique solution might not be known.

All applications considered can be formulated in a mixed setting. A unified representation is obtained in terms of the (bi)linear continuous forms

$$
\begin{aligned}
a &: Q \times Q \to \mathbb{R},\\
c &: V \times V \to \mathbb{R},\\
\ell_Q &: Q \to \mathbb{R},\\
\ell_V &: V \to \mathbb{R}
\end{aligned}
$$

plus a differential operator $\Lambda : V \to Q$. The space V always is a H^1, $H(\mathrm{div})$ or $H(\mathrm{curl})$ space, Q is usually an L^2 space and the bilinear form b is defined via

$$b : Q \times V \to \mathbb{R} \quad \text{with} \quad b(q,v) := a(q,\Lambda v) \quad \text{for all } (q,v) \in Q \times V.$$

With this, we define

$$\mathscr{A}(p,u)(q,v) := a(p,q) + b(p,v) - b(q,u) + c(u,v), \tag{2.1a}$$

$$\ell(q,v) := \ell_Q(q) + \ell_V(v). \tag{2.1b}$$

Then, the problem allows the following split: Seek $(p,u) \in Q \times V$ such that

$$\forall\, q \in Q \qquad a(p,q) - b(q,u) = \ell_Q(q), \tag{2.2a}$$

$$\forall\, v \in V \qquad b(p,v) + c(u,v) = \ell_V(v). \tag{2.2b}$$

2.2. Poisson problem

For a function $f \in L^2(\Omega)$, the *Poisson problem* reads: Seek $u \in H^1_0(\Omega)$ with

$$-\Delta u = f. \tag{2.3}$$

The Poisson equation arises in a variety of physical phenomena with the density u of some quantity in equilibrium, such as a chemical concentration, temperature or electrostatic potential.

In order to obtain error estimators of (2.3) with arbitrary data f, it is possible to examine the *mixed formulation* of the Poisson problem with two partial differential equations of first order instead of one equation of second order, namely, seek a tuple $(p,u) \in Q \times V := L^2(\Omega;\mathbb{R}^n) \times H_0^1(\Omega)$, such that

$$\nabla u = p \quad \text{and} \quad -\operatorname{div} p = f.$$

For the Poisson model equation, the mappings a, b, c, Λ and the right-hand sides ℓ_Q and ℓ_V from the definitions of the operator $\mathscr{A}$ and the right-hand side ℓ in (2.1) read, for all $p,q \in Q$ and $u,v \in V$,

$$\begin{aligned} a(p,q) &:= \int_\Omega p \cdot q \, dx, & b(q,u) &:= \int_\Omega q \cdot \nabla u \, dx, \\ c(u,v) &:= 0, & \Lambda u &:= \nabla u, \\ \ell_Q(q) &:= 0, & \ell_V(v) &:= \int_\Omega f v \, dx. \end{aligned}$$

Some function $u \in V$ solves the *weak form* of the mixed Poisson equation if and only if

$$\begin{aligned} &\forall\, q \in Q \quad a(p,q) - b(q,u) = \int_\Omega (p - \nabla u) \cdot q \, dx \overset{!}{=} 0 = \ell_Q(q), \\ &\forall\, v \in V \quad b(p,v) = \int_\Omega p \cdot \nabla v \, dx = -\int_\Omega \operatorname{div} p \, v \, dx \overset{!}{=} \int_\Omega f v \, dx = \ell_V(v). \end{aligned}$$

Hence, problem (2.3) is recast as: Seek $(p,u) \in Q \times V$ with $\mathscr{A}(p,u) = \ell_Q + \ell_V$.

Remark 2.1. The operator $\mathscr{A}$ belonging to the Poisson problem is bounded, linear and bijective: An inf-sup property can be shown immediately because for any $(p,u) \in Q \times V$, with $(q,v) := (p - \nabla u, 2u) \in Q \times V$ and the H^1 seminorm on $V = H_0^1(\Omega)$, it holds

$$\begin{aligned} &1/5 \|(p,u)\|_{Q\times V} \|(q,v)\|_{Q\times V} \\ \le& 1/5 \left(\|p\|_Q + \|u\|_V\right) \left(\|p\|_Q + 3\|u\|_V\right) \\ \le& \|p\|_Q^2 + \|u\|_V^2 = \left(\mathscr{A}(p,u)\right)(q,v). \end{aligned}$$

Thus, the (generalized) Lax-Milgram lemma yields bijectivity of $\mathscr{A}$, cf. [12,14].

2.3. Stokes problem

Stationary incompressible fluid flow in a two- or three-dimensional domain can be modeled by the *non-symmetric Stokes equations*

$$\begin{aligned} -\Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

for a velocity field u and a pressure p on a domain $\Omega \subset \mathbb{R}^n$ ($n \in \{2,3\}$). The equations are naturally stated in a mixed form with a vector valued solution $u \in V := H_0^1(\Omega;\mathbb{R}^n)$ and with the pressure $p \in Q := L_0^2(\Omega)$. In the *symmetric formulation* of the *Stokes equations*, one uses the symmetric part of the gradient of a vector valued function u written as

$$\varepsilon(u) := \frac{1}{2}(\nabla u + \nabla u^T) \in L^2\left(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}\right).$$

The *symmetric Stokes problem with viscosity parameter* $\mu > 0$ reads: Find u and p with

$$\begin{aligned} -\operatorname{div}\mu\varepsilon(u) + \nabla p &= f && \text{in } \Omega,\\ \operatorname{div} u &= 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The two formulations are equivalent in case $\mu = 1$ assuming homogeneous boundary conditions for the velocity on the entire boundary $\partial\Omega$, cf. [12, 15].

The corresponding mappings read

$$\begin{aligned} a(p,q) &:= -\int_\Omega pq\,dx, & b(q,v) &:= -\int_\Omega q \operatorname{div} v\,dx,\\ c_{\mathrm{sym}}(u,v) &:= \int_\Omega 2\mu\,\varepsilon(u) : \varepsilon(v)\,dx, & &\\ c_{\mathrm{asym}}(u,v) &:= \int_\Omega \nabla u : \nabla v\,dx, & \Lambda u &:= \operatorname{div} u,\\ \ell_Q(p,q) &:= -\int_\Omega pq\,dx, & \ell_V(v) &:= \int_\Omega f\cdot v\,dx. \end{aligned}$$

Note that only for formal reasons ℓ_Q is a bilinear form which depends on the solution p as well as on the test function q. It cancels out in the mixed formulation with $a(p,q)$. In the variational form, the symmetric Stokes equation reads: Given $f \in L^2(\Omega;\mathbb{R}^n)$, seek $(u,p) \in V \times Q$ such that

$$\forall\, q \in Q \quad -b(q,u) = \int_\Omega q \operatorname{div} u\,dx \stackrel{!}{=} 0, \tag{2.4a}$$

$$\begin{aligned} \forall\, v \in V \quad b(p,v) + c_{\mathrm{sym}}(u,v) &= \int_\Omega 2\mu\varepsilon(u) : \varepsilon(v)\,dx - \int_\Omega p \operatorname{div} v\,dx\\ &\stackrel{!}{=} \int_\Omega f\cdot v\,dx. \end{aligned} \tag{2.4b}$$

The variational unsymmetric Stokes equations are identical with c_{asym} instead of c_{sym}.

Remark 2.2. The Stokes equations (2.4) exhibit a unique solution (p,u), cf. [12,14,18,23].

2.4. Lamé problem

In linear elasticity theory, the displacement u and the symmetric stress tensor σ of a body Ω under the influence of applied forces f satisfy the *Navier-Lamé equations*. The linear stress-strain relation $\mathbb{C} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ reads

$$\mathbb{C}E := \lambda\, \mathrm{tr}(E)\, \mathbb{1} + 2\mu E \quad \text{for strain matrices } E \in \mathbb{R}^{n\times n}$$

with Lamé parameters λ, $\mu > 0$. The inverse relation reads

$$\mathbb{C}^{-1}\sigma = 1/(2\mu)\,\sigma + \lambda/(2\mu(n\lambda + 2\mu))\, \mathrm{tr}(\sigma)\, \mathbb{1} \quad \text{for stress matrices } \sigma \in \mathbb{R}^{n\times n}.$$

In the continuous model with the stress-strain relation $\sigma = \mathbb{C}\varepsilon(u)$, the resulting model problem reads: Given $f \in L^2(\Omega;\mathbb{R}^n)$, find $u \in H^1_0(\Omega;\mathbb{R}^n)$ such that

$$f + \mathrm{div}\, \mathbb{C}\, \varepsilon(u) = 0 \quad \text{in } \Omega. \tag{2.5}$$

To employ the unified theory for residuals with respect to discrete approximations of the solution, let $V := H^1_0(\Omega;\mathbb{R}^n)$, $Q := L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and set, for σ, $\tau \in Q$ and for u, $v \in V$,

$$\begin{aligned} a(\sigma,\tau) &:= \int_\Omega (\mathbb{C}^{-1}\sigma) : \tau\, dx, & b(\tau,u) &:= \int_\Omega \tau : \varepsilon(u)\, dx,\\ c(u,v) &:= 0, & \Lambda(u) &:= \mathbb{C}\varepsilon(u),\\ \ell_V(v) &:= \int_\Omega f \cdot v\, dx, & \ell_Q(\tau) &:= 0. \end{aligned}$$

A pair of functions $\sigma \in Q$ and $u \in V$ solves the mixed formulation of the Lamé problem if

$$\begin{aligned} &\forall\, \tau \in Q \qquad a(\sigma,\tau) - b(\tau,u) = \int_\Omega (\mathbb{C}^{-1}\sigma - \varepsilon(u)) : \tau\, dx \overset{!}{=} 0,\\ &\forall\, v \in V \qquad b(\sigma,v) = \int_\Omega \sigma : \varepsilon(v)\, dx = -\int_\Omega \mathrm{div}\,\sigma \cdot v\, dx \overset{!}{=} \int_\Omega f \cdot v\, dx = \ell_V(v). \end{aligned}$$

Remark 2.3. The elliptic PDE (2.5) has a unique solution u, see [27]. The operator $\mathscr{A} : (Q\times V) \to (Q\times V)^*$ from (2.1) is linear, bounded and bijective, and the operator norms of $\mathscr{A}$ and $\mathscr{A}^{-1}$ are λ-independent, cf. [13].

Remark 2.4. The method which is used to solve the problem depends on the representation of the problem. In Section 8, some mixed methods vary the spaces Q and V, and the mappings $\mathscr{A}$ and ℓ.

2.5. Eddy current problem

We consider the semi-discrete *eddy current equations* [36]

$$\operatorname{curl}\mu^{-1}\operatorname{curl}u + \sigma u = f \qquad \text{in } \Omega, \tag{2.6a}$$

$$u \wedge \nu = 0 \qquad \text{on } \partial\Omega \tag{2.6b}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with data $f \in L^2(\Omega;\mathbb{R}^3)$ and parameters μ, $\sigma > 0$. The variational form of (2.6) reads: Seek $u \in H_0(\operatorname{curl},\Omega)$ such that

$$\int_\Omega \left(\mu^{-1}\operatorname{curl}u \cdot \operatorname{curl}v + \sigma u \cdot v\right) dx = \int_\Omega f \cdot v\, dx \qquad \text{for all } v \in H_0(\operatorname{curl},\Omega).$$

The mixed formulation of (2.6) reads

$$\mu p - \operatorname{curl}u = 0 \quad \text{in } \Omega,$$
$$\operatorname{curl}p + \sigma u = f \quad \text{in } \Omega.$$

In accordance with our unified notation from Section 2.1 we set $V := H_0(\operatorname{curl},\Omega)$, $Q := L^2(\Omega;\mathbb{R}^3)$ and, for all $p, q \in Q$ and $u, v \in V$,

$$a(p,q) := \int_\Omega \mu p \cdot q\, dx, \qquad b(q,u) := \int_\Omega \operatorname{curl}u \cdot q\, dx, \tag{2.7a}$$

$$c(u,v) := \int_\Omega \sigma u \cdot v\, dx, \qquad \Lambda(u) := \mu^{-1}\operatorname{curl}u, \tag{2.7b}$$

$$\ell_Q(q) := 0, \qquad \ell_V(v) := \int_\Omega f \cdot v\, dx. \tag{2.7c}$$

The associated mixed formulation reads: Seek $(p,u) \in Q \times V$ such that

$$\forall q \quad \in Q a(p,q) - b(q,u) = \int_\Omega (\mu p - \operatorname{curl}u) \cdot q\, dx \overset{!}{=} 0,$$

$$\forall v \quad \in V b(p,v) + c(u,v) = \int_\Omega \operatorname{curl}v \cdot p + \sigma u \cdot v\, dx \overset{!}{=} \int_\Omega f \cdot v\, dx.$$

Remark 2.5. The operator $\mathscr{A} : (Q\times V) \to (Q\times V)^*$ defined by (2.1) and (2.7) is continuous, linear and bijective, and thus allows a bounded inverse, cf. [22].

3. Errors and residuals

3.1. Concept

For the exact solution $(p,u) \in Q \times V$ of a well-posed problem (2.2) and some approximation $(p_\ell, u_\ell) \in Q_\ell \times V_\ell$, the error e is defined as

$$e := (p,u) - (p_\ell, u_\ell) = (p - p_\ell, u - u_\ell). \tag{3.1}$$

Note that both, (p,u) and (p_ℓ,u_ℓ) map from Ω to some $\mathbb{R}$-vector-space. Thus, e is well defined even if the approximation p_ℓ or u_ℓ does not belong to the space Q or V, respectively. In general, it is of course not possible to calculate the exact error e, since the exact solution (p,u) would read

$$(p,u) = e + (p_\ell,u_\ell).$$

For non-conforming discretisations, since (p_ℓ,u_ℓ) does not need to belong to $Q\times V$, the error norm $\|e\|_{Q\times V}$ may not be well-defined and hence can not be estimated properly. The split (p,u) is always done in such a way that $Q_\ell\subset Q$. However, it might be that $V_\ell\not\subset V$. In order to allow for an estimation of the error, u_ℓ thus may have to be further approximated by some $\tilde{u}_\ell\in V$. We will discuss the strategy to find an appropriate $\tilde{u}_\ell$ in Section 5.1 by means of the central Theorem 5.1.

For a solution (p,u) and an approximation $(p_\ell,\tilde{u}_\ell)\in Q\times V$, the residual $\mathscr{R}es$ measures the image of $(p-p_\ell,u-\tilde{u}_\ell)$ under $\mathscr{A}$,

$$\mathscr{R}es := \mathscr{A}(p-p_\ell,u-\tilde{u}_\ell) = \mathscr{A}(p,u)-\mathscr{A}(p_\ell,\tilde{u}_\ell)\in(Q\times V)^*. \tag{3.2}$$

With practical calculations in mind, one aims at an estimation of the residual (in its operator norm) by some a posteriori error estimators which are reliable and efficient. The continuity and boundedness of $\mathscr{A}$ is inherited to $\mathscr{R}es$ and therefore, the error $(p-p_\ell,u-\tilde{u}_\ell)$ is equivalent to the residual,

$$\|\mathscr{R}es\|_{(Q\times V)^*}\approx\|(p-p_\ell,u-\tilde{u}_\ell)\|_{Q\times V}.$$

Recall that (p,u) solves $\mathscr{A}(p,u)=\ell_Q+\ell_V$. The identity (2.1) for $\mathscr{A}$ allows us to write the residual $\mathscr{R}es$ in terms of a, b, c, ℓ_Q and ℓ_V as

$$\mathscr{R}es(q,v)=\ell_Q(q)+\ell_V(v)-a(p_\ell,q)-b(p_\ell,v)+b(q,\tilde{u}_\ell)-c(\tilde{u}_\ell,v).$$

This is the sum of the partial residuals $\mathscr{R}es_Q\in Q^*$ and $\mathscr{R}es_V\in V^*$, namely

$$\mathscr{R}es_Q=\ell_Q-a(p_\ell,\cdot)+b(\cdot,\tilde{u}_\ell)=\ell_Q-a(p_\ell-\Lambda\tilde{u}_\ell,\cdot)\in Q^*, \tag{3.3a}$$

$$\mathscr{R}es_V=\ell_V-b(p_\ell,\cdot)-c(\tilde{u}_\ell,\cdot)\in V^*. \tag{3.3b}$$

The well-posedness and boundedness of $\mathscr{A}$ implies

$$\|p-p_\ell\|_Q+\|u-\tilde{u}_\ell\|_V\approx\|\mathscr{R}es_Q\|_{Q^*}+\|\mathscr{R}es_V\|_{V^*}.$$

In case $\ell_Q=0$ and if Q is a Hilbert space with scalar product $a(\cdot,\cdot)$, the fact that $\mathscr{R}es_Q\in Q^*$ implies

$$\|\mathscr{R}es_Q\|_{Q^*}=\|p_\ell-\Lambda\tilde{u}_\ell\|_Q.$$

In Sections 6-9, error estimators for the different problems defined above will be derived. For this, a consistency residual $\mathscr{R}es_{\mathrm{cons}}$ and an equilibrium residual $\mathscr{R}es_{\mathrm{eq}}$ are defined and analysed in each case. Usually these coincide with the residuals $\mathscr{R}es_V$ and $\mathscr{R}es_Q$ with a few terms swapped to either of the residuals $\mathscr{R}es_{\mathrm{cons}}$ or $\mathscr{R}es_{\mathrm{eq}}$ whenever it seems more natural (cf. Sections 3.3-3.6 below).

Remark 3.1. This paper does neither aim at a convergence analysis nor at quasi-optimality of adaptive FEM. We note that these issues have been initiated in [31] and subsequently studied in [10,42] and [26].

3.2. Hilbert space case

The following Theorem 3.1 gives a characterisation for $\mathscr{A}$ being a scalar product on $\mathscr{H} = Q \times V$. This is the condition for the result in Theorem 3.2 where we derive a perturbation result which is illustrative to consider although it is limited to certain applications only. In their mixed formulation, all applications in this paper have $b \neq 0$, which means, with the first part of Theorem 3.1 in mind, that Theorem 3.2 cannot be applied. However, since condition $b(\Lambda u, v) = b(\Lambda v, u)$ in the second part of Theorem 3.1 is always satisfied, Theorem 3.2 can be applied in all conforming non-mixed finite element methods in this paper.

Theorem 3.1.

1. *A bilinear form* $\mathscr{A} : (Q \times V) \times (Q \times V) \to \mathbb{R}$ *defined as in* (2.1) *with bilinear forms* a, b *and* c *is symmetric if and only if the mappings* a *and* c *are symmetric and if it holds* $b = 0$.
2. *A bilinear form* $\mathscr{B} : V \times V \to \mathbb{R}$ *defined by* $\mathscr{B}(u, v) = \mathscr{A}(\Lambda u, u)(\Lambda v, v)$ *with bilinear forms* a, b *and* c *is symmetric if and only if* a *and* c *are symmetric and if for all* $u, v \in V$ *it holds* $b(\Lambda u, v) = b(\Lambda v, u)$.

Proof. If $\mathscr{A}$ is symmetric, then for $p = q = 0 \in Q$ and for all $u, v \in V$ it holds

$$c(u, v) = \mathscr{A}(p, u)(q, v) = \mathscr{A}(q, v)(p, u) = c(v, u),$$

which means c is symmetric. Furthermore, for all $p, q \in Q$ and $u = v = 0 \in V$ it holds

$$a(p, q) = \mathscr{A}(p, u)(q, v) = \mathscr{A}(q, v)(p, u) = a(q, p).$$

Thus, a is symmetric. Using the fact that $\mathscr{A}$, a, b and c are bilinear, one easily realizes that for all $(p, u), (q, v) \in Q \times V$ it holds

$$a(p, q) + b(p, v) - b(q, u) + c(u, v) = a(q, p) + b(q, u) - b(p, v) + c(v, u).$$

Subtracting $a(p, q) = a(q, p)$ and $c(u, v) = c(v, u)$ yields

$$b(p, v) = b(q, u).$$

To prove part 1 of the theorem, set for example $q = 0$ to show for all $(p, v) \in Q \times V$ that $b(p, v) = 0$ which implies $b = 0$. To prove part 2, assume $u, v \in V$ and set $p = \Lambda u$ and $q = \Lambda v$ to see $b(\Lambda v, u) = b(\Lambda u, v)$. The implications in the other directions are obvious. □

Given a right-hand side $\ell \in \mathscr{H}^*$ and its Riesz representation u, i.e., $\mathscr{A} u = \ell$ in $\mathscr{H}$, suppose that u_ℓ is an approximation for u. Define the error e and the residual $\mathscr{R}es$ as in (3.1)-(3.2) by

$$e = u - u_\ell \quad \text{and} \quad \mathscr{R}es = \mathscr{A}(e) = \ell - \mathscr{A} u_\ell.$$

Theorem 3.2 (Error Approximation/Characterisation). *Let $\mathscr{A}$ be the scalar-product in the Hilbert space $\mathscr{H}$ and define $\mathscr{R}es = \mathscr{A}(e,\cdot) = \mathscr{A}(e)$ for any $e \in \mathscr{H}$. Then, for all $v \in \mathscr{H}$ with $\|v\|_{\mathscr{H}} = 1$, it holds*

$$\frac{\|e\|_{\mathscr{H}} - \mathscr{R}es(v)}{\|e\|_{\mathscr{H}}} = \frac{1}{2}\left\| v - \frac{e}{\|e\|_{\mathscr{H}}} \right\|^2_{\mathscr{H}}.$$

Proof. Some elementary algebraic calculations yield

$$\begin{aligned}
&\frac{\|e\|_{\mathscr{H}} - \mathscr{R}es(v)}{\|e\|_{\mathscr{H}}} \\
=&1 - \mathscr{A}\left(\frac{e}{\|e\|_{\mathscr{H}}}\right)(v) \\
=&\frac{1}{2}\mathscr{A}\left(\frac{e}{\|e\|_{\mathscr{H}}}\right)\left(\frac{e}{\|e\|_{\mathscr{H}}}\right) - \mathscr{A}\left(\frac{e}{\|e\|_{\mathscr{H}}}\right)(v) + \frac{1}{2}\mathscr{A}(v)(v) \\
=&\frac{1}{2}\left\| v - \frac{e}{\|e\|_{\mathscr{H}}} \right\|^2_{\mathscr{H}}.
\end{aligned}$$

□

Theorem 3.2 concerns the task to obtain a good approximation of $\|e\|_{\mathscr{H}}$ by the residual $\mathscr{R}es(v)$ for some test function v. It reveals that the function v needs to be close to $e/\|e\|_{\mathscr{H}}$. Hence, the evaluation of the operator norm $\|\mathscr{R}es\|_{\mathscr{H}^*}$ is, in general, equivalent in terms of complexity to the evaluation of the exact solution u. Consequently, alternative approaches for the calculation of upper and lower bounds for $\|\mathscr{R}es\|_{\mathscr{H}^*}$ are required and it appears advisable to compute good upper and lower bounds instead of the exact norm of the residual.

3.3. Poisson problem

With the definitions of Section 2.2, the residuals of the Poisson problem (2.3) result from the residual representation formula (3.3). For $q \in Q$, the residual $\mathscr{R}es_Q$ depends on the choice of $\tilde{u}_\ell$,

$$\mathscr{R}es_Q(q) = \ell_Q(q) - a(p_\ell - \Lambda\tilde{u}_\ell, q) = \int_\Omega (\nabla\tilde{u}_\ell - p_\ell)\cdot q\,dx. \tag{3.4}$$

Since $L^2(\Omega)$ is isometrically isomorphic to its dual $L^2(\Omega)^*$, this yields the equality

$$\|\mathscr{R}es_Q\|_{Q^*} = \|p_\ell - \nabla\tilde{u}_\ell\|_Q.$$

The minimisation with respect to $\tilde{u}_\ell$ leads to the consistency part of the residual as discussed in Section 5.1 below.

For $v \in V$, the equilibrium residual for the Poisson problem $\mathscr{R}es_V$ can be written as

$$\mathscr{R}es_V(v) = \ell_V(v) - b(p_\ell, v) - c(\tilde{u}_\ell, v) = \int_\Omega f\,v\,dx - \int_\Omega \nabla v\cdot p_\ell\,dx.$$

For some general subset $Q_\ell \subset L^2(\Omega;\mathbb{R}^n)$, the divergence $\operatorname{div} p_\ell$ is merely understood as some distribution in $H^{-1}(\Omega)$. An integration by parts shows that $f + \operatorname{div} p_\ell$ is the Riesz-representative of $\mathscr{R}es_V$ which we write as

$$\|\mathscr{R}es_V\|_{V^*} = \|f + \operatorname{div} p_\ell\|_{H^{-1}(\Omega)}.$$

3.4. Stokes problem

With an approximation $(p_\ell, \tilde{u}_\ell)$ of the exact solution (p, u) of the Stokes equations (2.4), the calculation

$$\mathscr{R}es_Q(q) = \ell_Q(p_\ell, q) - a(p_\ell - \Lambda\tilde{u}_\ell, q) = -\int_\Omega q \operatorname{div} \tilde{u}_\ell \, dx$$

yields the following part of the consistency residual,

$$\|\mathscr{R}es_Q\|_{Q^*} = \|\operatorname{div} \tilde{u}_\ell\|_{L^2(\Omega)}.$$

For the residual $\mathscr{R}es_V$ of the symmetric equations, it holds

$$\begin{aligned}\mathscr{R}es_V(v) &= \ell_V(v) - b(p_\ell, v) - c_{\mathrm{sym}}(\tilde{u}_\ell, v)\\ &= \int_\Omega f \cdot v \, dx - \int_\Omega 2\mu\varepsilon(\tilde{u}_\ell) : \varepsilon(v) \, dx + \int_\Omega p_\ell \operatorname{div} v \, dx\\ &= \int_\Omega f \cdot v \, dx - \int_\Omega (2\mu\varepsilon(\tilde{u}_\ell) - p_\ell \mathbb{1}) : \varepsilon(v) \, dx.\end{aligned}$$

Notice that the symmetry of the discrete stress tensor

$$\sigma_\ell := 2\mu\varepsilon(\tilde{u}_\ell) - p_\ell \mathbb{1}$$

allows $\sigma_\ell : \varepsilon(v) = \sigma_\ell : \nabla v$. This suggests a split of the last term

$$\int_\Omega (2\mu\varepsilon(\tilde{u}_\ell) - p_\ell \mathbb{1}) : \varepsilon(v) \, dx = \int_\Omega 2\mu \left(\varepsilon(\tilde{u}_\ell) - \varepsilon_\ell(u_\ell)\right) : \varepsilon(v) \, dx + \int_\Omega \sigma_\ell : \nabla v \, dx$$

and yields

$$\begin{aligned}\|\mathscr{R}es_V(v)\| \approx{}& 2\mu \|\varepsilon_\ell(u_\ell) - \varepsilon(\tilde{u}_\ell)\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \|\varepsilon(v)\|_{L^2(\Omega;\mathbb{R}^{n\times n})}\\ &+ \left| \int_\Omega f \cdot v \, dx + \int_\Omega \sigma_\ell : \nabla v \, dx \right|.\end{aligned}$$

The first term $\|\varepsilon_\ell(u_\ell) - \varepsilon(\tilde{u}_\ell)\|_{L^2(\Omega;\mathbb{R}^n)}$ may be treated with the methods for consistency error estimators in Section 5.1. The remaining two terms are treated as the equilibrium residual according to Section 5.2. The non-symmetric case can be analysed in an analogous way.

3.5. Lamé problem

An approximation $(\sigma_\ell, \tilde{u}_\ell)$ of the true solution (σ, u) of the Lamé equations (2.5) leads to

$$\mathscr{R}es_Q(\tau) = \ell_Q(\tau) - a(\sigma_\ell - \Lambda\tilde{u}_\ell, \tau) = \int_\Omega \left(\varepsilon(\tilde{u}_\ell) - \mathbb{C}^{-1}\sigma_\ell\right) : \tau \, dx. \tag{3.5}$$

Hence, the dual norm of the consistency residual reads

$$\|\mathscr{R}es_Q\|_{Q^*} = \|\varepsilon(\tilde{u}_\ell) - \mathbb{C}^{-1}\sigma_\ell\|_{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}.$$

For any $v \in V$, the resulting equilibrium residual in V^* is given by

$$\mathscr{R}es_V(v) = \ell_V(v) - b(\sigma_\ell, v) - c(\tilde{u}_\ell, v) = \int_\Omega f \cdot v \, dx - \int_\Omega \varepsilon(v) : \sigma_\ell \, dx. \tag{3.6}$$

Note that for all symmetric matrices $E, F \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ it holds

$$\mathbb{C}E : F = (\lambda \operatorname{tr}(E)\mathbb{1} + 2\mu E) : F = \lambda \operatorname{tr}(E)\mathbb{1} : F + 2\mu E : F = \lambda \operatorname{tr}(E) \operatorname{tr} F + 2\mu E = \mathbb{C}F : E.$$

Hence, the differential operator Λ is symmetric in the form b for the non-mixed formulation (cf. Theorem 3.1.2) with

$$b(\Lambda u, v) = \int_\Omega \mathbb{C}\varepsilon(u) : \varepsilon(v) \, dx = \int_\Omega \mathbb{C}\varepsilon(v) : \varepsilon(u) \, dx = b(\Lambda v, u).$$

3.6. Eddy current problem

With an approximation $(p_\ell, \tilde{u}_\ell)$ of the exact solution (p, u) of the eddy current equations (2.6), for all $q \in Q$ the residual $\mathscr{R}es_Q$ reads

$$\mathscr{R}es_Q(q) = \ell_Q(q) - a(p_\ell - \Lambda\tilde{u}_\ell, q) = \int_\Omega \left(\operatorname{curl}\tilde{u}_\ell - \mu p_\ell\right) \cdot q \, dx. \tag{3.7}$$

For the dual norm of this consistency residual one obtains

$$\|\mathscr{R}es_Q\|_{Q^*} = \|\mu p_\ell - \operatorname{curl}\tilde{u}_\ell\|_{L^2(\Omega;\mathbb{R}^n)}.$$

The other part $\mathscr{R}es_V$ reads for all $v \in V$,

$$\mathscr{R}es_V(v) = \ell_V(v) - b(p_\ell, v) - c(\tilde{u}_\ell, v) = \int_\Omega \left(f - \sigma\tilde{u}_\ell\right) \cdot v \, dx - \int_\Omega p_\ell \cdot \operatorname{curl} v \, dx. \tag{3.8}$$

The involved analysis of this equilibrium type residual $\mathscr{R}es_V$ will be studied in Section 9 below.

4. Finite element spaces

This section is devoted to the most common finite element spaces. After some preliminary notation in Section 4.1, they are introduced in Section 4.2. The subsequent Section 4.3 defines (quasi-)interpolation operators mapping from $V \times Q$ to $V_\ell \times Q_\ell$ in a generic way for a wide range of applications. Moreover, lifting operators needed in the analysis of discontinuous Galerkin (dG) methods are introduced. The section is completed by a formal reformulation of the conforming, non-conforming and mixed discrete problems in the unified notation (Section 4.4).

4.1. Notation

The construction of finite element spaces is based on the discretisation of a Lipschitz domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) which is a bounded, simply connected set with piecewise affine boundary $\partial\Omega$. Let $\mathcal{T}_\ell$ be a set of simplices in Ω, i.e. in 2D, a set of closed triangles $T \subset \overline{\Omega}$ of positive area $|T|$ respectively in 3D, a set of closed tetrahedrons $T \subset \Omega$ with positive volume $|T|$. The set $\mathcal{T}_\ell$ is called *triangulation of* Ω if $\bigcup \mathcal{T}_\ell = \overline{\Omega}$ and if for simplices $T_1, T_2 \in \mathcal{T}_\ell$, it holds $T_1 = T_2$ or the intersection of T_1 and T_2 has zero volume, that is, $|T_1 \cap T_2| = 0$. We denote the set of all nodes of simplices in $\mathcal{T}_\ell$ with $\mathcal{N}_\ell$, the set of inner nodes with $\mathcal{K}_\ell = \mathcal{N}_\ell \cap \Omega$, the set of all edges with $\mathcal{E}_\ell$ and in the 3D case the set of faces of tetrahedrons in $\mathcal{T}_\ell$ with $\mathcal{F}_\ell$. Furthermore, for $\omega \subset \overline{\Omega}$ and $p \in \overline{\Omega}$ we abbreviate

$$
\begin{aligned}
\mathcal{N}_\ell(\omega) &:= \mathcal{N}_\ell \cap \omega;\\
\mathcal{K}_\ell(\omega) &:= \mathcal{K}_\ell \cap \omega;\\
\mathcal{E}_\ell(\omega) &:= \left\{ E \in \mathcal{E}_\ell \;\middle|\; E \subset \omega \right\};\\
\mathcal{F}_\ell(\omega) &:= \left\{ F \in \mathcal{F}_\ell \;\middle|\; F \subset \omega \right\};\\
\mathcal{T}_\ell(\omega) &:= \left\{ T \in \mathcal{T}_\ell \;\middle|\; \omega \subset T \right\} \quad \text{and} \quad \mathcal{T}_\ell(p) := \mathcal{T}_\ell(\{p\})
\end{aligned}
$$

and write $\operatorname{mid}(\omega)$ for the center of gravity for elements, faces or edges ω.

A triangulation $\mathcal{T}_\ell$ is called *regular* if it holds

$$
T_1 \cap T_2 \in \mathcal{T}_\ell \cup \mathcal{E}_\ell \cup \mathcal{F}_\ell \cup \mathcal{N}_\ell \cup \{\emptyset\} \qquad \text{for simplices } T_1 \text{ and } T_2 \in \mathcal{T}_\ell.
$$

For a triangle (tetrahedron) $T \in \mathcal{T}_\ell$ with area (volume) $|T|$ we define its size $h_T := |T|^{1/n} \approx \operatorname{diam}(T)$. All triangulations in this paper are shape-regular in the sense that the volume $|T|$ is equivalent to the diameter h_T of each simplex T. The piecewise constant mapping $h_\mathcal{T} : \overline{\Omega} \to \mathbb{R}$ is defined in an L^2 sense via $h_\mathcal{T}|_T := h_T$ for all $T \in \mathcal{T}_\ell$. Accordingly, for any edge $E \in \mathcal{E}_\ell$ (or face $F \in \mathcal{F}_\ell$), we denote its size by $h_E := |E|$ (or $h_F := |F|^{1/2}$) while ν_E (or, in 3D ν_F) and τ_E are the unit normal and unit tangential vectors. The mapping $h_\mathcal{E} : \bigcup \mathcal{E}_\ell \to \mathbb{R}$ is defined by $h_\mathcal{E}|_E := h_E$ for all $E \in \mathcal{E}_\ell$.

Additionally, for $z \in \mathcal{N}_\ell$, $E \in \mathcal{E}_\ell$, $F \in \mathcal{F}_\ell$, and $T \in \mathcal{T}_\ell$ we refer to Ω_z, Ω_E, Ω_F, and Ω_T as the patches

$$\Omega_z := \bigcup \mathcal{T}_\ell(z), \qquad \Omega_E := \bigcup \mathcal{T}_\ell(E),$$
$$\Omega_F := \bigcup \mathcal{T}_\ell(F), \qquad \Omega_T := \bigcup \{T' \in \mathcal{T}_\ell \mid T' \cap T \neq \emptyset\}.$$

In some cases it is convenient to keep the notation independent of the space dimension by definition of the set

$$\mathcal{C}_\ell := \begin{cases} \mathcal{E}_\ell & \text{for } n = 2, \\ \mathcal{F}_\ell & \text{for } n = 3. \end{cases} \tag{4.1}$$

Besides, we sometimes write "edges/faces" if we want to use edges in the 2D case and faces in 3D. We define the jump of a function v on $C \in \mathcal{C}_\ell$ by

$$[v]_C := v|_{T_1} - v|_{T_2}$$

with $T_1,\ T_2 \in \mathcal{T}_\ell(C),\ C = T_1 \cap T_2$. The extension to all edges/faces $[v]_{\mathcal{C}} : \bigcup \mathcal{C}_\ell \to \mathbb{R}^m$ is given for all $C \in \mathcal{C}_\ell$ by $[v]_{\mathcal{C}}|_C = [v]_C$. By $\{v\}_{\mathcal{C}_\ell}$ we denote the mean value of test function v in case of jumps at inner edges/faces and elsewhere the value of v. Jumps and mean values are defined analogously for vector and matrix valued functions. Details can be found in the dG literature such as [4,25,34].

4.2. Discrete spaces and basis functions

Problems considered in this paper employ the spaces $H^1(\Omega;\mathbb{R}^m)$, $H(\mathrm{curl},\Omega)$, $H(\mathrm{div},\Omega)$, and $L^2(\Omega;\mathbb{R}^m)$. With appropriate differential operators, these spaces form an exact sequence [5,6] which is shown in the next commuting diagram together with corresponding conforming discrete spaces. The respective quasi-interpolation operators $J_\ell^N, J_\ell^E, J_\ell^F, J_\ell^T$ are defined in Section 4.3, the discrete spaces in the bottom line are defined below.

$$\begin{array}{ccccccc} H^1(\Omega;\mathbb{R}^m) & \xrightarrow{\nabla} & H(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega;\mathbb{R}^m) \\ \downarrow J_\ell^N & & \downarrow J_\ell^E & & \downarrow J_\ell^F & & \downarrow J_\ell^T \\ P_1(\mathcal{T}_\ell;\mathbb{R}^m) & \xrightarrow{\nabla} & Nd_1(\mathcal{T}_\ell) & \xrightarrow{\mathrm{curl}} & RT_0(\mathcal{T}_\ell) & \xrightarrow{\mathrm{div}} & P_0(\mathcal{T}_\ell;\mathbb{R}^m) \end{array} \tag{4.2}$$

Conforming Discrete Spaces. $P_k(\omega;\mathbb{R}^m)$ denotes the space of polynomials of total degree $\leq k$ which map from $\omega \subset \mathbb{R}^n$ to $\mathbb{R}^m$. For a triangulation $\mathcal{T}_\ell$ of Ω,

$$P_k(\mathcal{T}_\ell;\mathbb{R}^m) := \left\{ v \in L^2(\Omega;R^m) \mid \forall T \in \mathcal{T}_\ell,\ v|_T \in P_k(T;\mathbb{R}^m) \right\}$$

denotes the space of piecewise polynomials of total degree $\leq k$. Some common affine conforming discrete spaces are the Courant spaces (P_1^c), Nédélec's first family of elements

(Nd_1) and Raviart-Thomas spaces (RT_0), as shown in (4.2).

$$\begin{aligned}
&P_{1,0}^c(\mathscr{T}_\ell;\mathbb{R}^m) := P_1(\mathscr{T}_\ell;\mathbb{R}^m)\cap H_0^1(\Omega;\mathbb{R}^m),\\
&\mathrm{Nd}_1(T) := \left\{ v : T \to \mathbb{R}^3 \;\middle|\; \exists a, b \in \mathbb{R}^3,\ \forall x \in T,\ v(x) = a + b \wedge x \right\},\\
&\mathrm{Nd}_1(\mathscr{T}_\ell) := \left\{ v \in H(\mathrm{curl},\Omega;\mathbb{R}^3) \;\middle|\; \forall T \in \mathscr{T}_\ell,\ v|_T \in \mathrm{Nd}_1(T) \right\},\\
&\mathrm{Nd}_{1,0}(\mathscr{T}_\ell) := \left\{ v \in H(\mathrm{curl},\Omega;\mathbb{R}^3) \;\middle|\; \forall T \in \mathscr{T}_\ell,\ v|_T \in \mathrm{Nd}_1(T),\ (v \wedge \nu)|_{\partial\Omega} = 0 \right\},\\
&\mathrm{RT}_0(T) := \left\{ p : T \to \mathbb{R}^n \;\middle|\; \exists a \in \mathbb{R}^n,\ b \in \mathbb{R},\ \forall x \in T,\ p(x) = a + b\,x, \right\},\\
&\mathrm{RT}_0(\mathscr{T}_\ell) := \left\{ p \in H(\mathrm{div},\Omega;\mathbb{R}^n) \;\middle|\; \forall T \in \mathscr{T}_\ell,\ p|_T \in \mathrm{RT}_0(T) \right\}.
\end{aligned}$$

For the spaces defined above, we denote by $\{\varphi_z \mid z \in \mathscr{N}_\ell\}$, $\{\varphi_E \mid E \in \mathscr{E}_\ell\}$, $\{\varphi_F \mid F \in \mathscr{F}_\ell\}$, and $\{\varphi_T \mid T \in \mathscr{T}_\ell\}$ the nodal basis of $P_1^c(\mathscr{T}_\ell)$, the edge basis of $\mathrm{Nd}_1(\mathscr{T}_\ell)$, the face basis of $\mathrm{RT}_0(\mathscr{T}_\ell)$, and the element basis of $P_0(\mathscr{T}_\ell)$, respectively. The basis functions are chosen to be orthonormal with regard to the canonical degrees of freedom, i.e., for $z_j, z_k \in \mathscr{N}_\ell,\ E_j, E_k \in \mathscr{E}_\ell,\ F_j, F_k \in \mathscr{F}_\ell$ and $T_j, T_k \in \mathscr{T}_\ell$,

$$\varphi_{z_j}(z_k) = \delta_{jk},\ \int_{E_j} \varphi_{E_k}\, ds = \delta_{jk},\ \int_{F_j} \varphi_{F_k}\, d\sigma = \delta_{jk}\ \text{ and } \int_{T_j} \varphi_{T_k}\, dx = \delta_{jk}.$$

For convenience, we recall the construction of the common P_1^c-basis in more detail. The nodal basis functions φ_z are defined for each node $z \in \mathscr{N}_h$ in two steps. First, the values at the nodes are given by

$$\varphi_z(z) = 1 \quad \text{and} \quad \varphi_z(x) = 0 \quad \text{for any other node } x \in \mathscr{N}_h \setminus \{z\}.$$

Then, given φ_z on $\mathscr{N}_h$, φ_z is defined on each triangle $T = \mathrm{conv}\{A, B, C\}$ via affine interpolation. More precisely, $x \in T$ can be represented by $x = \alpha A + \beta B + \gamma C$ with convex coefficients $0 \le \alpha, \beta, \gamma \le 1$ where $\alpha + \beta + \gamma = 1$. The evaluation of φ_z at x is obtained by

$$\varphi_z(x) = \alpha\varphi_z(A) + \beta\varphi_z(B) + \gamma\varphi_z(C).$$

Since the triangulation is regular in the sense of Ciarlet [28], this defines a globally continuous and piecewise affine function $\varphi_z \in H^1(\Omega)$. Figs. 1 and 2 show the degrees of freedom and a P_1 basis function.

The construction of Nédélec basis functions is explained in Figs. 3 and 4, Raviart-Thomas basis functions and the use of degrees of freedom are shown in Figs. 5 and 6.

Non-Conforming Discrete Spaces. We will use the common non-conforming discrete Crouzeix-Raviart spaces on $\Omega \subset \mathbb{R}^2$ defined by

$$\begin{aligned}
\mathrm{CR}_1(\mathscr{T}_\ell;\mathbb{R}^m) &:= \left\{ v \in P_1(\mathscr{T}_\ell;\mathbb{R}^m) \;\middle|\; v \text{ continuous in } \mathrm{mid}(\mathscr{E}_\ell) \right\},\\
\mathrm{CR}_{1,0}(\mathscr{T}_\ell;\mathbb{R}^m) &:= \Big\{ v \in P_1(\mathscr{T}_\ell;\mathbb{R}^m) \;\Big|\; v \text{ continuous in } \mathrm{mid}(\mathscr{E}_\ell),\\
&\qquad \forall E \in \mathscr{E}_\ell \text{ with } E \subset \partial\Omega,\ v(\mathrm{mid}(E)) = 0 \Big\}.
\end{aligned}$$

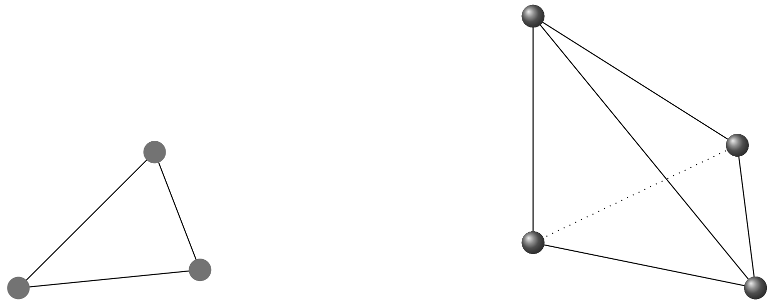

Figure 1: The P_1 functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ have three resp. four degrees of feedom. The values of the function can be chosen for each node of the simplex - the resulting functions are H^1-conforming.

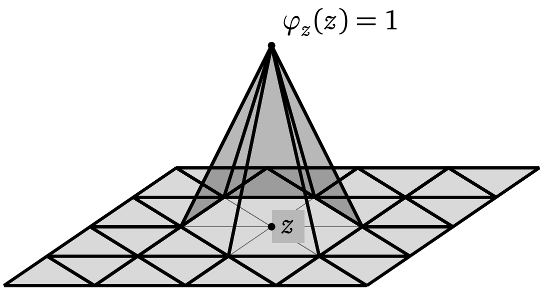

Figure 2: P_1 basis function φ_z in $\mathbb{R}^2$ belonging to node z. The function is continuous and therefore H^1-conforming.

Figure 3: The Nédélec functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ of the form $v(x_1,x_2) = (a_1 - b\,x_2, a_2 + b\,x_1)$ and $v(x) = a + b \wedge x$, respectively, have three resp. six degrees of feedom. For each edge, the tangential component can be chosen - the resulting functions are $H(\mathrm{curl})$-conforming.

According to this, basis functions for $\mathrm{CR}_1(\mathscr{T}_\ell;\mathbb{R})$ are defined for edges E of a triangle by

$$\psi_E \in P_1(\mathscr{T}_\ell) \quad \text{s.t.} \quad \psi_E(\mathrm{mid}(F)) = \delta_{EF} \quad \text{for } E, F \in \mathscr{E}_\ell.$$

Fig. 7 shows the use of the degrees of freedom and an example of a Crouzeix-Raviart basis function in 2D.

Modified Basis Functions and Oscillations. Assume that each triangle $T \in \mathscr{T}_\ell$ has at least one node in $\mathscr{K}_\ell \subset \Omega$ and install a mapping $\zeta : \mathscr{N}_\ell \to \mathscr{K}_\ell$ such that $\zeta|_{\mathscr{K}_\ell} = id|_{\mathscr{K}_\ell}$ and, for all

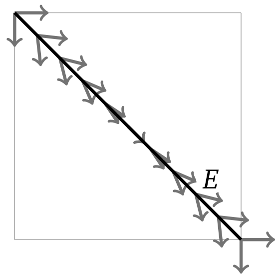

Figure 4: Nédélec basis function φ_E in $\mathbb{R}^2$ belonging to the edge E. The tangential jumps over all edges are zero and φ_E is therefore $H(\mathrm{curl})$-conforming.

Figure 5: The Raviart-Thomas functions in $\mathbb{R}^2$ and $\mathbb{R}^3$ of the form $v(x_1,x_2)=(a_1+b\,x_1,a_2+b\,x_2)$, have three resp. four degrees of feedom. For each face, the normal component can be chosen - the resulting functions are $H(\mathrm{div})$-conforming.

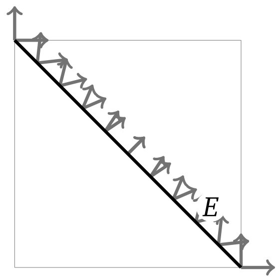

Figure 6: Raviart-Thomas basis function φ_E in $\mathbb{R}^2$ belonging to the edge E. The normal jumps over all edges are zero and φ_E is therefore $H(\mathrm{div})$-conforming.

$z\in\mathcal{N}_\ell\setminus\mathcal{K}_\ell$, $\zeta(z)\in\mathcal{K}_\ell(\mathcal{T}_\ell(z))$ is a neighboring free node. As ζ might not be injective, the inverse mapping ζ^{-1} is set valued, $\zeta^{-1}(z):=\{y\in\mathcal{N}_\ell:\zeta(y)=z\}$.

Recall that $(\varphi_z)_{z\in\mathcal{N}_\ell}$ denotes the nodal basis of $P_1(\mathcal{T}_\ell)\cap C(\overline{\Omega})$. Then, for $z\in\mathcal{K}_\ell$, define functions ψ_z via

$$\psi_z:=\sum_{y\in\zeta^{-1}(z)}\varphi_y\in P_1(\mathcal{T}_\ell)\cap C(\Omega)$$

and the corresponding support sets $\operatorname{supp}\psi_z:=\{x\in\Omega:\psi_z(x)>0\}$.

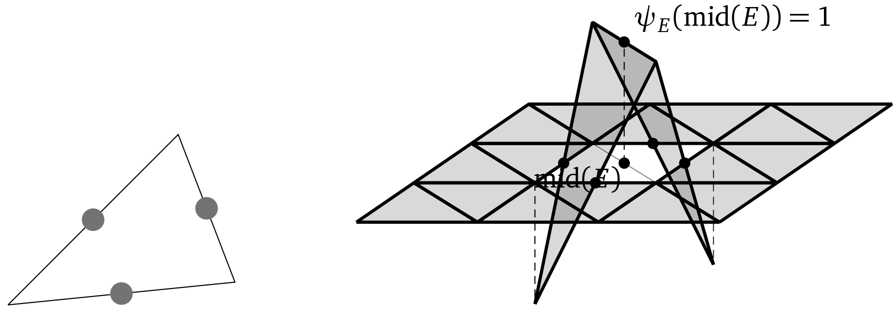

Figure 7: The CR_1 functions in $\mathbb{R}^2$ have three degrees of feedom. The values of the function can be chosen for the midpoints of the edges/faces (left). The figure on the right shows the CR_1 basis function ψ_E belonging to edge E (grey). The function is continuous in the midpoints of all edges.

Lemma 4.1.

1. *The family $(\psi_z : z \in \mathscr{K}_\ell)$ is a Lipschitz-continuous partition of unity on Ω,*

$$\sum_{z\in\mathscr{K}_\ell} \psi_z = 1 \text{ a.e. in } \Omega, \text{ and for all } z \in \mathscr{K}_\ell, \quad 0 \le \varphi_z \le \psi_z \le 1.$$

2. *From $\psi_z \neq \varphi_z$ it follows $\mathscr{E}_\ell(\mathscr{T}_\ell(z)) \cap \mathscr{E}_\ell(\partial\Omega) \neq \emptyset$ in 2D, i.e. at least one triangle with vertex z has at least one edge on the boundary $\partial\Omega$ of the domain Ω. The equivalent statement in 3D is $\mathscr{F}_\ell(\mathscr{T}_\ell(z)) \cap \mathscr{F}_\ell(\partial\Omega) \neq \emptyset$.*
3. *The supports* $\operatorname{supp}\psi_z$ *have finite overlap, i.e.*

$$\max_{x\in\Omega,\ell\in\mathbb{N}} \left| \{z \in \mathscr{K}_\ell \,:\, x \in \operatorname{supp}\psi_z\} \right| \lesssim 1.$$

Proof. Details of the proof are left to the reader, cf. [16, 17]. ☐

The following definition of oscillations and the Definition 4.1 below of the interpolation operator J_ℓ employ this partition of unity and allow the formulation of Theorem 5.2 which gives a posteriori estimators for equilibrium type residuals.

With the abbreviation $g_\omega := |\omega|^{-1} \int_\omega g(x)\,dx \in \mathbb{R}^m$ for the integral mean of a function $g \in L^2(\omega; \mathbb{R}^m)$ and any set $\mathscr{S}$ of measurable subsets ω of Ω of diameter $h_\omega^n := \mathrm{diam}(\omega)$, the *oscillation of g on $\mathscr{S}$* is defined by

$$\mathrm{osc}(g, \mathscr{S}) := \left(\sum_{\omega\in\mathscr{S}} h_\omega^2 \|g - g_\omega\|^2_{L^2(\omega)} \right)^{1/2}. \tag{4.4}$$

The data oscillation $\mathrm{osc}(g, \mathscr{T}_\ell)$ plays an important role. Additionally, we define oscillations subject to the set of free nodes $\mathscr{K}_\ell$ by

$$\mathrm{Osc}(g, \mathscr{K}_\ell) := \mathrm{osc}\left(g, \{\operatorname{supp}\psi_z : z \in \mathscr{K}_\ell\}\right).$$

One can easily derive that $\mathrm{osc}(g, \mathscr{T}_\ell) \le \mathrm{Osc}(g, \mathscr{K}_\ell)$.

4.3. Interpolation and lifting operators

Quasi-interpolation operators for the sequence of spaces satisfying the commuting diagram property as depicted in (4.2) have been constructed in [40] in the sense that

$$\nabla J_\ell^N = J_\ell^E \nabla, \qquad \operatorname{curl} J_\ell^E = J_\ell^F \operatorname{curl}, \qquad \operatorname{div} J_\ell^F = J_\ell^T \operatorname{div} . \tag{4.5}$$

The quasi-interpolation operators are defined as the compositions of the classical interpolation operators associated with the respective function spaces and appropriate smoothing operators. We emphasize that the incorporation of smoothing operators is mandatory, since the classical interpolation operators require continuous functions and, e.g., $H^1(\Omega) \subsetneq C(\overline{\Omega})$ for an open domain Ω.

We recall that, for sufficiently smooth functions, the classical interpolation operators

$$I_\ell^N : H^1(\Omega) \to P_1(\Omega; \mathbb{R}^m), \qquad I_\ell^E : H(\operatorname{curl}, \Omega) \to \mathrm{Nd}_1(\mathcal{T}_\ell),$$
$$I_\ell^F : H(\operatorname{div}, \Omega) \to \mathrm{RT}_0(\mathcal{T}_\ell), \qquad I_\ell^T : L^2(\Omega) \to P_0(\mathcal{T}_\ell)$$

are given by

$$I_\ell^N v := \sum_{z \in \mathcal{N}_\ell} v(z) \varphi_z, \tag{4.6a}$$

$$I_\ell^E v := \sum_{E \in \mathcal{E}_\ell} \left(\int_E v \cdot \tau_E \, ds \right) \varphi_E, \tag{4.6b}$$

$$I_\ell^F v := \sum_{F \in \mathcal{F}_\ell} \left(\int_F v \cdot \nu_F \, d\sigma \right) \varphi_F, \tag{4.6c}$$

$$I_\ell^T v := \sum_{T \in \mathcal{T}_\ell} \left(\int_T v \, dx \right) \varphi_T. \tag{4.6d}$$

For $x \in \overline{\Omega}$ and $T \in \mathcal{T}_\ell$ we refer to $(\lambda_z^T(x))_{z \in \mathcal{N}_\ell(T)}$ as the barycentric coordinates of x in T, such that x admits the representation $x = \sum_{z \in \mathcal{N}_\ell(T)} \lambda_z^T(x) z$ where all coordinates $\lambda_z^T \geq 0$ are non-negative and sum up to one, i.e. $\sum_{z \in \mathcal{N}(T)} \lambda_z^T = 1$. For every node $z \in \mathcal{N}_\ell$, we choose $\omega_z \subset \Omega_z$ as a simply-connected domain with $z \in \omega_z$, e.g., $\omega_z := B_r(z) \cap \Omega_z$ for some appropriately chosen $r > 0$, where $B_r(z)$ stands for the ball with radius r and center z.

For a triangle or tetrahedron $T \in \mathcal{T}_\ell$, $x \in \overline{\Omega}$, $y_z \in \omega_z$, $z \in \mathcal{N}_\ell(T)$ and $T' := \operatorname{conv}\{y_z : z \in \mathcal{N}_\ell(T)\}$, we consider the transformation

$$\hat{x}(x, (y_z)_{z \in \mathcal{N}_\ell(T)}) := \sum_{z \in \mathcal{N}_\ell(T)} \lambda_z^T(x) \, y_z, \tag{4.7}$$

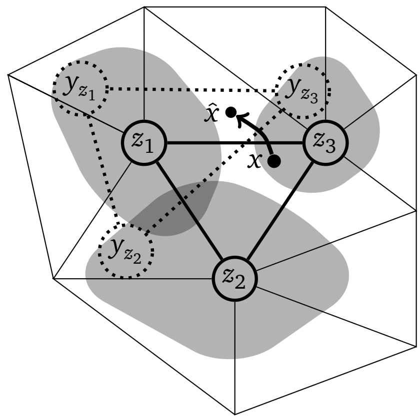

Figure 8: Illustration of the mapping $\hat{x}$ defined in equation (4.7). It maps a point x in a triangle T (printed bold) with nodes z_i and fixed points y_{z_i} in regions ω_{z_i} (grey shaded) around the triangle's nodes to a point $\hat{x}$ in the triangle $T' = \text{conv}\{y_z : z \in \mathcal{N}_\ell(T)\}$ (dotted).

see also Fig. 8, and define the smoothing operators S_ℓ^N on $H^1(\Omega)$, S_ℓ^E on $H(\text{curl},\Omega)$, S_ℓ^F on $H(\text{div},\Omega)$, and S_ℓ^T on $L^2(\Omega)$ by means of

$$(S_\ell^N v)(x) := \Big(\prod_{z\in\mathcal{N}_\ell(T)} |\omega_z|\Big)^{-1} \int_{\prod_{z\in\mathcal{N}_\ell(T)}\omega_z} v(\hat{x})\, d(y_z)_{z\in\mathcal{N}_\ell(T)}, \tag{4.8a}$$

$$(S_\ell^E v)(x) := \Big(\prod_{z\in\mathcal{N}_\ell(T)} |\omega_z|\Big)^{-1} \int_{\prod_{z\in\mathcal{N}_\ell(T)}\omega_z} \Big(\frac{d\hat{x}}{dx}\Big)^T v(\hat{x})\, d(y_z)_{z\in\mathcal{N}_\ell(T)}, \tag{4.8b}$$

$$(S_\ell^F v)(x) := \Big(\prod_{z\in\mathcal{N}_\ell(T)} |\omega_z|\Big)^{-1} \int_{\prod_{z\in\mathcal{N}_\ell(T)}\omega_z} \Big(\frac{d\hat{x}}{dx}\Big)^{-T} v(\hat{x}) \det\Big(\frac{d\hat{x}}{dx}\Big)\, d(y_z)_{z\in\mathcal{N}_\ell(T)}, \tag{4.8c}$$

$$(S_\ell^T v)(x) := \Big(\prod_{z\in\mathcal{N}_\ell(T)} |\omega_z|\Big)^{-1} \int_{\prod_{z\in\mathcal{N}_\ell(T)}\omega_z} v(\hat{x}) \det\Big(\frac{d\hat{x}}{dx}\Big)\, d(y_z)_{z\in\mathcal{N}_\ell(T)}. \tag{4.8d}$$

In terms of the classical interpolation operators (4.6a)–(4.6d) and the smoothing operators (4.8a)-(4.8d), the quasi-interpolation operators are given by

$$J_\ell^N := I_\ell^N \circ S_\ell^N, \qquad J_\ell^E := I_\ell^E \circ S_\ell^E, \tag{4.9a}$$

$$J_\ell^F := I_\ell^F \circ S_\ell^F, \qquad J_\ell^T := I_\ell^T \circ S_\ell^T. \tag{4.9b}$$

The commuting property (4.5) of diagram (4.2) follows from the corresponding properties of the classical interpolation operators and the smoothing operators (cf. Lemma 3 and Corollary 4 in [40]). The quasi-interpolation operator J_ℓ^N shares the H^1-stability and local approximation properties of Clément's quasi-interpolation operator [17, 29]. The same applies to J_ℓ^F compared to the quasi-interpolation operator from [44]. As far as J_ℓ^E is concerned, we have the following result.

Theorem 4.1. *Let $J_\ell^E : H_0(\mathrm{curl},\Omega) \to \mathrm{Nd}_{1,0}(\mathcal{T}_\ell)$ be the quasi-interpolation operator given by* (4.9a). *Then, for every $v \in H_0(\mathrm{curl},\Omega)$ there exist $\varphi \in H_0^1(\Omega)$ and $z \in H_0^1(\Omega)^3$ such that*

$$\begin{aligned} v - J_\ell^E v &= \nabla\varphi + z\,,\\ h_T^{-1}\,\|\varphi\|_{L^2(T)} + \|\nabla\varphi\|_{L^2(T)} &\lesssim \|v\|_{L^2(\widetilde{\Omega}_T)}, \qquad T \in \mathcal{T}_\ell\,,\\ h_T^{-1}\,\|z\|_{L^2(T)} + \|\nabla z\|_{L^2(T)} &\lesssim \|\,\mathrm{curl}\,v\|_{L^2(\widetilde{\Omega}_T)}, \qquad T \in \mathcal{T}_\ell\,, \end{aligned}$$

where the element patch $\widetilde{\Omega}_T$ is given by $\widetilde{\Omega}_T := \bigcup\{T' \in \mathcal{T}_\ell \mid T' \cap \Omega_T \neq \emptyset\}$.

Proof. We refer to the proof of Theorem 1 in [40]. □

Remark 4.1. A Scott-Zhang-type interpolation operator $\tilde{J}_\ell^E : H_0(\mathrm{curl},\Omega)\cap H^1(\Omega)^3 \to \mathrm{Nd}_{1,0}$ $(\mathcal{T}_\ell)$ that admits similar local approximation properties as in Theorem 4.1 has been derived in [9]. This result has been recently improved in [45] without requiring extra regularity.

For the proof of the reliability of the equilibrium estimator in Section 5.2 we make use of the following interpolation operator from [17].

Definition 4.1 (Weighted Interpolation Operator J_ℓ). *Given $g \in L^2(\Omega)$ set $J_\ell g \in P_{1,0}(\mathcal{T}_\ell)$ by*

$$J_\ell g := \sum_{z\in\mathcal{K}_\ell} \left(\int_\Omega g\psi_z\,dx \Big/ \int_\Omega \varphi_z\,dx \right) \varphi_z \tag{4.10}$$

to define an operator $J_\ell : L^2(\Omega) \to P_{1,0}(\mathcal{T}_\ell)$.

In addition, the operator J_ℓ fulfills some quasi-orthogonality and H^1-stability, see [17].

Theorem 4.2. *For all $f \in L^2(\Omega)$ and $g \in H^1(\Omega)$, the interpolation operator of* (4.10) *and the set of free nodes $\mathcal{K}_\ell$ of a triangulation $\mathcal{T}_\ell$ of domain Ω it holds*

$$\int_\Omega f(g - J_\ell g)\,dx \lesssim \mathrm{Osc}(f,\mathcal{K}_\ell)\|\nabla g\|_{L^2(\Omega)}. \tag{4.11}$$

Proof. For a free node $z \in \mathcal{K}_\ell$, set

$$g_z := \int_\Omega \psi_z g\,dx \Big/ \int_\Omega \varphi_z\,dx \quad \text{and} \quad f_{\mathrm{supp}\,\psi_z} := |\,\mathrm{supp}\,\psi_z|^{-1}\int_{\mathrm{supp}\,\psi_z} f\,dx.$$

Since $\int_\Omega (\psi_z g - g_z\varphi_z)\,dx = 0$ and $\sum_{z\in\mathcal{K}_\ell}\psi_z = 1$, the Cauchy inequality yields

$$\begin{aligned}\int_\Omega f(g - J_\ell g)\,dx &= \sum_{z\in\mathcal{K}_\ell}\int_\Omega f\,(\psi_z g - g_z\varphi_z)\,dx\\ &= \sum_{z\in\mathcal{K}_\ell}\int_{\operatorname{supp}\psi_z}(f - f_{\operatorname{supp}\psi_z})\,(\psi_z g - g_z\varphi_z)\,dx\\ &\le \operatorname{Osc}(f,\mathcal{K}_\ell)\left(\sum_{z\in\mathcal{K}_\ell}\operatorname{diam}(\operatorname{supp}\psi_z)^{-2}\|\psi_z g - g_z\varphi_z\|^2_{L^2(\operatorname{supp}\psi_z)}\right)^{1/2}.\end{aligned}$$

For every inner node $z \in \operatorname{supp}\psi_z$ the term $\operatorname{diam}(\operatorname{supp}\psi_z)^{-1}\|\psi_z g - g_z\varphi_z\|_{L^2(\operatorname{supp}\psi_z)}$ is bounded by $\|\nabla g\|_{L^2(\operatorname{supp}\psi_z)}$ and so (4.11) follows with Lemma 4.1. □

In the context of discontinuous Galerkin discretisations, integrals on the intersections of elements are crucial to penalise non-conformity. Appropriate lifting operators [4, 25] enable the error control on edges and faces. On $C \in \mathcal{C}_\ell$ and for some $k \in \mathbb{N}$ with the jump and the function average defined as in Section 4.1, the lifting operators $r_C : L^2(C;\mathbb{R}^{m\times n}) \to P_k(\mathcal{T}_\ell;\mathbb{R}^{m\times n})$ and $s_C : L^2(C;\mathbb{R}^m) \to P_k(\mathcal{T}_\ell;\mathbb{R}^{m\times n})$ read

$$\begin{aligned}\int_\Omega r_C(q) : r_\ell\,dx &= \int_C q : \{r_\ell\}\,ds \qquad \text{for all } r_\ell \in P_k(\mathcal{T}_\ell;\mathbb{R}^{m\times n}),\\ \int_\Omega s_C(v) : r_\ell\,dx &= \int_C v\cdot[r_\ell]\,ds \qquad \text{for all } r_\ell \in P_k(\mathcal{T}_\ell;\mathbb{R}^{m\times n}).\end{aligned}$$

Let $\mathcal{C}_{\ell,\Omega}$ be the interior edges or faces of $\mathcal{C}_\ell$. The global lifting operators $r : L^2(\mathcal{C}_\ell;\mathbb{R}^{m\times n}) \to L_\ell$ and $s : L^2(\mathcal{C}_{\ell,\Omega}) \to P_k(\mathcal{T}_\ell;\mathbb{R}^{m\times n})$ are given by

$$r := \sum_{C\in\mathcal{C}_\ell} r_C \quad \text{and} \quad s := \sum_{C\in\mathcal{C}_{\ell,\Omega}} s_E. \tag{4.12}$$

Note that the definition of the lifting operators for vector-valued functions is identical.

4.4. Discrete problems

Given the continuous problems $\mathcal{A}(p,u) = \ell_Q + \ell_V$ of (2.3), (2.5) and (2.6) with differential operator Λ and a regular triangulation $\mathcal{T}_\ell$ of the domain Ω, the generic discrete problem reads: Find $(p_\ell,u_\ell) \in Q_\ell \times V_\ell$ such that

$$\forall\ (q_\ell,v_\ell) \in Q_\ell \times V_\ell \quad \mathcal{A}_\ell(p_\ell,u_\ell)(q_\ell,v_\ell) = \ell_Q(q_\ell) + \ell_V(v_\ell).$$

The modified discrete operator $\mathcal{A}_\ell$ depends on the class of the finite element method. In conforming methods, one uses $\mathcal{A}_\ell = \mathcal{A}$ and $p_\ell = \Lambda u_\ell$ and therefore one solves

$$\mathcal{A}_\ell(\Lambda u_\ell, u_\ell) = a(\Lambda u_\ell, \Lambda\cdot) + c(u_\ell,\cdot) = \ell_Q + \ell_V.$$

In non-conforming methods, although the differential operator Λ and the bilinear form a is not defined on V_ℓ, it induces some $\mathscr{T}_\ell$-piecewise operator

$$\Lambda_\ell : V_\ell \to Q_\ell \quad \text{via} \qquad (\Lambda_\ell v_\ell)|_T = \Lambda|_T v_\ell|_T \text{ for all } T \in \mathscr{T}_\ell.$$

In the same way, it is assumed that c is extended to $V + V_\ell$ without being relabelled. With this operator Λ_ℓ, set $p_\ell = \Lambda_\ell u_\ell$ and solve

$$\mathscr{A}_\ell(\Lambda_\ell u_\ell, u_\ell) = a(\Lambda_\ell u_\ell, \Lambda_\ell \cdot) + c(u_\ell, \cdot) = \ell_Q + \ell_V.$$

Mixed methods are formulated explicitly to provide a tuple (p_ℓ, u_ℓ). In conforming mixed problems, one solves

$$\mathscr{A}_\ell(p_\ell, u_\ell) = a(p_\ell, \cdot) + a(p_\ell, \Lambda \cdot) - a(\cdot, \Lambda u_\ell) + c(u_\ell, \cdot) = \ell_Q + \ell_V.$$

In mixed non-conforming methods, Λ is replaced by the piecewise analogue Λ_ℓ which yields the formulation

$$\mathscr{A}_\ell(p_\ell, u_\ell) = a(p_\ell, \cdot) + a(p_\ell, \Lambda_\ell \cdot) - a(\cdot, \Lambda_\ell u_\ell) + c(u_\ell, \cdot) = \ell_Q + \ell_V.$$

Note that the natural formulation of the Stokes problem (2.4) is the mixed form. Moreover, for dG methods, additional flux functions F_Q and F_V specific to the dG method enter the discretisation which are evaluated with respect to u_ℓ and p_ℓ, see [25]. The general dG formulation reads

$$\begin{aligned}\mathscr{A}_\ell(p_\ell, u_\ell) =& a(p_\ell, \cdot) + a(p_\ell, \Lambda_\ell \cdot) + (F_Q u_\ell, \cdot)_Q - a(\cdot, \Lambda_\ell u_\ell) + c(u_\ell, \cdot) + (F_V p_\ell, \cdot)_V \\ =& \ell_Q + \ell_V.\end{aligned}$$

5. Abstract dual norm estimates

This section is devoted to estimators for the consistency and equilibrium errors. The results are general and can be applied to many applications and discretisation schemes discussed in this review as will be shown in subsequent sections.

5.1. Consistency error estimates

For a regular triangulation $\mathscr{T}_\ell$ of $\Omega \subset \mathbb{R}^2$ and functions $u_\ell \in V_\ell$ and $p_\ell = \nabla_\ell u_\ell \in Q_\ell$ with the piecewise gradient ∇_ℓ, our analysis of the consistency error is based on the minimum

$$\min \left\{ \|p_\ell - \nabla v\|_{L^2(\Omega)} \,|\, v \in V \right\}.$$

In many situations, the following theorem provides several equivalent computable a posteriori error estimators.

Theorem 5.1. *On some domain $\Omega \subset \mathbb{R}^2$, for a Crouzeix-Raviart function $u_\ell \in \mathrm{CR}_{1,0}(\mathscr{T}_\ell)$ and with $V = H_0^1(\Omega)$ and $Q = L^2(\Omega)$ it holds*

$$\min_{v\in V} \|\nabla_\ell u_\ell - \nabla v\|_{L^2(\Omega)} \le \min_{v_\ell\in P_{1,0}(\mathscr{T}_\ell)} \|\nabla_\ell u_\ell - \nabla v_\ell\|_{L^2(\Omega)} \tag{5.1a}$$

$$\lesssim \min_{v_\ell\in P_{1,0}(\mathscr{T}_\ell)} \|h_{\mathscr{T}}^{-1}(u_\ell - v_\ell)\|_{L^2(\Omega)} \tag{5.1b}$$

$$\le \|h_{\mathscr{T}}^{-1}(u_\ell - A_{\mathscr{T}_\ell} u_\ell)\|_{L^2(\Omega)} \tag{5.1c}$$

$$\lesssim \left\| h_{\mathscr{E}}^{-1/2} \left[u_\ell \right]_{\mathscr{E}_\ell} \right\|_{L^2(\bigcup \mathscr{E}_\ell)} \tag{5.1d}$$

$$\lesssim \left\| h_{\mathscr{E}}^{1/2} \left[\nabla_\ell u_\ell \cdot \tau_{\mathscr{E}} \right]_{\mathscr{E}_\ell} \right\|_{L^2(\bigcup \mathscr{E}_\ell)} \tag{5.1e}$$

$$\lesssim \min_{v\in V} \|\nabla_\ell u_\ell - \nabla v\|_{L^2(\Omega)}. \tag{5.1f}$$

The averaging operator $A_{\mathscr{T}_\ell} : \mathrm{CR}_1(\mathscr{T}_\ell) \to P_1(\mathscr{T}_\ell)$ employed in the theorem is defined by

$$A_{\mathscr{T}_\ell} u_\ell(z) := \sum_{T\in\mathscr{T}_\ell(z)} u_\ell|_T(z)/|\mathscr{T}_\ell(z)| \qquad \text{for all } z \in \mathscr{N}_\ell.$$

The following lemma is needed in the proof of Theorem 5.1.

Lemma 5.1. *For any finite index set J of length $|J| \in \mathbb{N}$ and a family $(a_j \mid j \in J)$ of real numbers with mean value $a := \sum_{j\in J} a_j/|J|$, it follows that*

$$\sum_{j,k\in J} (a_j - a_k)^2 \approx \sum_{j\in J} (a_j - a)^2.$$

Proof. The calculation

$$|J|^2 \sum_{j\in J} (a_j - a)^2 = \sum_{j\in J} \left(|J| a_j - \sum_{k\in J} a_k \right)^2 = \sum_{j\in J} \left(\sum_{k\in J} (a_j - a_k) \right)^2$$
$$\le |J|^2 \sum_{j\in J} \sum_{k\in J} (a_j - a_k)^2$$

shows the inequality "$\lesssim$". To prove the converse inequality, for $j, k \in J$ one observes that

$$(a_j - a_k)^2 = \big((a_j - a) - (a_k - a) \big)^2 \le 2(a_j - a)^2 + 2(a_k - a)^2.$$

The sum over all tuples $(j,k) \in J^2$ yields

$$\sum_{j,k\in J} (a_j - a_k)^2 \le 2|J|^2 \sum_{j\in J} (a_j - a)^2. \qquad \square$$

Proof of Theorem 5.1. The first estimate (5.1a) holds due to the inclusion $P_{1,0}(\mathscr{T}_\ell) \subset V$, while inverse inequalities on every element domain $T \in \mathscr{T}_\ell$ yield (5.1b). To prove (5.1c), define $v_\ell \in P_{1,0}(\mathscr{T}_\ell)$ as $\mathscr{T}_\ell$-affine interpolation of the mean values

$$v_\ell(z) := A_{\mathscr{T}_\ell} u_\ell(z) = \sum_{T\in\mathscr{T}_\ell(z)} u_\ell|_T(z)/|\mathscr{T}_\ell(z)|$$

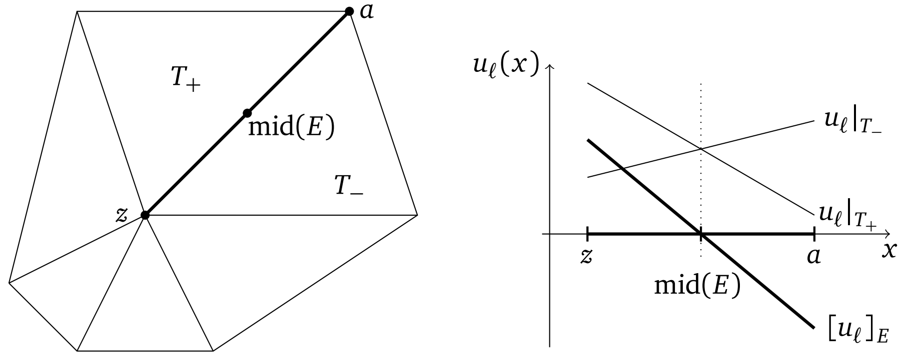

Figure 9: Illustration of the geometry and notation in the proof of (5.1d) of Theorem 5.1. To calculate the L^2-norm of the jump of the CR_1-function u_ℓ along the edge $E = T_+ \cap T_-$, merely $u_\ell|_{T_+}$ and $u_\ell|_{T_-}$ are evaluated in z.

at inner nodes $z \in \mathscr{K}_\ell$. (5.1c) follows from $A_{\mathscr{T}_\ell} u_\ell \in P_1(\mathscr{T}_\ell)$. Note that $v_\ell = \sum_{z\in\mathscr{K}_\ell} v_\ell(z)\varphi_z \in V$ vanishes on the boundary. On each domain $T \in \mathscr{T}_\ell$, $\nabla_\ell|_T = \nabla|_T$ and the argument in the L^2-norm is piecewise constant

$$h_T^{-2}\|u_\ell - v_\ell\|^2_{L^2(T)} \approx |T| h_T^{-2} \sum_{z\in\mathscr{N}_\ell(T)} \left|u_\ell|_T(z) - v_\ell|_T(z)\right|^2.$$

Notice $|T|h_T^{-2} \approx 1$. The sum over all triangles/tetrahedrons and a change of summation order yields

$$\|h_{\mathscr{T}}^{-1}(u_\ell - v_\ell)\|^2_{L^2(\Omega)} \lesssim \sum_{z\in\mathscr{N}_\ell} \sum_{T\in\mathscr{T}_\ell(z)} \left|u_\ell|_T(z) - v_\ell|_T(z)\right|^2.$$

For any $z \in \mathscr{N}_\ell$, Lemma 5.1 leads to

$$\sum_{T\in\mathscr{T}_\ell(z)} \left|u_\ell|_T(z) - v_\ell|_T(z)\right|^2 \approx \sum_{T,S\in\mathscr{T}_\ell(z)} \left|u_\ell|_T(z) - u_\ell|_S(z)\right|^2.$$

Recall that u_ℓ is piecewise affine and continuous in the midpoints mid(E) of edges $E \in \mathscr{E}_\ell(z)$. On the common edge E of two adjacent triangles $T_+, T_- \in \mathscr{T}_\ell(z)$, $E = T_+ \cap T_-$, the $L^2(E)$ norm of the jump $[u_\ell]_E$ as shown in Fig. 9 can therefore be calculated by

$$\begin{aligned}\|[u_\ell]_{\mathscr{E}_\ell}\|^2_{L^2(E)} &= \int_0^{h_E} \left(\frac{2(u_\ell|_{T_+}(z) - u_\ell|_{T_-}(z))}{h_E} s - (u_\ell|_{T_+}(z) - u_\ell|_{T_-}(z))\right)^2 ds \\ &\approx h_E \left|u_\ell|_{T_+}(z) - u_\ell|_{T_-}(z)\right|^2.\end{aligned}$$

The combination of the aforementioned estimates results in

$$\min_{v_\ell\in P_1(\mathscr{T}_\ell)\cap V} \|h_{\mathscr{T}}^{-1}(u_\ell - v_\ell)\|_{L^2(\Omega)} \lesssim \|h_{\mathscr{E}}^{-1/2}[u_\ell]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell)}.$$

This proves (5.1d).

For functions $v_\ell \in \mathrm{CR}_{1,0}$ for all edges $E \in \mathcal{E}_\ell$ it holds $\int_E [v_\ell]_{\mathcal{E}_\ell} ds = 0$ and therefore the Poincaré inequality along E yields the estimate (5.1e).

A standard argument with edge-bubble functions b_E allows the proof of (5.1f), namely

$$\|h_{\mathcal{E}}^{1/2} \left[\nabla_\ell u_\ell \cdot \tau_{\mathcal{E}}\right]_{\mathcal{E}_\ell}\|_{L^2(\bigcup \mathcal{E}_\ell)} \lesssim \min_{v \in V} \|\nabla_\ell u_\ell - \nabla v\|_{L^2(\Omega)},$$

as follows, cf. [43]. The quadratic edge-bubble function $b_E := 4\varphi_a \varphi_b$ is defined for an edge $E := \mathrm{conv}\{a, b\}$ with end-points $a, b \in \mathcal{N}_\ell$, subject to the affine nodal functions $\varphi_a, \varphi_b \in P_1(\mathcal{T}_\ell)$. The gradient $\nabla_\ell u_\ell$ is piecewise constant and on an edge $E \in \mathcal{E}_\ell$ it holds $\int_E b_E \, ds = \frac{2}{3}|E|$. Therefore,

$$h_E^{1/2} \left\| \left[\nabla_\ell u_\ell \cdot \tau_E\right]_{\mathcal{E}_\ell} \right\|_{L^2(E)} = |E| \left| \left[\nabla_\ell u_\ell \cdot \tau_E\right]_{\mathcal{E}_\ell} \right| \approx \left| \int_E b_E \left[\nabla_\ell u_\ell \cdot \tau_E\right]_{\mathcal{E}_\ell} \, ds \right|.$$

With $\nabla_\ell u_\ell \cdot \tau_E = \mathrm{Curl}_\ell \, u_\ell \cdot \nu_E$, an integration by parts leads to

$$\int_E b_E \left[\nabla_\ell u_\ell \cdot \tau_E\right]_{\mathcal{E}_\ell} \, ds = \int_{\Omega_E} \nabla b_E \cdot \mathrm{Curl}_\ell \, u_\ell \, dx + \int_{\Omega_E} b_E \, \mathrm{div}_\ell \, \mathrm{Curl}_\ell \, u_\ell \, dx.$$

The last term vanishes because $u_\ell \in P_1(\mathcal{T}_\ell)$ is piecewise affine. Recall that Ω_E denotes the patch of the edge E. For all $v \in V$, the orthogonality $\nabla v \perp_{L^2(\Omega)} \mathrm{Curl}_\ell \, u_\ell$ leads to

$$h_E^{1/2} \|[\nabla u_\ell \cdot \tau_E]_{\mathcal{E}_\ell}\|_{L^2(E)} \approx \left| \int_{\omega_E} \mathrm{Curl} \, b_E (\nabla_\ell u_\ell - \nabla v) \, dx \right|.$$

The Cauchy inequality implies

$$h_E^{1/2} \|[\nabla u_\ell \cdot \tau_E]_{\mathcal{E}_\ell}\|_{L^2(E)} \lesssim \|\mathrm{Curl} \, b_E\|_{L^2(\Omega_E)} \|\nabla_\ell u_\ell - \nabla v\|_{L^2(\Omega_E)}.$$

The sum over all edges results in

$$\left(\sum_{E \in \mathcal{E}_\ell} h_E \|[\nabla_\ell u_\ell \cdot \tau_E]_{\mathcal{E}_\ell}\|_{L^2(E)}^2 \right)^{1/2}$$
$$\lesssim \left(\sum_{E \in \mathcal{E}_\ell} \|\nabla_\ell u_\ell - \nabla v\|_{L^2(\Omega_E)}^2 \right)^{1/2} \leq \sqrt{3} \|\nabla_\ell u_\ell - \nabla v\|_{L^2(\Omega)}.$$

Since this is valid for all $v \in H_0^1(\Omega)$, the proof is finished. □

Remark 5.1. For dG discretisations, the consistency residual involves jumps along edges and faces related to some flux functions. It can be estimated by means of the lifting operators defined above. As shown in [25], for the jump estimator η_ℓ there holds

$$\|\mathcal{R}es_{\mathrm{cons}}\|_{Q^*}^2 \lesssim \sum_{C \in \mathcal{C}_\ell} 1/h_C \|[u_\ell]\|_C^2 =: \zeta_\ell^2. \tag{5.2}$$

5.2. Explicit equilibration error estimates

This section is devoted to the estimation of residuals of the form

$$R(v) = \int_\Omega R_{\mathcal{T}} \cdot v \, dx + \int_{\bigcup \mathcal{C}_\ell} R_{\mathcal{C}} \cdot \{v\}_{\mathcal{C}_\ell} \, ds \tag{5.3}$$

for arbitrary functions $R_{\mathcal{T}} \in L^2(\Omega;\mathbb{R}^m)$ and $R_{\mathcal{C}} \in L^2(\bigcup \mathcal{C}_\ell;\mathbb{R}^m)$ defined for a regular triangulation $\mathcal{T}_\ell$ of Ω with the associated set $\mathcal{C}_\ell$. Set $R_T := R_{\mathcal{T}}|_T$ for triangles/tetrahedrons $T \in \mathcal{T}_\ell$ and $R_C := R_{\mathcal{C}}|_C$ for $C \in \mathcal{C}_\ell$. Recall that C can either be an edge (in 2D) or a face (in 3D) as defined in (4.1).

As mentioned before, the estimation of the dual norm of residuals structured like R has been subject to intense research in the last decades, cf. [1, 8, 43]. For any conforming discretisation considered here, it holds $V_\ell^{\mathrm{c}} \subset \ker(\mathcal{R}es_V)$. Moreover, we require the following condition for the non-conforming discrete finite element space $V_\ell^{\mathrm{nc}} \subset H^1(\mathcal{T}_\ell;\mathbb{R}^m) := \{v \in L^2(\Omega;\mathbb{R}^m) \mid \forall T \in \mathcal{T}_\ell, v|_T \in H^1(\Omega;\mathbb{R}^m)\}$ and its conforming counterpart $V_\ell^{\mathrm{c}} \subseteq V := H_0^1(\Omega;\mathbb{R}^m)$.

Assumption 5.1. *There exists an operator* $\Pi : V_\ell^c \to V_\ell^{nc}$ *such that for all* $v_\ell \in V_\ell^c$ *and all* $T \in \mathcal{T}_\ell$ *it holds*

$$\|\nabla_\ell(\Pi v_\ell)\|_{L^2(\Omega)} \lesssim \|\nabla v_\ell\|_{L^2(\Omega)} \quad \textit{and} \quad \int_T v_\ell \, dx = \int_T \Pi v_\ell \, dx. \tag{5.4}$$

Furthermore, for the discrete approximation $p_\ell \in L^2(\Omega;\mathbb{R}^{m\times n})$*, it holds*

$$\int_\Omega p_\ell : \nabla v_\ell \, dx = \int_\Omega p_\ell : \nabla_\ell(\Pi v_\ell) \, dx.$$

Remark 5.2. Assumption 5.1 was introduced in [23] called *(H3)* to generalise results for the equilibration residual estimation to a huge set of nonstandard finite elements. The non-conforming spaces V_ℓ^{nc} in our examples contain the conforming spaces $V_\ell^{\mathrm{c}} \subset V_\ell^{\mathrm{nc}}$. Hence, the assumption holds with $\Pi = \mathrm{id}|_{V_\ell^{\mathrm{c}}}$. Assumptions *(H1)*,*(H2)* from that reference are also automatically fulfilled in all examples of the present paper.

Indeed, for each triangle $T \in \mathcal{T}_\ell$ and $v_\ell \in V_\ell^{\mathrm{c}} := P_1(\mathcal{T}_\ell;\mathbb{R}^m)$, $\int_T (v_\ell - \Pi v_\ell)\, dx = 0$ by (5.4) and so Poincaré's inequality provides

$$\|h_T^{-1}(v_\ell - \Pi v_\ell)\|_{L^2(T)} \lesssim \|\nabla v_\ell\|_{L^2(T)} + \|\nabla \Pi v_\ell\|_{L^2(T)}.$$

The sum over all elements $T \in \mathcal{T}_\ell$ plus (5.4) implies

$$\|h_{\mathcal{T}}^{-1}(v_\ell - \Pi v_\ell)\|_{L^2(\Omega)} \lesssim \|\nabla v_\ell\|_{L^2(\Omega)}. \tag{5.5}$$

Theorem 5.2. *Suppose the linear functional $R : V + V_\ell^{nc} \to \mathbb{R}$ satisfies $V_\ell^{nc} \subset \ker R$ for the non-conforming finite element space V_ℓ^{nc} and can be written in the form* (5.3). *Moreover, suppose there is an operator Π according to Assumption* 5.1 *and consider the estimator*

$$\eta_\ell := \left(\sum_{C \in \mathscr{C}_\ell} h_C \|R_C\|^2_{L^2(C)} \right)^{1/2} = \|h_{\mathscr{C}}^{1/2} R_{\mathscr{C}}\|_{L^2(\bigcup \mathscr{C}_\ell)}$$

associated to some triangulation $\mathscr{T}_\ell$ with the set of free nodes $\mathscr{K}_\ell$. Then η_ℓ is a reliable and efficient estimator for $\|R\|_{V^*}$, i.e., it holds

$$\eta_\ell \lesssim \|R\|_{V^*} + \operatorname{osc}(R_{\mathscr{T}}, \mathscr{T}_\ell) + \operatorname{osc}(R_E, \mathscr{E}_\ell), \tag{5.6a}$$

$$\|R\|_{V^*} \lesssim \eta_\ell + \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell). \tag{5.6b}$$

Proof. [of (5.6b)] Let $\Pi : V_\ell^{\rm c} \to V_\ell^{\rm nc}$ be the operator of Assumption 5.1 and recall the approximation operator J_ℓ from Definition 4.1. For any $v \in V$, the linearity of R and $\Pi J_\ell v \in V_\ell^{\rm nc} \subset \ker R$ imply

$$R(v) = R(v - \Pi J_\ell v) = \int_\Omega R_{\mathscr{T}} \cdot (v - \Pi J_\ell v)\, dx + \int_{\bigcup \mathscr{C}_\ell} R_{\mathscr{C}} \cdot \{v - \Pi J_\ell v\}_{\mathscr{C}_\ell}\, ds.$$

The volume term can be estimated with the Cauchy inequality, estimation (4.11) in Theorem 4.2, and with the oscillation (4.4),

$$\begin{aligned} &\int_\Omega R_{\mathscr{T}} \cdot (v - \Pi J_\ell v)\, dx \\ &= \sum_{T \in \mathscr{T}_\ell} \left(\int_T R_T \cdot (v - J_\ell v)\, dx + \int_T (R_T - \overline{R}_T) \cdot (J_\ell v - \Pi J_\ell v)\, dx \right) \\ &\lesssim \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell) \|\nabla v\|_{L^2(\Omega)} + \operatorname{osc}(R_{\mathscr{T}}, \mathscr{T}_\ell) \|h_{\mathscr{T}}^{-1}(J_\ell v - \Pi J_\ell v)\|_{L^2(\Omega)}. \end{aligned}$$

With the estimate (5.5) and the stability of J_ℓ, the last norm can be further bounded by

$$\|h_{\mathscr{T}}^{-1}(J_\ell v - \Pi J_\ell v)\|_{L^2(\Omega)} \lesssim \|\nabla J_\ell v\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)}.$$

Hence,

$$\int_\Omega R_{\mathscr{T}} \cdot (v - \Pi J_\ell v)\, dx \lesssim \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell) \|v\|_{H^1(\Omega)}. \tag{5.7}$$

To analyze the edge/face terms, Cauchy's inequality leads to

$$\int_{\bigcup \mathscr{C}_\ell} R_{\mathscr{C}} \cdot \{v - \Pi J_\ell v\}_{\mathscr{C}_\ell}\, ds \le \sum_{C \in \mathscr{C}_\ell} \|R_C\|_{L^2(C)} \|\{v - \Pi J_\ell v\}_C\|_{L^2(C)}.$$

For each interior edge/face $C = T_+ \cap T_- \in \mathscr{C}_\ell$ with $T_+,\, T_- \in \mathscr{T}_\ell$, the trace inequality shows

$$\begin{aligned}\|\{v - \Pi J_\ell v\}_C\|_{L^2(C)} &\le \frac{1}{2}\|(v - \Pi J_\ell v)|_{T_+}\|_{L^2(C)} + \frac{1}{2}\|(v - \Pi J_\ell v)|_{T_-}\|_{L^2(C)} \\ &\lesssim h_C^{1/2}\|\nabla(v - \Pi J_\ell v)\|_{L^2(\Omega_C)} + h_C^{-1/2}\|v - \Pi J_\ell v\|_{L^2(\Omega_C)}.\end{aligned}$$

On boundary edges/faces $C \in \partial\Omega \cap \partial T$, the norm

$$\|\{v - \Pi J_\ell v\}_C\|_{L^2(C)} = \|v - \Pi J_\ell v\|_{L^2(C)}$$

is estimated by the same bound with $\Omega_C = \operatorname{int}(T)$. The sum over all edges/faces of $\mathscr{C}_\ell$ reads

$$\begin{aligned}&\int_{\bigcup \mathscr{C}_\ell} R_{\mathscr{C}} \cdot \{v - \Pi J_\ell v\}_{\mathscr{C}_\ell} ds \\ &\lesssim \|h_{\mathscr{C}}^{1/2} R_{\mathscr{C}}\|_{L^2(\bigcup \mathscr{C}_\ell)} \Big(\|\nabla(v - \Pi J_\ell v)\|_{L^2(\Omega)} + \|h_{\mathscr{T}}^{-1}(v - \Pi J_\ell v)\|_{L^2(\Omega)} \Big).\end{aligned}$$

The approximation and stability property of the last terms is controlled via the split

$$v - \Pi J_\ell v = (v - J_\ell v) + (J_\ell v - \Pi J_\ell v).$$

First, Theorem 4.2 implies

$$\|\nabla(v - J_\ell v)\|_{L^2(\Omega)} + \|h_{\mathscr{T}}^{-1}(v - J_\ell v)\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)}.$$

Second, inequality (5.5) reveals

$$\|\nabla(J_\ell v - \Pi J_\ell v)\|_{L^2(\Omega)} + \|h_{\mathscr{T}}^{-1}(J_\ell v - \Pi J_\ell v)\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)}.$$

The combination of the two estimates plus (5.7) implies

$$\|\nabla(v - \Pi J_\ell v)\|_{L^2(\Omega)} + \|h_{\mathscr{T}}^{-1}(v - \Pi J_\ell v)\|_{L^2(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)}.$$

Therefore,

$$\|R\|_{V^*} \lesssim \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{T}_\ell) + \|h_{\mathscr{C}}^{1/2} R_{\mathscr{C}}\|_{L^2(\bigcup \mathscr{C}_\ell)}. \qquad \square$$

Proof. [of (5.6a)] With $\Omega \subset \mathbb{R}^m$, for an edge or face $C = T_+ \cap T_- \in \mathscr{C}_\ell$ and simplices $T_+, T_- \in \mathscr{T}_\ell$, recall that for the bubble functions

$$b_C = \prod_{z \in \mathscr{N}(C)} \varphi_z \quad \text{and} \quad b_T = \prod_{z \in \mathscr{N}(T)} \varphi_z$$

it holds for all $D \in \mathscr{C}_\ell$ and $D \neq C$

$$\int_D b_C \, ds = 0 \quad \text{and} \quad \int_D b_T \, ds = 0.$$

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For $C = T_+ \cap T_- \in \mathscr{C}_\ell,\ T_+, T_- \in \mathscr{T}_\ell$, define the function

$$v_C = \alpha_{\mathscr{C}} b_C + \alpha_{\mathscr{T}} b_{T_+} + \alpha_{\mathscr{T}} b_{T_-}$$

and choose the two real numbers

$$\alpha_{\mathscr{C}} = |C| \left(\int_C b_C \, ds \right)^{-1} \quad \text{and} \quad \alpha_{\mathscr{T}} = -\frac{|C| \int_{T_+} b_C \, dx}{\int_C b_C \, ds \int_T b_T \, dx}.$$

In 2D we obtain $\alpha_{\mathscr{C}} = 6$ and $\alpha_{\mathscr{T}} = -30$, in 3D a simple calculation reveals $\alpha_{\mathscr{C}} = 120$ and $\alpha_{\mathscr{T}} = -840$. Then, for all $C, D \in \mathscr{C}_\ell$,

$$\int_D v_C \, ds = |C| \delta_{CD} \quad \text{and, for all } T \in \mathscr{T}_\ell, \quad \int_T v_C \, dx = 0.$$

Observe that

$$h_C^{1/2} \|R_C\|_{L^2(C)} \le h_C^{1/2} \|R_C - \overline{R}_C\|_{L^2(C)} + h_C^{1/2} \|\overline{R}_C\|_{L^2(C)} = \operatorname{osc}(R_C, C) + h_C^{1/2} \|\overline{R}_C\|_{L^2(C)}$$

and set $\varsigma := \operatorname{sign}(\int_C \overline{R}_C \, ds)$ to obtain

$$\varsigma h_C^{1/2} \|\overline{R}_C\|_{L^2(C)} = \int_C \overline{R}_C \, ds = \int_C \overline{R}_C v_C \, ds = \int_C (\overline{R}_C - R_C) v_C \, ds + \int_C R_C v_C \, ds.$$

The first term of the last line equals $\operatorname{osc}(R_C, C)$. The second term reads

$$\int_C R_C v_C \, ds = R(v_C) - \int_{\omega_C} R_{\mathscr{T}} v_C \, dx = R(v_C) - \int_{\omega_C} (R_{\mathscr{T}} - \overline{R}_{\mathscr{T}}) v_C \, dx.$$

With $\gamma_C := h_C^{1/2} \|R_C\|_{L^2(C)}$ the estimator η_ℓ is bounded by

$$\begin{aligned}
\eta_\ell^2 &= \sum_{C \in \mathscr{C}_\ell} h_C^{1/2} \gamma_C \|R_C\|_{L^2(C)} \\
&\le \sum_{C \in \mathscr{C}_\ell} R(\gamma_C v_C) + 2 \sum_{C \in \mathscr{C}_\ell} \gamma_C \operatorname{osc}(R_C, C) + \sum_{C \in \mathscr{C}_\ell} \gamma_C \operatorname{osc}(R_{\mathscr{T}}, \mathscr{T}_\ell(\omega_C)) \\
&\lesssim \|R\|_{V^*} \|\sum_{C \in \mathscr{C}_\ell} \gamma_C v_C\|_V + \left(\sum_{C \in \mathscr{C}_\ell} \gamma_C^2 \right)^{1/2} \left(\operatorname{osc}(R_{\mathscr{C}}, \mathscr{C}_\ell) + \operatorname{osc}(R_{\mathscr{T}}, \mathscr{T}_\ell) \right).
\end{aligned}$$

Noting that $\|\sum_{C \in \mathscr{C}_\ell} \gamma_C v_C\|_V \lesssim \eta_\ell = \left(\sum_{C \in \mathscr{C}_\ell} \gamma_C^2 \right)^{1/2}$ yields the desired result

$$\eta_\ell \lesssim \|R\|_{V^*} + \operatorname{osc}(R_{\mathscr{C}}, \mathscr{C}_\ell) + \operatorname{osc}(R_{\mathscr{T}}, \mathscr{T}). \qquad \square$$

6. Poisson problem

This section is devoted to conforming, non-conforming and mixed finite element discretisations for the Poisson model problem of Section 2.2.

6.1. Conforming finite element methods

Conforming methods with a discrete space $V_\ell \subset V = H_0^1(\Omega)$ approximate the flux $p = \nabla u$ by $p_\ell := \nabla u_\ell$. The discrete problem reads: Find $u_\ell \in V_\ell$ such that for all $v_\ell \in V_\ell$

$$\mathscr{A}(\nabla u_\ell, u_\ell)(\nabla v_\ell, v_\ell) = \int_\Omega \nabla u_\ell \cdot \nabla v_\ell \, dx = \int_\Omega f v_\ell \, dx = \ell(\nabla v_\ell, v_\ell).$$

Note that the consistency residual vanishes, i.e., $\mathscr{R}es_{\text{Cons}} = \mathscr{R}es_Q = 0$.

In order to estimate the equilibrium residual (3.4), we use Theorem 5.2. Let $V_\ell = P_k(\mathscr{T}_\ell) \cap V$. An elementwise integration by parts and the product rule

$$[v\, p_\ell \cdot \nu_E]_E = [p_\ell \cdot \nu_E]_E \{v\}_E + \{p_\ell \cdot \nu_e\}_E [v]_E = [p_\ell \cdot \nu_E]_E \{v\}_E$$

imply

$$\begin{aligned} \mathscr{R}es_{\text{eq}}(v) = \mathscr{R}es_V(v) &= \int_\Omega f v \, dx - \int_\Omega p_\ell \cdot \nabla v \, dx \\ &= \sum_{T \in \mathscr{T}_\ell} \int_T (f + \operatorname{div} p_\ell) v \, dx + \sum_{E \in \mathscr{E}_\ell} \int_E [p_\ell \cdot \nu_E]_E \{v\}_E \, ds. \end{aligned}$$

Hence, the residual allows the form (5.3) with the local volume and edge terms of the explicit residual on triangles $T \in \mathscr{T}_\ell$ and edges $E \in \mathscr{E}_\ell$ by

$$R_T := f|_T + \operatorname{div} p_\ell|_T \quad \text{and} \quad R_E := [p_\ell \cdot \nu_E]_E.$$

The global residuals then read

$$R_{\mathscr{T}} := f + \operatorname{div}_\ell p_\ell \quad \text{and} \quad R_{\mathscr{E}} := [p_\ell \cdot \nu_{\mathscr{E}}]_{\mathscr{E}}. \tag{6.1}$$

The equilibrium residual $\mathscr{R}es_V$ has the form of Theorem 5.2 and $V_\ell \subset \ker \mathscr{R}es_V$. Moreover, Assumption 5.1 holds with $\Pi = \mathrm{id}_{V_\ell}$. This leads to the error estimator

$$\eta_\ell = \|h_{\mathscr{E}}^{1/2} R_{\mathscr{E}}\|_{L^2(\bigcup \mathscr{E}_\ell)} \tag{6.2}$$

and to the error estimations

$$\begin{aligned} \|u - u_\ell\|_V \approx \|\mathscr{R}es_{\text{eq}}\|_{V^*} &\lesssim \eta_\ell + \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell), \\ \eta_\ell &\lesssim \|u - u_\ell\|_V + \operatorname{osc}(R_{\mathscr{C}}, \mathscr{C}_\ell) + \operatorname{osc}(R_{\mathscr{T}}, \mathscr{T}_\ell). \end{aligned}$$

In case of the affine discrete space $V_\ell = P_1(\mathscr{T}_\ell)$ for $m = 1$, $\operatorname{div} \nabla u_\ell = \overline{\operatorname{div} p_\ell}|_T = 0$ and the error estimator simplifies to

$$\|u - u_\ell\| \lesssim \eta_\ell + \operatorname{Osc}(f, \mathscr{K}_\ell) \quad \text{and} \quad \eta_\ell \lesssim \|u - u_\ell\|_V + \operatorname{osc}(R_{\mathscr{C}}, \mathscr{C}_\ell) + \operatorname{osc}(f, \mathscr{T}_\ell).$$

6.2. Non-conforming finite element methods ($\mathrm{CR}_1(\mathcal{T}_\ell)$)

In the non-conforming Crouzeix-Raviart FEM, the discrete solution $u_\ell \in V_\ell^{\mathrm{nc}} = \mathrm{CR}_1(\mathcal{T}_\ell)$ solves

$$\mathcal{A}_\ell(\nabla_\ell u_\ell, u_\ell) = \ell_Q + \ell_V.$$

This is equivalent to the formulation: Seek $u_\ell \in \mathrm{CR}_1(\mathcal{T}_\ell)$ with

$$\int_\Omega \nabla_\ell u_\ell \cdot \nabla_\ell v_\ell \, dx = \int_\Omega f v_\ell \, dx \quad \text{for all } v_\ell \in V_\ell^{\mathrm{nc}}(\mathcal{T}_\ell). \tag{6.3}$$

Recall that the ∇_ℓ denotes the piecewise action of the gradient operator with respect to the triangulation $\mathcal{T}_\ell$.

Theorem 5.1 yields equivalence of the explicit consistency error estimators for $\Omega \subset \mathbb{R}^2$,

$$\mu_1 := \|h_{\mathcal{T}}^{-1}(u_\ell - A_{\mathcal{T}_\ell} u_\ell)\|_{L^2(\Omega)} = \left(\sum_{T \in \mathcal{T}_\ell} \|h_T^{-1}(u_\ell - A_{\mathcal{T}_\ell} u_\ell)\|_{L^2(T)}^2 \right)^{1/2},$$

$$\mu_2 := \|h_{\mathcal{E}}^{1/2} \left[\nabla u_\ell \cdot \tau_{\mathcal{E}}\right]_{\mathcal{E}_\ell}\|_{L^2(\bigcup \mathcal{E}_\ell)} = \left(\sum_{E \in \mathcal{E}_\ell} h_E \| \left[\nabla u_\ell \cdot \tau_E\right]_E \|_{L^2(E)}^2 \right)^{1/2},$$

$$\mu_3 := \|h_{\mathcal{E}}^{-1/2} [u_\ell]_{\mathcal{E}_\ell}\|_{L^2(\bigcup \mathcal{E}_\ell)} = \left(\sum_{E \in \mathcal{E}_\ell} h_E^{-1} \|[u_\ell]_E\|_{L^2(E)}^2 \right)^{1/2}.$$

The equilibrium estimators from the previous section in the conforming case can be used for $V_\ell = \mathrm{CR}_1(\mathcal{T}_\ell)$ since all conditions of Theorem 5.2 are satisfied: The residual $\mathcal{R}es_V$ can be written in the form (5.3) with $R_{\mathcal{T}}$ and $R_{\mathcal{E}}$ from (6.1) and it holds $\mathrm{CR}_1(\mathcal{T}_\ell) \subset \ker \mathcal{R}es_V$.

Remark 6.1. The references [23, 24] list examples for non-conforming FEMs where Π is not the identity, e.g. the Rannacher-Turek nonconforming FEM for rotated polynomials on parallelograms where $V_\ell^{\mathrm{c}} \not\subset V_\ell^{\mathrm{nc}}$.

6.3. Mixed finite element methods (RT_0)

In the mixed Raviart-Thomas FEM of the Poisson model problem (2.3), one seeks the discrete solution $(p_\ell, u_\ell) \in Q_\ell \times V_\ell := \mathrm{RT}_0(\mathcal{T}_\ell) \times P_0(\mathcal{T}_\ell)$ for the solution $p := \nabla u$, i.e.

$$\begin{aligned} \forall q_\ell \in Q_\ell \qquad & \int_\Omega p_\ell \cdot q_\ell \, dx - \int_\Omega \mathrm{div}_\ell \, q_\ell u_\ell \, dx = 0, \\ \forall v_\ell \in V_\ell \qquad & \int_\Omega v_\ell \, \mathrm{div}\, p_\ell \, dx = \int_\Omega f v_\ell \, dx. \end{aligned} \tag{6.4}$$

By the Helmholtz decomposition in $n = 2$ or $n = 3$ dimensions, there exist functions $\alpha \in V = H_0^1(\Omega)$ and $\beta \in H^1(\Omega)$ with

$$p_\ell = \nabla \alpha + \mathrm{Curl}\, \beta \quad \text{and} \quad \mathrm{dist}_{L^2(\Omega)}(p_\ell, \nabla H_0^1(\Omega)) = \|\mathrm{Curl}\, \beta\|_{L^2(\Omega)}.$$

We utilize the different definitions of the curl differential operator given in Section 1. With $\tilde{u}_\ell = \alpha$ it holds, for any $q_\ell \in Q_\ell$,

$$\mathscr{R}es_Q(q_\ell) = -\int_\Omega (p_\ell - \nabla\alpha)\cdot q_\ell \, dx = \int_\Omega \operatorname{Curl}\beta \cdot q_\ell \, dx. \tag{6.5}$$

The residual (6.5) does not satisfy the conditions in Theorem 5.1 since $p_\ell \in \mathrm{RT}_0(\mathscr{T}_\ell)$ does not belong to the space of piecewise gradients $P_0(\mathscr{T}_\ell)$. It holds

$$\|\mathscr{R}es_Q\|^2_{Q_\ell^*} = \|\operatorname{Curl}\beta\|^2_{L^2(\Omega)}$$

and the orthogonality $\operatorname{Curl}\beta \perp_{L^2(\Omega)} \nabla\alpha$ implies

$$\|\operatorname{Curl}\beta\|^2_{L^2(\Omega)} = \int_\Omega \operatorname{Curl}\beta\cdot(\operatorname{Curl}\beta + \nabla\alpha)\,dx = \int_\Omega \operatorname{Curl}\beta\cdot p_\ell\,dx.$$

For any function $\beta_\ell \in P_{1,0}(\mathscr{T}_\ell)$ with $\operatorname{Curl}\beta_\ell \in P_0(\mathscr{T}_\ell;\mathbb{R}^n)$ it holds

$$\operatorname{div}\operatorname{Curl}\beta_\ell = 0 \quad \text{a.e. in } \Omega$$

and it is an admissible test function in the RT-FEM. To obtain an explicit consistency error estimator, calculate

$$\begin{aligned}\int_\Omega \operatorname{Curl}\beta\cdot p_\ell &= \int_\Omega \operatorname{Curl}(\beta-\beta_\ell)\cdot p_\ell\,dx\\ &= \sum_{T\in\mathscr{T}_\ell}\left\{-\int_T \operatorname{curl} p_\ell\,(\beta-\beta_\ell)\,dx + \int_{\partial T} p_\ell\,(\beta-\beta_\ell)\cdot\tau_{\partial T}\,ds\right\}\\ &= -\int_\Omega \operatorname{curl}_\ell p_\ell(\beta-\beta_\ell)\,dx + \sum_{E\in\mathscr{E}_\ell}\int_E (\beta-\beta_\ell)\,[p_\ell\cdot\tau_E]_E\,ds\end{aligned}$$

and choose $\beta_\ell = J_\ell\beta$ for the approximation operator J_ℓ (cf. Definition 4.1 and [17]) to obtain

$$\|\mathscr{R}es_{\mathrm{cons}}\|^2_{Q^*} \lesssim \|\nabla\beta\|_{L^2(\Omega)}\left(\|h_{\mathscr{T}}\operatorname{curl}_\ell p_\ell\|_{L^2(\Omega)} + \|h_{\mathscr{E}}^{1/2}[p_\ell\cdot\tau_{\mathscr{E}}]_{\mathscr{E}_\ell}\|_{L^2(\bigcup\mathscr{E}_\ell)}\right).$$

With $\|\beta\|_{H^1(\Omega)} \lesssim \|p-p_\ell\|_{L^2(\Omega)}$, it follows that

$$\|p-p_\ell\|_{L^2(\Omega)} \lesssim \mu_\ell := \|h_{\mathscr{T}}\operatorname{curl}_\ell p_\ell\|_{L^2(\Omega)} + \|h_{\mathscr{E}}^{1/2}[p_\ell\cdot\tau_{\mathscr{E}}]_{\mathscr{E}_\ell}\|_{L^2(\bigcup\mathscr{E}_\ell)}.$$

For the analysis of the efficiency of the error estimator we refer to [16].

For the evaluation of $\mathscr{R}es_V$, note that $p_\ell \in H(\operatorname{div},\Omega)$ and define $\overline{f}_\ell \in P_0(\mathscr{T}_\ell)$ satisfying $\overline{f}_\ell|_T = |T|^{-1}\int_T f\,dx$ on every triangle $T \in \mathscr{T}_\ell$. Then, for all $v \in H_0^1(\Omega)$ and $\overline{v}_\ell \in P_0(\mathscr{T}_\ell)$ analogous to $\overline{f}_\ell$, it follows

$$\begin{aligned}\mathscr{R}es_{\mathrm{eq}}(v) = \mathscr{R}es_V(v) &:= \int_\Omega f v\,dx - \int_\Omega p_\ell\cdot\nabla v\,dx\\ &= \int_\Omega (f + \operatorname{div} p_\ell)v\,dx = \int_\Omega (f-\overline{f}_\ell)(v-\overline{v}_\ell)\,dx.\end{aligned} \tag{6.6}$$

For the norms it then holds

$$\|\mathscr{R}es_{\mathrm{eq}}(v)\|_{L^2(\Omega)} \leq \sum_{T\in\mathscr{T}} \|f - \overline{f}_\ell\|_{L^2(T)} \|v - \overline{v}_\ell\|_{L^2(T)}.$$

On every $T \in \mathscr{T}$, a Poincaré inequality with the Payne-Weinberger constant $1/\pi$ [39] for $v - \overline{v} \in H_0^1(T)$ yields

$$\|v - \overline{v}_\ell\|_{L^2(T)} \leq \frac{h}{\pi}\|\nabla v\|_{L^2(T)} \leq h\|v\|_{H^1(T)}.$$

These considerations lead to the result

$$\|\mathscr{R}es_{\mathrm{eq}}\|_{V^*} \lesssim \mathrm{osc}(f, \mathscr{T}_\ell).$$

Alternatively, from (6.6) it follows with $R_{\mathscr{T}} := f + \mathrm{div}\, p_\ell$ and $R_{\mathscr{E}} = [p_\ell \cdot \nu_E]_{\mathscr{E}} = 0$ for $p_\ell \in \mathrm{RT}_0(\mathscr{T}_\ell)$ that Theorem 5.2 yields an equilibrium estimator $\eta_\ell = \|h_{\mathscr{E}}^{1/2} R_{\mathscr{E}}\|_{L^2(\bigcup \mathscr{E}_\ell)} = 0$ with

$$\|\mathscr{R}es_{\mathrm{eq}}\|_{V^*} \lesssim \mathrm{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell).$$

6.4. Discontinuous Galerkin methods

A unified formulation for all common discontinuous Galerkin (dG) schemes was presented in [4], see also [25]. dG methods are characterized by suitably chosen local numerical flux functions $\hat{u}_T$ and $\hat{p}_T$ which are linear functionals in u_ℓ and give rise to global flux functions $\hat{u}$ and $\hat{p}$.

The unified problem for the Poisson problem reads: Find $u_\ell \in V_\ell$ and $p_\ell \in Q_\ell$ such that, for all $q_\ell \in Q_\ell$ and $v_\ell \in V_\ell$, it holds

$$a(p_\ell, q_\ell) = -\int_\Omega u_\ell \,\mathrm{div}_\ell\, q_\ell \,dx + \sum_{T\in\mathscr{T}_\ell} \int_{\partial T} \hat{u}_T (\nu_T \cdot q_\ell)\,ds,$$

$$\int_\Omega p_\ell \cdot \nabla_\ell v_\ell \,dx = \int_\Omega g v_\ell \,dx + \sum_{T\in\mathscr{T}_\ell} \int_{\partial T} (\hat{p}_T \cdot \nu_T) v_\ell \,ds.$$

The solution (u_ℓ, p_ℓ) satisfies [4]

$$p_\ell = \nabla_\ell u_\ell + r\left([\hat{u} - u_\ell]\right) + s\left([\hat{u} - u_\ell]\right).$$

With the equilibrium error estimator η_ℓ from (6.2) and the consistency error estimator ζ_ℓ from (5.2) it holds

$$\|p - p_\ell\|_Q \lesssim \eta_\ell + \zeta_\ell.$$

In case of the Interior Penalty (IP) dG method, the local flux functions are chosen according to $\hat{u}_T = \{u_\ell\}$ and $\hat{p}_T = \{\nabla u_\ell\} - \alpha([u_\ell])$, where $\alpha > 0$ is a sufficiently large penalty parameter. A convergence analysis and quasi-optimality of the IP dG scheme has been established in [11]. In particular, it has been shown that the consistency error ζ_ℓ can be controlled by the estimator η_ℓ in the sense that $\alpha\,\zeta_\ell \lesssim \eta_\ell$ for sufficiently large α.

7. Stokes problem

7.1. Mixed conforming finite element methods

The conforming discrete problem of the Stokes equations in the symmetric formulation reads: Find $u_\ell \in V_\ell$ and $p_\ell \in Q_\ell$ such that

$$\forall\, v_\ell \in V_\ell \quad \int_\Omega 2\mu\, \varepsilon(u_\ell) : \varepsilon(v_\ell)\, dx - \int_\Omega p_\ell \operatorname{div} v_\ell\, dx = \int_\Omega f \cdot v_\ell\, dx,$$
$$\forall\, q_\ell \in Q_\ell \quad \int_\Omega q_\ell \operatorname{div} u_\ell\, dx = 0.$$

Recall from Section 2.3 that

$$\|\mathscr{R}es_{\text{cons}}\|_{Q_\ell^*} \approx 2\mu \|\varepsilon_\ell(u_\ell) - \varepsilon_\ell(\tilde{u}_\ell)\|_{L^2(\Omega;\mathbb{R}^{n\times n})} + \|\operatorname{div} \tilde{u}_\ell\|_{L_0^2(\Omega)}.$$

In the conforming case, $\tilde{u}_\ell = u_\ell$ implies $\|\varepsilon(u_\ell) - \varepsilon(\tilde{u}_\ell)\|_{L^2(\Omega)} = 0$ (the same holds in the non-symmetrical formulation), and thus, $\|\mathscr{R}es_{\text{cons}}\|_{Q_\ell^*} = \|\operatorname{div} u_\ell\|_{L_0^2(\Omega)}$.

The equilibrium residual reads

$$\mathscr{R}es_{\text{eq}}(v) := \int_\Omega f \cdot v\, dx + \int_\Omega \sigma_\ell : \varepsilon(v)\, dx.$$

A piecewise integration by parts argument yields

$$\begin{aligned} \mathscr{R}es_{\text{eq}}(v) &= \int_\Omega f \cdot v\, dx - \int_\Omega \varepsilon(v) : \sigma_\ell\, dx \\ &= \int_\Omega f \cdot v\, dx - \int_\Omega \sigma_\ell : \nabla v\, dx \\ &= \int_\Omega f \cdot v\, dx + \sum_{T\in\mathscr{T}_\ell} \left(\int_T \operatorname{div}_\ell \sigma_\ell \cdot v\, dx - \int_{\partial T} \sigma_\ell \cdot \nu_T v\, ds \right) \\ &= \sum_{T\in\mathscr{T}_\ell} \int_T (f + \operatorname{div}_\ell \sigma_\ell) \cdot v\, dx + \sum_{E\in\mathscr{E}_\ell} \int_E [\sigma_\ell \cdot \nu_E]_E \cdot v\, ds. \end{aligned} \tag{7.1}$$

Hence, in order to apply Theorem 5.2, we set

$$R_{\mathscr{T}} := f + \operatorname{div}_\ell \sigma_\ell \quad \text{and} \quad R_{\mathscr{E}} := [\sigma_\ell \cdot \nu_{\mathscr{E}}]_{\mathscr{E}_\ell}. \tag{7.2}$$

With the identity Π, one derives the reliable equilibrium estimator

$$\eta_{\text{eq}} := \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell) + \|h_{\mathscr{E}}^{1/2} [\sigma_\ell \cdot \nu_{\mathscr{E}}]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell)} \tag{7.3}$$

for the symmetrical as well as for the non-symmetrical formulation of the Stokes problem.

Remark 7.1 (Taylor-Hood Elements in 2D)**.** The Taylor-Hood FEM for the non-symmetrical Stokes equations uses P_2-elements for both components of u_ℓ and P_1-elements for the approximated pressure p_ℓ [12, 14]. The error estimator

$$\eta_\ell := \eta_{\mathrm{eq}} + \|\operatorname{div} u_\ell\|_{L^2(\Omega)} = \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell) + \|h_{\mathscr{E}}^{1/2}[\sigma_\ell \cdot \nu_{\mathscr{E}}]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell)} + \|\operatorname{div} u_\ell\|_{L^2(\Omega)}$$

is reliable and efficient for the overall error $\|\mathscr{R}es_{\mathrm{eq}}\| + \|\mathscr{R}es_{\mathrm{cons}}\|$.

Remark 7.2 (MINI elements in 2D)**.** For the affine nodal basis functions φ_z (cf. Section 5) the triangle bubble function b_T for $T = \operatorname{conv}\{a, b, c\} \in \mathscr{T}_\ell$ is defined by

$$b_T := \varphi_a \varphi_b \varphi_c = \prod_{z \in \mathscr{N}(T)} \varphi_z.$$

The MINI-FEM in two dimensions uses the discrete spaces [12, 14]

$$V_\ell = \left(P_{1,0}(\mathscr{T}_\ell) + \operatorname{span}\{b_T \mid T \in \mathscr{T}_\ell\}\right)^2 \quad \text{and} \quad Q_\ell = P_{1,0}(\mathscr{T}_\ell; \mathbb{R}^2).$$

The term $\|\operatorname{div} u_\ell\|_{L^2(\Omega)}$ is not necessarily zero and therefore, its norm has to be added to the error estimator. The value of

$$\eta_\ell := \eta_{\mathrm{eq}} + \|\operatorname{div} u_\ell\|_{L^2(\Omega)} = \operatorname{Osc}(R_{\mathscr{T}}, \mathscr{K}_\ell) + \|h_{\mathscr{E}}^{1/2}[\sigma_\ell \cdot \nu_{\mathscr{E}}]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell)} + \|\operatorname{div} u_\ell\|_{L^2(\Omega)}$$

is a reliable and efficient estimator for the error norm.

7.2. Non-conforming finite element methods

The equilibrium error estimator of Section 7.1 can also be employed in the non-conforming case.

For the norm of the consistency residual in the non-symmetric formulation of the Stokes equations as well as (up to a constant) in the symmetric case (7.1), we obtain

$$\|\mathscr{R}es_{\mathrm{cons}}\|_{V^*} \approx \|\nabla_\ell(u_\ell) - \nabla(\tilde{u}_\ell)\|_Q + \|\operatorname{div} \tilde{u}_\ell\|_Q.$$

In the non-conforming Crouzeix-Raviart finite element method for the non-symmetric Stokes problem in two dimensions, one chooses the discrete spaces $V_\ell = \mathrm{CR}_{1,0}(\mathscr{T}_\ell; \mathbb{R}^2)$ and $Q_\ell = P_0(\mathscr{T}_\ell)$. In this setting it holds that $\operatorname{div}_\ell u_\ell = 0$ and the second term can be estimated by the first term as follows,

$$\|\operatorname{div} \tilde{u}_\ell\|_Q = \|\operatorname{div}_\ell u_\ell - \operatorname{div} \tilde{u}_\ell\|_Q = \|\operatorname{tr}(\nabla_\ell \tilde{u}_\ell - \nabla u_\ell)\|_Q \lesssim \|\nabla_\ell(u_\ell) - \nabla(\tilde{u}_\ell)\|_Q.$$

Then, component-wise application of Theorem 5.1 provides the error estimates

$$\mu_1 := \|h_{\mathscr{E}}^{1/2}\left[\nabla u_\ell \cdot \tau_{\mathscr{E}}\right]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell; \mathbb{R}^2)}, \qquad \mu_2 := \|h_{\mathscr{E}}^{-1/2}[u_\ell]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell; \mathbb{R}^2)}.$$

Another example is the finite element $(P_{1,0}(\mathscr{T}_\ell; \mathbb{R}) \times \mathrm{CR}_{1,0}(\mathscr{T}_\ell, \mathbb{R})) \times P_0(\mathscr{T}_\ell)$ of Kouhia and Stenberg [35] for non-symmetric Stokes which is only non-conforming in one component. Hence, Theorem 5.1 again yields error estimates, with $u_\ell = (u_{\ell,1}, u_{\ell,2})$ for example

$$\mu_1 := \|h_{\mathscr{E}}^{1/2}\left[\nabla u_{\ell,2} \cdot \tau_{\mathscr{E}}\right]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell)}, \qquad \mu_2 := \|h_{\mathscr{E}}^{-1/2}[u_{\ell,2}]_{\mathscr{E}_\ell}\|_{L^2(\bigcup \mathscr{E}_\ell)}.$$

7.2.1. Discontinuous Galerkin Methods

Given some flux functions $\hat{u}_{T,\sigma}, \hat{u}_{T,p}, \hat{\sigma}_T$ and the space

$$P_k^0(\mathscr{T}_\ell; R^m) := \left\{ v \in P_k(\mathscr{T}_\ell; \mathbb{R}^m) \;\Big|\; \int_\Omega v\, dx = 0 \right\},$$

the unified dG formulation of the Stokes problem reads: Find $(u_\ell, \sigma_\ell, p_\ell) \in P_k(\mathscr{T}_\ell; \mathbb{R}^m) \times P_k(\mathscr{T}_\ell; \mathbb{R}^{m\times n}) \times P_{k-1}^0(\mathscr{T}_\ell; \mathbb{R}^m) := X_\ell$ such that for all $(v, w, q) \in X_\ell$,

$$\begin{aligned}
\int_\Omega \sigma_\ell : w\, dx &= -\mu \int_\Omega u_\ell \cdot (\operatorname{div}_\ell w)\, dx + \mu \sum_{T\in\mathscr{T}_\ell} \int_{\partial T} \hat{u}_{T,\sigma} \cdot (w \cdot \nu_T)\, ds \\
&\quad - \int_\Omega p_\ell \operatorname{tr}(w)\, dx, \\
\int_\Omega \sigma_\ell : \nabla_\ell v\, dx &= \int_\Omega f \cdot v\, dx + \sum_{T\in\mathscr{T}_\ell} \int_{\partial T} \hat{\sigma}_T : (v \otimes \nu_T)\, ds, \\
\int_\Omega u_\ell \cdot \nabla_\ell q\, dx &= \sum_{T\in\mathscr{T}_\ell} \int_{\partial T} (\hat{u}_{T,p} \cdot \nu_T) q\, ds
\end{aligned}$$

are satisfied. As an example, the IP dG scheme is obtained for $\hat{u}_{T,\sigma} = \hat{u}_{T,p} = \{u_\ell\}$ and $\hat{\sigma}_T = \{\nabla_\ell u_\ell\} - \alpha_j([u_\ell])$. The selection of the parameter α_j is described in [41].

The discrete solution $(u_\ell, \sigma_\ell, p_\ell) \in X_\ell$ satisfies the identity [25]

$$\sigma_\ell = \mu \nabla_\ell u_\ell - p_\ell \mathbb{1} + \mu r([\hat{u}_\sigma - u_\ell]) + \mu s([\hat{u}_\sigma - u_\ell]).$$

Suppose that u and $\sigma := \mu\nabla u - p\mathbb{1}$ solve the Stokes problem and (u_ℓ, σ_ℓ) is the solution of the previous discretisation. From the unified analysis, with the equilibrium error estimator $\eta_\ell = (7.3)$ and with the consistency error estimator ζ_ℓ from (5.2) it holds

$$\|\sigma - \sigma_\ell\|_Q \lesssim \eta_\ell + \zeta_\ell + \|\operatorname{div}_\ell u_\ell\|_{L^2(\Omega)} \lesssim \eta_\ell + \zeta_\ell.$$

8. Lamé problem

8.1. Conforming finite element methods

Conforming methods with a discrete finite element space $V_\ell \subset V$ approximate the stress matrix $\sigma = \mathbb{C}\varepsilon(u)$ by $\sigma_\ell := \mathbb{C}\varepsilon(u_\ell)$ and the discrete problem therefore turns out to solve the weak formulation of the Lamé problem: Find the Galerkin solution $u_\ell \in V_\ell$ such that

$$\mathscr{A}(\mathbb{C}\varepsilon(u_\ell), u_\ell) = \ell_Q + \ell_V.$$

This is equivalent to the mixed formulation of the problem: Find $u_\ell \in V_\ell$ and $p_\ell \in Q_\ell$ such that

$$\forall\ \tau_\ell \in Q_\ell \quad \int_\Omega (\mathbb{C}^{-1}\sigma_\ell - \varepsilon(u_\ell)) : \tau_\ell \, dx = 0, \tag{8.1a}$$

$$\forall\ v_\ell \in V_\ell \quad \int_\Omega \varepsilon(v_\ell) : \sigma_\ell \, dx = \int_\Omega f \cdot v_\ell \, dx. \tag{8.1b}$$

In the conforming case considered first, the previous formulation is identical to

$$\int_\Omega \mathbb{C}\varepsilon(u_\ell) : \varepsilon(v_\ell)\, dx = \int_\Omega f \cdot v_\ell \, dx \qquad \text{for all } v_\ell \in V_\ell.$$

The consistency residual vanishes for any conforming finite element method, i.e. $\mathscr{R}es_{\text{cons}} = \mathscr{R}es_Q = 0$.

Let $V_\ell = P_k(\mathscr{T}_\ell; \mathbb{R}^n)$. In analogy to the derivation of the explicit estimator in Section 7.1, noting that $\sigma_\ell := \mathbb{C}\varepsilon(u_\ell)$ is symmetric and using integration by parts, the equilibrium residual $\mathscr{R}es_{\text{eq}} = \mathscr{R}es_V$ is defined by (3.6) and can be written as in (5.3) with

$$R_{\mathscr{T}} := f + \operatorname{div}_\ell \sigma_\ell \quad \text{and} \quad R_{\mathscr{E}} := [\sigma_\ell \cdot \nu_{\mathscr{E}}]_{\mathscr{E}_\ell}.$$

This is identical to (7.2) and again can be used with Theorem 5.2 and $\Pi = \mathrm{id}_{V_\ell}$ which yields the estimator (7.3). Moreover, from $\|e\|_{H^1(\Omega)} \lesssim \|\sigma - \sigma_\ell\|_{L^2(\Omega)}$ one obtains

$$\|\sigma - \sigma_\ell\|_{L^2(\Omega)} \approx \eta_\ell$$

independent of the material parameter λ.

Remark 8.1 ($P_k \times P_k$)**.** For any conforming finite element method of arbitrary polynomial degree the usual equilibrium estimator given above is applicable.

8.2. Non-conforming finite element methods

There are different robust non-conforming finite element methods known for elasticity problems. As an example we consider a construction due to Kouhia and Stenberg [35] which leads to the discrete space $V_\ell := P_{1,0}(\mathscr{T}_\ell) \times \mathrm{CR}_{1,0}(\mathscr{T}_\ell)$. The discrete problem reads: Find $u_\ell \in V_\ell$ such that

$$\int_\Omega \varepsilon(v_\ell) : \mathbb{C}\varepsilon_\ell(u_\ell)\, dx = \int_\Omega f \cdot v_\ell \, dx \qquad \text{for all } v_\ell \in V_\ell.$$

The discrete stress is given by $\sigma_\ell := \mathbb{C}\varepsilon_\ell(u_\ell)$ and while the equilibrium error is estimated by η_ℓ as in the conforming case above, the consistency residual $\mathscr{R}es_{\text{cons}} = \mathscr{R}es_Q = \varepsilon_\ell(u - \tilde{u}_\ell)$

does not vanish in the second component and also has to be considered. Since u_ℓ satisfies the requirements of Theorem 5.1, two equivalent reliable consistency estimators are

$$\mu_1 := \left\| h_{\mathcal{E}}^{1/2} \left[\nabla u_\ell \cdot \tau_{\mathcal{E}} \right]_{\mathcal{E}_\ell} \right\|_{L^2(\bigcup \mathcal{E}_\ell)} = \left(\sum_{E \in \mathcal{E}_\ell} h_E \| \left[\nabla u_\ell \cdot \tau_E \right]_E \|_{L^2(E)}^2 \right)^{1/2},$$

$$\mu_2 := \left\| h_{\mathcal{E}}^{-1/2} [u_\ell]_{\mathcal{E}_\ell} \right\|_{L^2(\bigcup \mathcal{E}_\ell)} = \left(\sum_{E \in \mathcal{E}_\ell} h_E^{-1} \| [u_\ell]_E \|_{L^2(E)}^2 \right)^{1/2}.$$

8.3. Mixed finite element methods

8.3.1. Plane Elasticity Element with Reduced Symmetry (PEERS)

Some λ-robust approximation of the solution of the mixed formulation concerns discrete spaces of reduced symmetry. For a possibly non-symmetric matrix-valued L^2-function A, we denote the skew-symmetric part of A by

$$\operatorname{skew} A = A - \operatorname{sym} A = (A - A^T)/2 \text{ and } \mathbb{R}^{d\times d}_{\text{skew}} = \left\{ B \in \mathbb{R}^{d\times d} \,|\, B + B^T = 0 \right\}.$$

Notice that for a symmetric matrix $A \in \mathbb{R}^{d\times d}_{\text{sym}}$ and for a skew-symmetric matrix $B \in \mathbb{R}^{d\times d}_{\text{skew}}$ it holds $A : B = 0$.

The idea behind the PEERS [3] is to allow a non-symmetric stress matrix σ. In our notation, a tuple (σ, γ) denotes such a possibly non-symmetric stress matrix σ and a skew-symmetric matrix γ. The equation

$$\int_\Omega \sigma : \gamma \, dx = 0 \quad \text{for all } \gamma \text{ in some subset of } L^2\left(\Omega; \mathbb{R}^{d\times d}_{\text{skew}}\right)$$

guarantees the symmetry in weak form of σ. The spaces

$$Q := L^2\left(\Omega; \mathbb{R}^2\right) \times W, \;\; W := L^2\left(\Omega; \mathbb{R}^{2\times 2}_{\text{skew}}\right) \;\text{ and }\; V := H\left(\operatorname{div}, \Omega; \mathbb{R}^{2\times 2}\right)$$

and the bilinear forms

$$a(u, \gamma; v, \delta) := \int_\Omega \left(u \cdot v + \gamma : \delta \right) \, dx,$$

$$b(u, \gamma; \sigma) = \int_\Omega \left(u \cdot \operatorname{div} \sigma + \operatorname{skew} \sigma : \gamma \right) \, dx,$$

$$c(\sigma, \tau) := \int_\Omega \mathbb{C}^{-1} \sigma : \tau \, dx, \qquad \Lambda(\sigma) := (\operatorname{div} \sigma, \operatorname{skew} \sigma), \qquad \ell_V(\sigma) := 0,$$

$$\ell_Q(u, \gamma; v, \delta) := \int_\Omega \left(f \cdot v + u \cdot v + \gamma : \delta \right) \, dx$$

lead to a mixed formulation which includes an additional equation for the weak symmetry condition, i.e.

$$\forall v \in L^2\left(\Omega;\mathbb{R}^2\right) \quad -\int_\Omega \operatorname{div}\sigma\cdot v\,dx = \int_\Omega f\cdot v\,dx, \tag{8.2a}$$

$$\forall \delta \in W \quad \int_\Omega \operatorname{skew}\sigma : \delta\,dx = 0, \tag{8.2b}$$

$$\forall \tau \in V \quad \int_\Omega \left(\operatorname{div}\tau\cdot u + \gamma : \operatorname{skew}\tau + \mathbb{C}^{-1}\sigma : \tau\right)\,dx = 0. \tag{8.2c}$$

PEERS employ the discrete spaces

$$\begin{aligned}
Q_\ell &:= P_0\left(\mathcal{T}_\ell;\mathbb{R}^2\right)\times W_\ell,\\
W_\ell &:= \left\{\gamma_\ell \in L^2\left(\Omega;\mathbb{R}^{2\times 2}_{\text{skew}}\right)\cap C^0\left(\Omega;\mathbb{R}^{2\times 2}_{\text{skew}}\right)\;\middle|\;\forall T\in\mathcal{T}_\ell,\quad \gamma_\ell|_{\mathcal{T}}\in P_1\left(T;\mathbb{R}^{2\times 2}_{\text{skew}}\right)\right\},\\
V_\ell &:= \left\{\sigma_\ell \in H\left(\operatorname{div},\Omega;\mathbb{R}^{2\times 2}\right)\;\middle|\;\forall T\in\mathcal{T}_\ell,\quad \sigma_\ell|_T \in \mathrm{RT}_0(T)\oplus B_0(T)\right\}.
\end{aligned}$$

Here, RT_0 is the lowest order Raviart-Thomas space and B_0 is the curl of the bubble-function space for $b_T = \prod_{z\in\mathcal{N}(T)}\varphi_z$ defined on any $T\in\mathcal{T}_\ell$ by

$$\begin{aligned}
\mathrm{RT}_0(T) &:= \left\{\sigma \in L^2\left(T;\mathbb{R}^{2\times 2}\right)\;\middle|\;\sigma(x) = \tau + a\cdot x^T, \tau\in\mathbb{R}^{2\times 2},\ a\in\mathbb{R}^2\right\},\\
B_0(T) &:= \left\{\sigma \in L^2\left(T;\mathbb{R}^{2\times 2}\right)\;\middle|\;\sigma(x) = a\cdot \operatorname{Curl} b_T(x)^T,\ a\in\mathbb{R}^2\right\}.
\end{aligned}$$

Note that, in general, $\operatorname{skew}\sigma_\ell \not\equiv 0$ for the PEERS stress field σ_ℓ. The following equivalence is due to the residuals from Section 3.5 for any $\tilde{u}_\ell \in H^1\left(\Omega;\mathbb{R}^2\right)$,

$$\|\sigma - \operatorname{sym}\sigma_\ell\|_Q + \|u - \tilde{u}_\ell\|_V \approx \|\varepsilon(\tilde{u}_\ell) - \mathbb{C}^{-1}\operatorname{sym}\sigma_\ell\|_{Q^*} + \|\mathcal{R}es_V\|_{V^*}. \tag{8.3}$$

We remark that in essence the derivation of bounds for the residuals is analogous to the treatment in Section 6.3. To estimate the consistency term

$$\mu := \min_{\tilde{u}_\ell\in V}\|\varepsilon(\tilde{u}_\ell) - \mathbb{C}^{-1}\operatorname{sym}\sigma_\ell\|_Q, \tag{8.4}$$

a Helmholtz decomposition for some symmetric stress in $L^2(\Omega;\mathbb{R}^{2\times 2}_{\text{sym}})$ has been derived in [19] and results in

$$\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell = \varepsilon(\alpha) + \operatorname{Curl}\operatorname{Curl}\beta \quad \text{for some } \alpha\in H^1_0(\Omega)^2 \text{ and } \beta\in H^2(\Omega),$$

cf. [19,20] for details in slightly different notation. Therefore, the choice of $\tilde{u}_\ell = \alpha$ leads to the minimum in (8.4) and leads to orthogonality of $B := \operatorname{Curl}\beta \in \operatorname{Curl}(H^2(\Omega))$ and $\nabla(H^1_0(\Omega)^2)$

$$\mu = \min_{\tilde{u}_\ell\in V}\|\varepsilon(\tilde{u}_\ell) - \mathbb{C}^{-1}\operatorname{sym}\sigma_\ell\|_Q = \|\operatorname{Curl}\operatorname{Curl}\beta\|_Q.$$

Let $J_\ell : H^1(\Omega) \to P_1(\mathcal{T}_\ell)$ be some quasi-interpolation operator from Section 4.3, set $B := \operatorname{Curl}\beta$ and notice that $\operatorname{Curl} B_\ell := \operatorname{Curl} J_\ell B \in \mathrm{RT}_0(\mathcal{T}_\ell)$ is an admissible test tensor in (8.2c) and hence

$$\int_\Omega \operatorname{Curl} B_\ell : (\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell + \gamma_\ell)\,dx = 0.$$

This, the aforementioned orthogonality and an element-wise integration by parts lead to

$$\begin{aligned}\mu^2 = \|\operatorname{Curl} B\|_Q^2 &= \int_\Omega \operatorname{Curl}(B - B_\ell) : (\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell + \gamma_\ell)\,dx \\ &\le \sum_{T\in\mathcal{T}_\ell} \left\|B - B_\ell\right\|_{L^2(T)} \left\|\operatorname{curl}(\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell + \gamma_\ell)\right\|_{L^2(T)} \\ &\quad + \sum_{E\in\mathcal{E}_\ell} \left\|B - B_\ell\right\|_{L^2(E)} \left\|\left[\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell + \gamma_\ell\right]\cdot\tau_E\right\|_{L^2(E)}.\end{aligned}$$

Since $\|B\|_{H^1(\Omega)} \lesssim \|\operatorname{Curl} B\|_Q$ and with the interpolation estimates for $\left\|B - B_\ell\right\|$ on triangles and edges, it follows eventually that

$$\begin{aligned}\mu \lesssim &\left(\sum_{T\in\mathcal{T}_\ell} h_T^2 \left\|\operatorname{curl}(\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell + \gamma_\ell)\right\|_{L^2(T)}^2\right)^{1/2} \\ &+ \left(\sum_{E\in\mathcal{E}_\ell} h_E \left\|\left[\mathbb{C}^{-1}\operatorname{sym}\sigma_\ell + \gamma_\ell\right]\cdot\tau_E\right\|_{L^2(E)}^2\right)^{1/2}.\end{aligned}$$

Recall from (8.2a) that, for $\sigma_\ell \in V_\ell$, it holds

$$\int_\Omega v_\ell\cdot(f + \operatorname{div}\sigma_\ell)\,dx = 0 \qquad \text{for all } v_\ell \in P_0\left(\mathcal{T}_\ell;\mathbb{R}^2\right).$$

The equilibrium residual $\mathcal{R}es_V$ is split in the symmetric and the skew-symmetric part of σ_ℓ and the observation that $\varepsilon(v) : \sigma_\ell = \nabla v : \operatorname{sym}\sigma_\ell$ plus an integration by parts lead, for all $v \in Q$, to

$$\begin{aligned}\mathcal{R}es_V(v) &= \int_\Omega \left(f\cdot v - \varepsilon(v) : \sigma_\ell\right)\,dx \\ &= \int_\Omega (f + \operatorname{div}\sigma_\ell)\cdot v\,dx - \int_\Omega \operatorname{skew}(\sigma_\ell) : \nabla v\,dx.\end{aligned}$$

Note that $\sigma_\ell \in H(\operatorname{div},\Omega)$ and thus there are no jumps across inter-element edges. Let $\overline{v}_\ell$ and $\overline{f}_\ell|_T = |T|^{-1}\int_T \operatorname{div}\sigma_\ell\,dx$ denote the $\mathcal{T}$-piecewise constant averages of v and f. Then, following the same arguments as for (6.6), Poincaré inequalities result in

$$\begin{aligned}\int_\Omega (f + \operatorname{div}\sigma_\ell)\cdot v\,dx &= -\int_\Omega (v - \overline{v}_\ell)(f - \overline{f}_\ell)\,dx \\ &\le \|h_{\mathcal{T}}^{-1}(v - \overline{v}_\ell)\|_{L^2(\Omega)}\|h_{\mathcal{T}}(f - \overline{f}_\ell)\|_{L^2(\Omega)} \\ &\le \|\nabla v\|_{L^2(\Omega)}\operatorname{osc}(f,\mathcal{T}_\ell)/\pi.\end{aligned}$$

With higher-order oscillations $\operatorname{osc}(f,\mathcal{T}_\ell)$ for $f \in H^1(\mathcal{T}_\ell)^2$, it follows

$$\|\mathcal{R}es_V\|_{V^*} \le \|\operatorname{skew}\sigma_\ell\|_{L^2(\Omega)} + \operatorname{osc}(f,\mathcal{T}_\ell)/\pi.$$

The non-symmetric stress error of σ_ℓ is split into

$$\|\sigma - \sigma_\ell\|_Q^2 = \|\sigma - \operatorname{sym}\sigma_\ell\|_Q^2 + \|\operatorname{skew}\sigma_\ell\|_Q^2.$$

This amounts to the final a posteriori error estimate

$$\|\sigma - \sigma_\ell\|_Q \lesssim \mu + \|\mathcal{R}es_Q\|_{V^*} + \|\operatorname{skew}\sigma_\ell\|_Q$$

with a reliability constant (behind the notation $\lesssim$) which does not depend on λ or $h_{\mathcal{T}}$.

8.3.2. Arnold-Winther finite elements

Arnold and Winther proposed a mixed finite element method with symmetric stress field (cf. [7]). These elements are locking-free in numerical experiments and satisfy the predicted convergence rates, cf. [21].

One seeks the displacement field $u \in L^2\left(\Omega;\mathbb{R}^2\right)$ and the stress tensor

$$\sigma \in H\left(\operatorname{div},\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}}\right)$$

satisfying the mixed problem (8.1). We consider the spaces

$$V := H\left(\operatorname{div},\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}}\right), \quad Q := L^2\left(\Omega;\mathbb{R}^2\right),$$

and their discrete counterparts, on any triangle $T \in \mathcal{T}_\ell$,

$$\begin{aligned} V_T &:= \left\{\tau \in P_3\left(T;\mathbb{R}^{2\times 2}_{\mathrm{sym}}\right) \;\middle|\; \operatorname{div}\tau \in P_1(T;\mathbb{R}^2)\right\},\\ Q_\ell &:= P_1\left(\mathcal{T}_\ell;\mathbb{R}^2\right). \end{aligned}$$

Given the data $f \in L^2(\Omega;\mathbb{R}^2)$, the weak formulation of the linear Lamé problem reads: Find $(\sigma,u) \in V \times Q$ such that

$$\begin{aligned} \int_\Omega \sigma : \mathbb{C}^{-1}\tau\,dx + \int_\Omega u\cdot\operatorname{div}\tau\,dx &= 0 \qquad \text{for all } \tau \in V,\\ \int_\Omega v\cdot\operatorname{div}\sigma\,dx = -\int_\Omega f\cdot v\,dx & \qquad\qquad \text{for all } v \in Q. \end{aligned} \tag{8.5}$$

The corresponding discrete solution in the discrete spaces defined above is denoted (σ_ℓ, u_ℓ). We refer to Section 3.5 and stress that σ_ℓ is symmetric. The analysis follows that of the previous section and starts with (8.3). The equilibrium residual reads

$$\mathcal{R}es_V(v) := \int_\Omega f\cdot v\,dx - \int_\Omega \varepsilon(v):\sigma_\ell = \int_\Omega (f + \operatorname{div}\sigma_\ell)\cdot v\,dx.$$

Analogous to Section 8.3.1, it can be estimated by $\operatorname{osc}(f,\mathcal{T}_\ell) := \min_{f_\ell \in P_1(\mathcal{T}_\ell)^2} \|h_{\mathcal{T}}(f - f_\ell)\|$.

The consistency residual reads

$$\mathcal{R}es_Q(\tau) = \int_\Omega (\varepsilon(\tilde{u}_\ell)) - \mathbb{C}^{-1}\sigma_\ell) : \tau \, dx.$$

8.3.3. Discontinuous Galerkin methods

Given some numerical fluxes $\hat{u}_{T,\varepsilon}, \hat{u}_{T,p}, \hat{\varepsilon}_T, \hat{p}_T$ and the space $P_k^0(\Omega;\mathbb{R}^m)$ as in Section 7.2.1, the unified dG formulation of the Lamé problem reads: Find $(\varepsilon_\ell, u_\ell, p_\ell) \in P_k(\Omega;\mathbb{R}^{m\times n}_{\text{sym}}) \times P_k(\Omega;\mathbb{R}^m) \times P_k^0(\Omega;\mathbb{R}^m) =: Q_\ell$ such that, for all $(\tau, v, q) \in Q_\ell$, it holds

$$\begin{aligned}
&\int_\Omega \varepsilon_\ell : \tau \, dx = -\int_\Omega u_\ell \operatorname{div}_\ell \tau \, dx + \sum_{T\in\mathcal{T}_\ell} \int_{\partial T} \hat{u}_{T,\varepsilon} \cdot (\tau \nu_T) \, ds,\\
&\int_\Omega (2\mu\varepsilon_\ell - p_\ell \mathbb{1}) : D_\ell v \, dx = \int_\Omega g \cdot v \, dx + \sum_{T\in\mathcal{T}_\ell} \int_{\partial T} (2\mu\hat{\varepsilon}_T - \hat{p}_T \mathbb{1}) : v \otimes \nu_T \, ds,\\
&\int_\Omega \frac{1}{\lambda} p_\ell q \, dx = \int_\Omega u_\ell \cdot \nabla_\ell q \, dx - \sum_{T\in\mathcal{T}_\ell} \int_{\partial \mathcal{T}_\ell} (\hat{u}_{T,p} \cdot \nu_T) q \, ds.
\end{aligned}$$

We obtain the IP dG scheme with the fluxes $\hat{u}_{T,\varepsilon} = \hat{u}_{T,p} = \{u_\ell\}$, $\hat{\varepsilon}_T = \{\varepsilon(u_\ell)\}$ and $\hat{p}_T = -\lambda\{\operatorname{div} u_\ell\}$, for instance.

Set $\sigma_\ell := 2\mu\varepsilon_\ell - p_\ell \mathbb{1}$ and assume a discrete solution $(\varepsilon_\ell, u_\ell, p_\ell)$. Then, with parameters κ_p, d, β, c_1, c_2 from [30,32] and [25],

$$L_1 = -r\left([u_\ell]\right) + c_1 s\left([u_\ell]\cdot\beta\right) \quad \text{and} \quad L_2 = r\left([u_\ell]\right) + c_2 s\left(d\cdot[u_\ell] + \kappa_p[p_\ell]\right)$$

satisfy

$$\mathbb{C}^{-1}\sigma_\ell = \varepsilon(u_\ell) + L_1 - \frac{\lambda}{n\lambda + 2\mu}(L_2 + \operatorname{tr}(L_1))\mathbb{1}$$

for some $c_1, c_2 \in \{-1, 0, 1\}$. Moreover, the following error bound can be shown

$$\|\sigma - \sigma_\ell\|_V \lesssim \eta_\ell + \zeta_\ell + \left(\sum_{C\in\mathcal{C}_{\ell,\Omega}} 1/h_c \|[p_\ell]^2\|_{L^2(C)}\right)^{1/2}.$$

9. Eddy current problem

9.1. Conforming edge element methods

The curl-conforming approximation of the eddy current equation (2.6) by means of the lowest order edge elements of Nédélec's first family [37,38] with respect to a simplicial triangulation $\mathcal{T}_\ell$ of the computational domain $\Omega \subset \mathbb{R}^3$ amounts to the computation of $u_\ell \in \mathrm{Nd}_{1,0}(\mathcal{T}_\ell)$ such that

$$\int_\Omega \left(\mu^{-1} \operatorname{curl} u_\ell \cdot \operatorname{curl} v_\ell + \sigma u_\ell \cdot v_\ell\right) dx = \int_\Omega f \cdot v_\ell \, dx \quad \text{for all } v_\ell \in \mathrm{Nd}_{1,0}(\mathcal{T}_\ell).$$

Consequently, in (3.7),(3.8) we may choose $\tilde{u}_\ell = u_\ell$ and $p_\ell = \mu^{-1} \operatorname{curl} u_\ell$. For the consistency residual $\mathcal{R}es_{\text{cons}} = \mathcal{R}es_Q$ it follows that

$$\mathcal{R}es_Q(q) = 0 \quad \text{for all } q \in \mathrm{Nd}_{1,0}(\mathcal{T}_\ell).$$

On the other hand, the equilibrium residual $\mathcal{R}es_{\text{eq}} = \mathcal{R}es_V$ reads

$$\mathcal{R}es_V(v) = \int_\Omega (f - \sigma u_\ell) \cdot v \, dx - \int_\Omega p_\ell \cdot \operatorname{curl} v \, dx \quad \text{for all } v \in V.$$

Following [2], we may decompose $v \in V$ by means of

$$v = z + \nabla\varphi \ \text{ for } \ z \in \ker(\operatorname{curl})^\perp \ \text{ and } \ \varphi \in H^1_0(\Omega),$$

so that the equilibrium residual splits accordingly,

$$\mathcal{R}es_V^{(1)}(z) := \int_\Omega (f - \sigma u_\ell) \cdot z \, dx - \int_\Omega p_\ell \cdot \operatorname{curl} z \, dx \qquad \text{for } z \in \ker(\operatorname{curl})^\perp, \tag{9.1a}$$

$$\mathcal{R}es_V^{(2)}(\varphi) := \int_\Omega (f - \sigma u_\ell) \cdot \nabla\varphi \, dx \qquad \text{for } \varphi \in H^1_0(\Omega). \tag{9.1b}$$

An elementwise application of Stokes' theorem resolves the second integral on the right-hand side of (9.1a) to

$$\int_T p_\ell \cdot \operatorname{curl} z \, dx = -\int_{\partial T} \nu \wedge (p_\ell \wedge \nu) \cdot (z \wedge \nu) \, d\sigma + \int_T \operatorname{curl} p_\ell \cdot z \, dx.$$

Hence,

$$\begin{aligned}\mathcal{R}es_V^{(1)}(z) = &\sum_{T \in \mathcal{T}_\ell} \int_T (f - \sigma u_\ell - \operatorname{curl} p_\ell) \cdot z \, dx \\ &+ \sum_{F \in \mathcal{F}_\ell} \int_F [\nu \wedge (p_\ell \wedge \nu)]_F \cdot (z \wedge \nu) \, d\sigma \text{ for } z \in \ker(\operatorname{curl})^\perp.\end{aligned}$$

On the other hand, for $\varphi \in H^1_0(\Omega)$ an elementwise Green's formula on the right-hand side in (9.1b) results in

$$\mathcal{R}es_V^{(2)}(\varphi) = \sum_{T \in \mathcal{T}_\ell} \int_T \operatorname{div}(\sigma u_\ell - f) \ \varphi \, dx + \sum_{F \in \mathcal{F}_\ell} \int_F [\nu \cdot (f - \sigma u_\ell)]_F \ \varphi \, d\sigma.$$

Introducing the element residuals

$$R_T^{(1)} := \left(f - \sigma u_\ell - \operatorname{curl} p_\ell\right)\Big|_T \quad \text{and} \quad R_T^{(2)} := \operatorname{div}\left(\sigma u_\ell - f\right)\Big|_T,$$

and the face residuals

$$R_F^{(1)} := \left[\nu \wedge (p_\ell \wedge \nu)\right]_F \quad \text{and} \quad R_F^{(2)} := \left[\nu \cdot \left(f - \sigma u_\ell\right)\right]_F,$$

and applying Theorem 4.1 and Theorem 5.2 yield the equilibrium error estimator (cf. [33, 40])

$$\eta_\ell = \left(\sum_{j=1}^{2}\left(\sum_{T\in\mathcal{T}_\ell} h_T^2 \|R_T^{(j)}\|_{L^2(T)}^2 + \sum_{F\in\mathcal{F}_\ell} h_F \|R_F^{(j)}\|_{L^2(F)}^2\right)\right)^{1/2}. \tag{9.2}$$

9.2. Discontinuous Galerkin methods

We consider an *Interior Penalty Discontinuous Galerkin* (IP dG) method for the eddy current equations (2.6). Given a geometrically conforming, shape-regular simplicial triangulation $\mathcal{T}_\ell$ of the computational domain $\Omega \subset \mathbb{R}^3$, the discrete spaces V_ℓ and Q_ℓ are chosen as elementwise polynomials of degree $\leq p$,

$$V_\ell := P_p(\mathcal{T}_\ell; \mathbb{R}^3) \quad \text{and} \quad Q_\ell := P_p(\mathcal{T}_\ell; \mathbb{R}^3).$$

For $u_\ell, v_\ell \in V_\ell$ and $q_\ell \in Q_\ell$ we set

$$J_V(u_\ell, v_\ell) := \sum_{F\in\mathcal{F}_\ell} \int_F \Big(\{\nu \wedge (\operatorname{curl}\, u_\ell \wedge \nu)\}_F - \alpha\,[u_\ell \wedge \nu]_F\Big) \cdot [v_\ell \wedge \nu]_F \, d\sigma,$$

$$J_Q(v_\ell, q_\ell) := \sum_{F\in\mathcal{F}_\ell} \int_F \{\nu \wedge q_\ell \wedge \nu\}_F \cdot [v_\ell \wedge \nu]_F \, d\sigma,$$

where $\alpha \geq \alpha_{\min} > 0$ is some suitably chosen penalty parameter. The mixed formulation of the *Interior Penalty Discontinuous Galerkin Method* reads: Find $(u_\ell, p_\ell) \in V_\ell \times Q_\ell$ such that

$$a(p_\ell, q_\ell) - b(u_\ell, q_\ell) = \ell_Q(q_\ell) + J_Q(u_\ell, q_\ell) \quad \text{for all } q_\ell \in Q_\ell, \tag{9.3a}$$

$$b(v_\ell, p_\ell) + c(u_\ell, v_\ell) = \ell_V(v_\ell) + J_V(u_\ell, v_\ell) \quad \text{for all } v_\ell \in V_\ell. \tag{9.3b}$$

We note that the classical formulation in the primal variable reads: Find $u_\ell \in V_\ell$ such that, for all $v_\ell \in V_\ell$, there holds

$$\begin{aligned} &c(u_\ell, v_\ell) + \sum_{T\in\mathcal{T}_\ell} (\mu^{-1} \operatorname{curl} u_\ell, \operatorname{curl} v_\ell)_{L^2(T)} \\ =&\ell_Q(\mu^{-1}\ \operatorname{curl} v_\ell) + \ell_V(v_\ell) + J_Q(u_\ell, \mu^{-1}\ \operatorname{curl} v_\ell) + J_V(u_\ell, v_\ell). \end{aligned} \tag{9.4}$$

Remark 9.1. It is easy to see that the formulations (9.3)–(9.4) are formally equivalent in the following sense. If $(u_\ell, p_h) \in V_\ell \times Q_\ell$ solves (9.3), then $u_\ell \in V_\ell$ solves (9.4). Conversely, if $u_\ell \in V_\ell$ solves (9.4), then there exists some $p_\ell \in Q_\ell$ such that (u_ℓ, p_ℓ) solves (9.3) (cf. Theorem 4.1 in [22]).

The *consistency error* associated with the solution $(u_\ell, p_\ell) \in V_\ell \times Q_\ell$ of (9.3) is given by

$$\mu_\ell := \min_{\tilde{v}_\ell \in V} \Big(\|u_\ell - \tilde{v}_\ell\|^2_{L^2(\Omega)} + \|\operatorname{curl}_\ell u_\ell - \operatorname{curl} \tilde{v}_\ell\|^2_{L^2(\Omega)} \Big)^{1/2},$$

where curl_ℓ stands for the elementwise curl. The minimum is attained with a minimizer $\tilde{u}_\ell \in V$, i.e.,

$$\mu_\ell^2 = \|u_\ell - \tilde{u}_\ell\|^2_{L^2(\Omega)} + \|\operatorname{curl}_\ell u_\ell - \operatorname{curl} \tilde{u}_\ell\|^2_{L^2(\Omega)}.$$

Choosing $p_\ell := \mu^{-1} \operatorname{curl} \tilde{u}_\ell \in Q$, the residuals $\mathscr{R}es_V$ and $\mathscr{R}es_Q$ are given by

$$\begin{aligned} \mathscr{R}es_V(v) &= \int_\Omega \Big(f \cdot v - \mu^{-1} \operatorname{curl}_\ell \tilde{u}_\ell \cdot \operatorname{curl} v - \sigma\, \tilde{u}_\ell \cdot v \Big) dx, \quad v \in V, \\ \mathscr{R}es_Q(q) &= 0, \quad q \in Q. \end{aligned}$$

Since $J_V(u_\ell, v_\ell) = 0$ for $v_\ell \in \mathrm{Nd}_{1,0}(\mathscr{T}_\ell)$, an application of Stokes' theorem shows, for all $v_\ell \in \mathrm{Nd}_{1,0}(\mathscr{T}_\ell)$,

$$\mathscr{R}es_V(v_\ell) = c(u_\ell - \tilde{u}_\ell, v_\ell).$$

The unified theory leads to

$$\|(u - \tilde{u}_\ell, p - p_\ell\|_{V \times Q} \lesssim \eta_\ell + \mu_\ell,$$

where the estimator η_ℓ is as in (9.2) with the element and face residuals $R_T^{(i)}, R_F^{(i)}, 1 \le i \le 2$, given by

$$\begin{aligned} R_T^{(1)} &:= h_T \Big(f - \sigma u_\ell - \operatorname{curl} \mu^{-1} \operatorname{curl} u_\ell \Big)\Big|_T, \qquad & R_T^{(2)} &:= h_T \operatorname{div}(f - \sigma u_\ell)\Big|_T, \\ R_F^{(1)} &:= h_F^{1/2} \Big[\nu \wedge (\mu^{-1} \operatorname{curl} u_\ell \wedge \nu) \Big]_F, & R_F^{(2)} &:= h_F^{1/2} \Big[\nu \cdot \big(f - \sigma u_\ell \big) \Big]_F. \end{aligned}$$

An estimate $\bar{\mu}_\ell$ for the consistency error μ_ℓ has been provided in Proposition 4.1 of [34] according to

$$\mu_\ell^2 \lesssim \bar{\mu}_\ell^2 := \alpha \sum_{F \in \mathscr{F}_\ell(\Omega)} h_F^{-1} \|[u_\ell \wedge \nu]_F\|^2_{L^2(F)},$$

which yields

$$\|(u - \tilde{u}_\ell, p - p_\ell)\|_{V \times Q} \lesssim \eta_\ell + \bar{\mu}_\ell.$$

As in the case of the IP dG scheme for second order elliptic boundary value problems (cf., e.g., Lemma 3.6 in [11]), it is not difficult to see that $\bar{\mu}_\ell$ can be controlled by the estimator η_ℓ. In fact, for sufficiently large penalty parameter α it holds

$$\alpha \left(\sum_{F \in \mathscr{F}_\ell(\Omega)} h_F^{-1} \|[u_\ell \wedge \nu]_F\|^2_{L^2(F)} \right)^{1/2} \lesssim \eta_\ell,$$

so that we arrive at

$$\|(u - \tilde{u}_\ell, p - p_\ell)\|_{V\times Q} \lesssim \eta_\ell.$$

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