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On Stability of L-Fuzzy Mappings with Related Fixed Point Results

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Abstract In this paper, a general idea of Presic type L-fuzzy fixed point results using some weakly contractive conditions in the setting of metric space is initiated. Stability of L-fuzzy mappings and associated new concepts are proposed herein to complement their corresponding notions related to crisp multi-valued and single-valued mappings. Illustrative nontrivial examples are provided to support the assertions of our main results.

Keywords L-fuzzy set, L-fuzzy fixed point, metric space, multi-valued mapping, stability

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1. Introduction

In fixed point theory with metric structure, the contractive inequalities on underlying mappings play a significant role in solving fixed point problems. The Banach contraction mapping principle (see [6]) is one of the first well-known results in metric fixed point theory. Meanwhile, various extensions and generalizations of Banach contraction principle abound in the literature (e.g., see [2,15,17] and the references therein). One of the well-celebrated results in this field is attributed to Presic [25], who established an interesting generalization of the Banach contraction principle with significant applications in the study of global asymptotic stability of equilibriums of nonlinear difference equations arising in dynamic systems and related areas. For some articles related to Presic type results, we refer to [4,10,24] and references therein.

On the other hand, the real world is filled with uncertainty, vagueness and imprecisions. The notions we meet in everyday life are vague rather than precise. In practical, if a model asserts that conclusions drawn from it have some bearings on reality, then two major complications are obvious, namely, real situations are often not crisp and deterministic; a complete description of real systems often requires more detailed data than human beings can recognize simultaneously, process and understand. Conventional mathematical tools, which require all inferences to be exact, are not always sufficient for handling imprecisions in a wide variety of practical fields. Thus, to reduce these shortcomings inherent with the earlier mathematical concepts, the introduction of fuzzy sets were introduced in 1965 by Zadeh [29]. Consequently, various areas of mathematics, social sciences and engineering witnessed

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tremendous revolutions. In the mean time, the basic notions of fuzzy sets have been improved and applied in different directions. In 1981, Heilpern [14] availed the idea of fuzzy set to initiate a class of fuzzy set-valued mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of fixed point theorems by Nadler [23] and Banach [6]. Afterwards, several authors have studied the existence of fixed points of fuzzy set-valued maps. For example, see [7, 13, 19–22]. One of the useful generalizations of fuzzy sets by replacing the interval $[0, 1]$ of the range set by a complete distributive lattice was initiated by Goguen [12] and was called L-fuzzy sets. Not long ago, Rashid et al., [26] came up with the notion of L-fuzzy mappings and established a common fixed point theorem through β_{FL}-admissible pair of L-fuzzy mappings. As an improvement of the notion of Hausdorff distance and σ_∞-metric for fuzzy sets, Rashid et al., [27] defined the concepts of $D_{\widetilde{\alpha}L}$ and σ_L^∞ distances for L-fuzzy sets and generalized some known fixed point theorems for fuzzy and multi-valued mappings.

Following the above chain of developments, we initiate in this paper a general examination of Presic type L-fuzzy fixed point results by employing weakly contractive conditions in the bodywork of metric space. Stability of L-fuzzy mappings and associated novel notions are proposed to complement their corresponding concepts related to multi-valued and point-to-point-valued mappings. In the case where the L-fuzzy set valued map is reduced to its crisp counterparts, our results improve a number of significant metric fixed point theorems in the related literature.

2. Preliminaries

Hereafter, the sets $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$, represent the set of real numbers, positive real numbers and natural numbers respectively.

Definition 2.1. Let $(\widetilde{X}, \sigma)$ be a metric space. A mapping $\vartheta : \widetilde{X} \longrightarrow \widetilde{X}$ is said to be weakly contractive, if for all $x, y \in \widetilde{X}$,

$$\sigma(\vartheta(x), \vartheta(y)) \leq \sigma(x, y) - \varphi(\sigma(x, y)),$$

where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and non-decreasing function such that $\varphi(0) = 0$ and $\varphi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Alber and Guerre-Delabriere [3] showed that every weakly contractive mapping on a Hilbert space is a Picard operator. Rhoades [28] proved that the corresponding theorem on a complete metric space is also true. Dutta et al., [11] extended the idea of weak contractive condition and obtained a fixed point result which improved the main results in [3, 28].

Definition 2.2. Let $l \geq 1$ be a positive integer. A point $u \in \widetilde{X}$ is called a fixed point of $\vartheta : \widetilde{X}^l \longrightarrow \widetilde{X}$, if $\vartheta(u, \cdots, u) = u$.

Consider the lth-order nonlinear difference equation given by

$$x_{n+l} = \vartheta(x_n, \cdots, x_{n+l-1}), \ n \in \mathbb{N} \tag{2.1}$$

with initial values $x_1, \cdots, x_l \in \widetilde{X}$. Equation (2.1) becomes a fixed point problem in the sense that $u \in \widetilde{X}$ is a solution of (2.1), if and only if u is a fixed point of $\rho : \widetilde{X} \longrightarrow \widetilde{X}$ defined as

$$\rho(u) = \vartheta(u, \cdots, u), \ \text{for all } u \in \widetilde{X}.$$

Presic [25] established a very significant result in the light of (2.1) as follows.

Theorem 2.1 ([25]). *Let $(\widetilde{X},\sigma)$ be a complete metric space, l a positive integer and $\vartheta:\widetilde{X}^l\longrightarrow\widetilde{X}$ be a mapping satisfying the condition*

$$\sigma(\vartheta(x_1,\cdots,x_l),\vartheta(x_2,\cdots,x_{l+1}))\leq\lambda\max\{\sigma(x_1,x_2),\cdots,\sigma(x_l,x_{l+1})\},$$

for all $x_1,\cdots,x_l\in\widetilde{X}$, where $\lambda\in(0,1)$. Then, there exists $u\in\widetilde{X}$ such that $\vartheta(u,\cdots,u)=u$. Moreover, for any arbitrary point $x_1,\cdots,x_l\in\widetilde{X}$, the sequence defined by (2.1) *converges to u and*

$$\lim_{n\longrightarrow\infty}x_n=\vartheta(\lim_{n\longrightarrow\infty}x_n,\cdots,\lim_{n\longrightarrow\infty}x_n).$$

Notice that by putting $l=1$, Theorem 2.1 reduces to the Banach fixed point theorem. Theorem 2.1 has attracted a lot of attention due to its importance in the study of global asymptotic stability for the equilibrium of the fixed point problem (2.1).

Not long ago, Abbas et al., [1] studied the convergence of a generalized weak Presic type l-step method for a certain family of operators fulfilling Presic type contractive conditions as follows.

Theorem 2.2 ([1]). *Let $(\widetilde{X},\sigma)$ be a complete metric space. If a mapping $\vartheta:\widetilde{X}^l\longrightarrow\widetilde{X}$, for a positive l, satisfies:*

$$\begin{aligned}\sigma(\vartheta(x_1,\cdots,x_l),\vartheta(x_2,\cdots,x_{l+1}))\leq\ &\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\}\\&-\varphi(\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\}),\end{aligned}$$

for all $(x_1,\cdots,x_{l+1})\in\widetilde{X}^{l+1}$, where $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a lower semi-continuous function with $\varphi(t)=0$, if and only if $t=0$. Then, for arbitrary point $x_1,\cdots,x_l\in\widetilde{X}$, the sequence defined by (2.1) *converges to $u\in\widetilde{X}$ and $\vartheta(u,\cdots,u)=u$. Moreover, if*

$$\sigma(\vartheta(x,\cdots,x),\vartheta(y,\cdots,y))\leq\sigma(x,y)-\varphi(\sigma(x,y))$$

holds for all $x,y\in\widetilde{X}$ with $x\neq y$, then ϑ has a unique fixed point in $\widetilde{X}$.

Let the set of all nonempty compact subsets of $\widetilde{X}$ be denoted by $\mathcal{K}(\widetilde{X})$, where $(\widetilde{X},\sigma)$ is a metric space. For $A,B\in\mathcal{K}(\widetilde{X})$, the function $\widetilde{H}:\mathcal{K}(\widetilde{X})\times\mathcal{K}(\widetilde{X})\longrightarrow\mathbb{R}_+$ defined by

$$\widetilde{H}(A,B)=\max\left\{\sup_{x\in A}\sigma(x,B),\sup_{x\in B}\sigma(x,A)\right\}$$

is called Hausdorff-Pompeiu metric on $\mathcal{K}(\widetilde{X})$ induced by the metric σ, where

$$\sigma(x,A)=\inf_{y\in A}\sigma(x,y).$$

The following Lemma due to Nadler [23] is useful for establishing our results.

Lemma 2.1. *Let $(\widetilde{X},\sigma)$ be a metric space and $A,B\in\mathcal{K}(\widetilde{X})$. Then, for each $a\in A$, there exists $b\in B$ such that*

$$\sigma(a,b)\leq\widetilde{H}(A,B).$$

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In the following, we recall specific concepts of fuzzy sets and L-fuzzy sets that are needed in the sequel. For these concepts, we follow [12,26,29].

Let $\widetilde{X}$ be a universal set. A fuzzy set in $\widetilde{X}$ is a function with domain $\widetilde{X}$ and values in $[0,1]=I$. If A is a fuzzy set in $\widetilde{X}$, then the function value $A(x)$ is called the grade of membership of x in A. The $\widetilde{\alpha}$-level set of a fuzzy set A is denoted by $[A]_{\widetilde{\alpha}}$, and is defined as follows

$$[A]_{\widetilde{\alpha}}=\begin{cases}\overline{\{x\in\widetilde{X}:A(x)>0\}}, & \text{if } \widetilde{\alpha}=0;\\ \{x\in\widetilde{X}:A(x)\geq\widetilde{\alpha}\}, & \text{if } \widetilde{\alpha}\in(0,1],\end{cases}$$

where by $\overline{M}$, we mean the closure of the crisp set M. We denote the family of fuzzy sets in $\widetilde{X}$ by $I^{\widetilde{X}}$.

A fuzzy set A in a metric space V is said to be an approximate quantity, if and only if $[A]_{\widetilde{\alpha}}$ is compact and convex in V and $\sup_{x\in V}A(x)=1$. Denote the collection of all approximate quantities in V by $W(V)$. If there exists an $\widetilde{\alpha}\in[0,1]$ such that $[A]_{\widetilde{\alpha}},[B]_{\widetilde{\alpha}}\in\mathcal{K}(\widetilde{X})$, then we define

$$D_{\widetilde{\alpha}}(A,B)=\widetilde{H}([A]_{\widetilde{\alpha}},[B]_{\widetilde{\alpha}}),$$

$$\sigma_{\infty}(A,B)=\sup_{\widetilde{\alpha}}D_{\widetilde{\alpha}}(A,B).$$

Definition 2.3. A relation $\preceq$ on a nonempty set L is called a partial order, if it is

(i) reflexive;

(ii) antisymmetric;

(iii) transitive.

A set L together with a partial ordering $\preceq$ is called a partially ordered set (poset for short), and is denoted by $(L,\preceq_L)$. Recall that partial orderings are used to provide an order for sets that may not have a natural one.

Definition 2.4. Let L be a nonempty set and $(L,\preceq)$ be a partially ordered set. Then, any two elements $x,y\in L$ are said to be comparable if either $x\preceq y$ or $y\preceq x$.

Definition 2.5. A partially ordered set $(L,\preceq_L)$ is called:

(i) a lattice , if $x\vee y\in L$ and $x\wedge y\in L$, for any $x,y\in L$;

(ii) a complete lattice, if $\bigvee A\in L$, and $\bigwedge A\in L$, for any $A\subseteq L$;

(iii) distributive lattice, if $x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)$ and $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$, for any $x,y,z\in L$.

A partially ordered set L is called a complete lattice, if for every doubleton $\{x,y\}$ in L, either $\sup\{x,y\}=x\bigvee y$ or $\inf\{x,y\}=x\bigwedge y$ exists.

Definition 2.6. Let L be a lattice with top element 1_L and bottom element 0_L, and let $x,y\in L$. Then, y is called a complement of x, if $x\vee y=1_L$ and $x\wedge y=0_L$. If $x\in L$ has a complement, then it is unique. We denote by x^c, the complement of x.

Definition 2.7. An L-fuzzy set A on a nonempty set $\widetilde{X}$ is a function with domain $\widetilde{X}$ and whose range lies in a complete distributive lattice L with top and bottom elements 1_L and 0_L respectively.

Remark 2.1. The class of L-fuzzy sets is larger than the class of fuzzy sets as an L-fuzzy set reduces to a fuzzy set, if $L = I = [0, 1]$.

Denote the class of all L-fuzzy sets on a nonempty set $\widetilde{X}$ by $L^{\widetilde{X}}$ (to mean a function: $\widetilde{X} \longrightarrow L$).

Definition 2.8. The $\widetilde{\alpha}_L$-level set of an L-fuzzy set A is denoted by $[A]_{\widetilde{\alpha}L}$, and is defined as follows

$$[A]_{\widetilde{\alpha}L} = \begin{cases} \overline{\{x \in \widetilde{X} : 0_L \preceq_L A(x)\}}, & \text{if } \widetilde{\alpha} = 0; \\ \{x \in \widetilde{X} : \widetilde{\alpha}_L \preceq_L A(x)\}, & \text{if } \widetilde{\alpha}_L \in L \setminus \{0_L\}. \end{cases}$$

Definition 2.9. Let $\widetilde{X}$ be an arbitrary nonempty set and Y a metric space. A mapping $\widetilde{S} : \widetilde{X} \longrightarrow L^Y$ is called an L-fuzzy mapping. The function value $\widetilde{S}(x)(y)$ is called the degree of membership of y in $\widetilde{S}(x)$. For any two L-fuzzy mappings $S, \widetilde{S} : \widetilde{X} \longrightarrow L^Y$, a point $u \in \widetilde{X}$ is called an L-fuzzy fixed point of S, if there exists $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $u \in [Su]_{\widetilde{\alpha}_L}$. A point u is known as a common L-fuzzy fixed point of S and $\widetilde{S}$, if $u \in [Su]_{\widetilde{\alpha}_L} \cap [\widetilde{S}u]_{\widetilde{\alpha}_L}$.

3. Main results

We begin this section by inaugurating the notion of stationary points (also called end points) for L-fuzzy mappings which is motivated by the uniqueness concept of fixed point of point-valued mappings. For related articles on end point results, we refer to Amini-Harandi [5] and Choudhury [8].

Definition 3.1. Let $\widetilde{X}$ be a nonempty set. An element $u \in \widetilde{X}$ is called a stationary point of an L-fuzzy set-valued map $\widetilde{S} : \widetilde{X} \longrightarrow L^{\widetilde{X}}$, if there exists an $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $[Tu]_{\widetilde{\alpha}_L} = \{u\}$. Similarly, for $l \in \mathbb{N}$, the point u is said to be a stationary point of $\widetilde{S}(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow L$, if there exists an $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $[\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L} = \{u\}$.

Theorem 3.1. *Let $(\widetilde{X}, \sigma)$ be a complete metric space, l a positive integer and $\widetilde{S}(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow L$ an L-fuzzy set-valued map. Assume that the following conditions hold*

(i) there exists an $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $[\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}$ is a nonempty compact subset of $\widetilde{X}$;

(ii) there exists a lower semi-continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\varphi(t) = 0$, if and only if $t = 0$ such that

$$\begin{aligned} \widetilde{H}([\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2, \cdots, x_{l+1})]_{\widetilde{\alpha}_L}) \leq\ & \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\} \\ & - \varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\}), \end{aligned} \tag{3.1}$$

for all $(x_1, \cdots, x_{l+1}) \in \widetilde{X}^{l+1}$. Then, for any arbitrary point $x_1, \cdots, x_l \in \widetilde{X}$, the sequence $\{x_{n+l}\}_{n \geq 1}$ defined by

$$x_{n+l} \in [\widetilde{S}(x_n, \cdots, x_{n+l-1})]_{\widetilde{\alpha}_L}, \ n \in \mathbb{N} \tag{3.2}$$

converges to $u \in \widetilde{X}$ *and* $u \in [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}$. *Moreover, if*

$$\widetilde{H}([\widetilde{S}(x_1, \cdots, x)]_{\widetilde{\alpha}_L}, [\widetilde{S}(y, \cdots, y)]_{\widetilde{\alpha}_L}) \leq \sigma(x, y) - \varphi(\sigma(x, y)) \tag{3.3}$$

holds for all $x, y \in \widetilde{X}$ *with* $x \neq y$, *then* $\widetilde{S}$ *has a stationary point in* $\widetilde{X}$.

Proof. Let $x_1, \cdots, x_l$ be arbitrary l elements in $\widetilde{X}$. Consider the sequence defined by (3.2). If there exists an $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $x_i = x_{i+1}$ for all $i = n, n+1, \cdots, n+l-1$, then $x_i \in [\widetilde{S}(x_i, \cdots, x_i)]_{\widetilde{\alpha}_L}$. That is, x_i is an L-fuzzy fixed point of $\widetilde{S}$, and the proof is finished. Hence, we assume that $x_i \neq x_{i+1}$ for all $i = n, n+1, \cdots, n+l-1$. For $l \geq n$, from (3.1) and Lemma 2.1, we have the following inequalities

$$\begin{aligned}\sigma(x_{l+n}, x_{l+n-1}) &\leq \widetilde{H}([\widetilde{S}(x_n, \cdots, x_{l+n-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_{n+1}, \cdots, x_{n+l})]_{\widetilde{\alpha}_L}) \\ &\leq \max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\} \\ &\quad -\varphi(\max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\}) \\ &< \max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\}.\end{aligned}$$

$$\begin{aligned}\sigma(x_{l+1}, x_{l+2}) &\leq \widetilde{H}([\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2, \cdots, x_{l+1})]_{\widetilde{\alpha}_L}) \\ &\leq \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\} \\ &\quad -\varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\}) \\ &< \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\}.\end{aligned}$$

$$\begin{aligned}\sigma(x_l, x_{l+1}) &\leq \widetilde{H}([\widetilde{S}(x_1, \cdots, x_{l-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2, \cdots, x_l)]_{\widetilde{\alpha}_L}) \\ &\leq \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l-1\} \\ &\quad -\varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l-1\}) \\ &< \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l-1\}.\end{aligned}$$

$$\begin{aligned}\sigma(x_{l-n}, x_{l-n+1}) &\leq \widetilde{H}([\widetilde{S}(x_1, \cdots, x_{l-n-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2, \cdots, x_{l-n})]_{\widetilde{\alpha}_L}) \\ &\leq \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l-n-1\} \\ &\quad -\varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l-n-1\}) \\ &< \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l-n-1\}.\end{aligned}$$

Hence, we conclude that the sequence $\{\sigma(x_{n+l-1}, x_{n+l})\}_{n \geq 1}$ is monotone non-increasing and bounded below. Therefore, there exists $\tau \geq 0$ such that

$$\lim_{n \longrightarrow \infty} \sigma(x_{n+l-1}, x_{n+l}) = \lim_{n \longrightarrow \infty} \max\{\sigma(x_{n+i}, x_{n+i+1}) : 1 \leq i \leq l-1\} = \tau. \tag{3.4}$$

We claim that $\tau = 0$. To see this, consider the following inequalities

$$\begin{aligned}\sigma(x_{l+n}, x_{l+n+1}) &\leq \widetilde{H}([\widetilde{S}(x_n, \cdots, x_{l+n-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_{n+1}, \cdots, x_{l+n})]_{\widetilde{\alpha}_L}) \\ &\leq \max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\} \\ &\quad - \varphi(\max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\}).\end{aligned} \tag{3.5}$$

Taking upper limit in (3.5) as $n \longrightarrow \infty$, we have $\tau \leq \tau - \varphi(\tau)$, which implies that $\varphi(\tau) \leq 0$. Hence, $\varphi(\tau) = 0$. Therefore, $\lim_{n\longrightarrow\infty} \sigma(x_{l+n}, x_{l+n+1}) = 0$.
Next, we show that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\widetilde{X}$. Let $m, n \in \mathbb{N}$ with $n \geq m$. Then, from (3.1) and Lemma 2.1, we obtain

$$\begin{aligned}
&\sigma(x_{l+n}, x_{l+m}) \\
\leq\ & \widetilde{H}([\widetilde{S}(x_n, \cdots, x_{l+n-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_m, \cdots, x_{l+m-1})]_{\widetilde{\alpha}_L}) \\
\leq\ & \widetilde{H}([\widetilde{S}(x_n, \cdots, x_{l+n-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_{n+1}, \cdots, x_{l+n})]_{\widetilde{\alpha}_L}) \\
& + \widetilde{H}([\widetilde{S}(x_{n+1}, \cdots, x_{l+n})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_{n+2}, \cdots, x_{l+n+1})]_{\widetilde{\alpha}_L}) \\
& + \cdots + \widetilde{H}([\widetilde{S}(x_{m-1}, \cdots, x_{l+m-2})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_m, \cdots, x_{l+m-1})]_{\widetilde{\alpha}_L}) \\
\leq\ & \max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\} \\
& - \varphi(\max\{\sigma(x_i, x_{i+1}) : n \leq i \leq l+n-1\}) \\
& + \max\{\sigma(x_i, x_{i+1}) : n+1 \leq i \leq l+n\} \\
& - \varphi(\max\{\sigma(x_i, x_{i+1}) : n+1 \leq i \leq l+n\}) \\
& + \cdots + \max\{\sigma(x_i, x_{i+1}) : m-1 \leq i \leq l+m-2\} \\
& - \varphi(\max\{\sigma(x_i, x_{i+1}) : m-1 \leq i \leq l+m-2\}).
\end{aligned} \tag{3.6}$$

Taking upper limit in (3.6) as $n \longrightarrow \infty$ gives $\lim_{n\longrightarrow\infty} \sigma(x_{l+n}, x_{l+m}) = 0$. This shows that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\widetilde{X}$. Hence, the completeness of this space guarantees the existence of $u \in \widetilde{X}$ such that

$$\lim_{n\longrightarrow\infty} \sigma(x_n, u) = 0. \tag{3.7}$$

Now, to show that u is an L-fuzzy fixed point of $\widetilde{S}$, let $n \in \mathbb{N}$, then consider

$$\begin{aligned}
\sigma(u, [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}) &\leq \sigma(u, x_{n+l}) + \sigma(x_{n+l}, [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}) \\
&\leq \sigma(u, x_{n+l}) + \widetilde{H}([\widetilde{S}(x_n, \cdots, x_{n+l-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}) \\
&\leq \sigma(u, x_{n+l}) + \widetilde{H}([\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}), [\widetilde{S}(u, \cdots, x_n)]_{\widetilde{\alpha}_L} \\
&\quad + \widetilde{H}([\widetilde{S}(u, \cdots, x_n)]_{\widetilde{\alpha}_L}, [\widetilde{S}(u, \cdots, x_n, x_{n+1})]_{\widetilde{\alpha}_L}) \\
&\quad + \cdots + \widetilde{H}([\widetilde{S}(u, x_n, \cdots, x_{n+l-2})]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_n, \cdots, x_{n+l-1})]_{\widetilde{\alpha}_L}). \\
&\leq \sigma(u, x_{n+l}) + \max\{\sigma(u, x_i) : 1 \leq i \leq n\} \\
&\quad - \varphi(\max\{\sigma(u, x_i) : 1 \leq i \leq n\}) \\
&\quad + \max\{\sigma(u, x_n), \sigma(x_n, x_{n+1})\} \\
&\quad - \varphi(\max\{\sigma(u, x_n), \sigma(x_n, x_{n+1})\}) \\
&\quad + \cdots + \max\{\sigma(u, x_n), \sigma(x_n, x_{n+1}), \cdots, \sigma(x_{n+l-2}, x_{n+l-1})\} \\
&\quad - \varphi(\max\{\sigma(u, x_n), \sigma(x_n, x_{n+1}), \cdots, \sigma(x_{n+l-2}, x_{n+l-1})\}).
\end{aligned} \tag{3.8}$$

Taking upper limit in (3.8) gives $\sigma(u, [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}) \leq 0$, which implies that $u \in [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}$. That is, u is an L-fuzzy fixed point of $\widetilde{S}$. Now, we prove that under condition (3.3), $\widetilde{S}$ has a stationary point in $\widetilde{X}$. For this purpose, assume that there exists $u^* \in [\widetilde{S}(u^*, \cdots, u^*)]_{\widetilde{\alpha}_L}$ with $u \neq u^*$ such that $[\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L} \neq \{u\}$. Then, by Lemma 2.1, we have

$$\sigma(u, u^*) \leq \widetilde{H}([\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}_L}, [\widetilde{S}(u^*, \cdots, u^*)]_{\widetilde{\alpha}_L})$$

$$\begin{aligned}&\leq \sigma(u,u^*) - \varphi(\sigma(u,u^*))\\&< \sigma(u,u^*)\end{aligned} \tag{3.9}$$

as a contradiction. Hence, $\widetilde{S}$ has a stationary point in $\widetilde{X}$. □

Example 3.1. Let $L = \{a,b,c,g,s,m,n,v\}$ be such that $a \preceq_L s \preceq_L c \preceq_L v$, $a \preceq_L g \preceq_L b \preceq_L v$, $s \preceq_L m \preceq_L v$, $g \preceq_L m \preceq_L v$, $n \preceq_L b \preceq_L v$, and each element of the doubletons $\{c,m\},\{m,b\},\{s,n\},\{n,g\}$ are not comparable. It follows that $(L,\preceq_L)$ is a complete distributive lattice. Let $\widetilde{X} = [0,\infty)$ and define $\sigma : \widetilde{X} \times \widetilde{X} \longrightarrow \mathbb{R}$ by $\sigma(x,y) = |x-y|$, for all $x,y \in \widetilde{X}$. Clearly, $(\widetilde{X},d)$ is a complete metric space. Let $\widetilde{\alpha}_L : \widetilde{X} \longrightarrow L \setminus \{0_L\}$ be a mapping. For all $x_1,\cdots,x_l \in \widetilde{X}$, consider an L-fuzzy set-valued map $\widetilde{S}(x_1,\cdots,x_l) : \widetilde{X}^l \longrightarrow L$ defined as follows

$$\widetilde{S}(x_1,\cdots,x_l)(t_1,\cdots,t_l) = \begin{cases} v, & \text{if } (t_1,\cdots,t_l) \in \left[0, \frac{x_1+\cdots+x_l}{5l^2}\right];\\ s, & \text{if } (t_1,\cdots,t_l) \in \left(\frac{x_1+\cdots+x_l}{5l^2}, \frac{x_1+\cdots+x_l}{4l^2}\right];\\ m & \text{if } (t_1,\cdots,t_l) \in \left(\frac{x_1+\cdots+x_l}{4l^2}, \frac{x_1+\cdots+x_l}{3l^2}\right];\\ g, & \text{if } (t_1,\cdots,t_l) \in \left(\frac{x_1+\cdots+x_l}{3l^2}, \infty\right).\end{cases}$$

Assume that $\widetilde{\alpha}_L = v$, then there exists $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that

$$[\widetilde{S}(x_1\cdots,x_l)]_{\widetilde{\alpha}_L} = \left[0, \frac{x_1+\cdots+x_l}{5l^2}\right].$$

Define the function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\varphi(t) = \begin{cases} \frac{t}{6}, & \text{if } t \in \left[0, \frac{5}{2}\right)\\ \frac{2^n(2^{n+1}t)}{2^{2n+1}-1}, & \text{if } t \in \left[\frac{2^{2n}+1}{2^n}, \frac{2^{2(n+1)}+1}{2^{n+1}}\right], n \in \mathbb{N}.\end{cases}$$

A direct calculation verifies that φ is lower semi-continuous on $\mathbb{R}_+$ and $\varphi(t) = 0$, if and only if $t = 0$. Now, for all $(x_1,\cdots,x_{l+1}) \in \widetilde{X}$, we have

$$\begin{aligned}&D_{\widetilde{\alpha}_L}(\widetilde{S}(x_1,\cdots,x_l),\widetilde{S}(x_2,\cdots,x_{l+1}))\\&= \widetilde{H}([\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L},[\widetilde{S}(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L})\\&= \widetilde{H}\left(\left[0,\frac{x_1+\cdots+x_l}{5l^2}\right],\left[0,\frac{x_2+\cdots+x_{l+1}}{5l^2}\right]\right)\\&\leq \frac{1}{5l}|x_1-x_{l+1}| \leq \frac{1}{5}\max\{|x_i-x_{i+1}| : 1 \leq i \leq l\}\\&\leq \frac{5}{6}\max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\}\\&= \max\{\sigma(x_i,x_{i+1} : 1 \leq i \leq l)\}\\&\ -\varphi(\max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\}).\end{aligned}$$

Moreover, for all $x,y \in \widetilde{X}$, we have

$$\begin{aligned}\widetilde{H}([\widetilde{S}(x,\cdots,x)]_{\widetilde{\alpha}_L},[\widetilde{S}(y,\cdots,y)]_{\widetilde{\alpha}_L}) &\leq \frac{1}{20}|x-y|\\&\leq \frac{5}{6}\sigma(x,y)\end{aligned}$$

$$= \sigma(x,y) - \varphi(\sigma(x,y)).$$

Therefore, all the conditions of Theorem 3.1 are satisfied. Consequently, there exists $u = 0 \in \widetilde{X}$ such that $0 \in [\widetilde{S}(0,\cdots,0)]_{\widetilde{\alpha}_L}$. That is, 0 is an L-fuzzy fixed point of $\widetilde{S}$.

Corollary 3.1. *Let $(\widetilde{X},\sigma)$ be a complete metric space, l a positive integer and $\widetilde{S}(x_1,\cdots,x_l) : \widetilde{X}^l \longrightarrow L$ be an L-fuzzy set-valued map. Assume that the following conditions hold*

(i) there exists $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $[\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}$ is a nonempty compact subset of $\widetilde{X}$;

(ii) there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} &\widetilde{H}([\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L}) \\ &\leq \lambda \max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\}, \end{aligned} \tag{3.10}$$

for all $(x_1,\cdots,x_{l+1}) \in \widetilde{X}^{l+1}$. Then, for any arbitrary point $x_1,\cdots,x_l \in \widetilde{X}$, the sequence $\{x_{n+l}\}_{n\geq 1}$ defined by $x_{n+l} \in [\widetilde{S}(x_n,\cdots,x_{n+l-1})]_{\widetilde{\alpha}_L}$, $n \in \mathbb{N}$ converges to $u \in \widetilde{X}$ and $u \in [\widetilde{S}(u,\cdots,u)]_{\widetilde{\alpha}_L}$. Moreover, if for all $x,y \in \widetilde{X}$ with $x \neq y$,

$$\widetilde{H}([\widetilde{S}(x,\cdots,x)]_{\widetilde{\alpha}_L}[\widetilde{S}(y,\cdots,y)]_{\widetilde{\alpha}_L}) \leq \lambda\sigma(x,y),$$

then $\widetilde{S}$ has a stationary point in $\widetilde{X}$.

Proof. Put $\varphi(t) = (1-\lambda)t$, where $\lambda \in (0,1)$ and $t \geq 0$ in Theorem 3.1. □

Corollary 3.2. *Let $(\widetilde{X},\sigma)$ be a complete metric space, l a positive integer and $\widetilde{S}(x_1,\cdots,x_l) : \widetilde{X}^l \longrightarrow L$ be an L-fuzzy set-valued map. Assume that the following conditions are satisfied*

(i) there exists $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $[\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}$ is a nonempty compact subset of $\widetilde{X}$;

(ii) there exist non-negative constants $\lambda_1,\cdots,\lambda_l$ with $\sum_{i=1}^{l} \lambda_i < 1$ such that

$$\begin{aligned} \widetilde{H}([\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L}) \leq\ & \lambda_1\sigma(x_1,x_2) + \lambda_2\sigma(x_2,x_3) \\ & + \cdots + \lambda_l\sigma(x_l,x_{l+1}), \end{aligned} \tag{3.11}$$

for all $(x_1,\cdots,x_{l+1}) \in \widetilde{X}^{l+1}$. Then, for any arbitrary point $x_1,\cdots,x_l \in \widetilde{X}$, the sequence $\{x_{n+l}\}_{n\geq 1}$ defined by $x_{n+l} \in [\widetilde{S}(x_n,\cdots,x_{n+l-1})]_{\widetilde{\alpha}_L}$ converges to $u \in \widetilde{X}$ and $u \in [\widetilde{S}(u,\cdots,u)]_{\widetilde{\alpha}_L}$. Moreover, if

$$\begin{aligned} &\widetilde{H}([\widetilde{S}(x,\cdots,x)]_{\widetilde{\alpha}_L}, [\widetilde{S}(y,\cdots,y)]_{\widetilde{\alpha}_L}) \\ &\leq \sum_{i=1}^{l} \lambda_i\sigma(x,y) \end{aligned} \tag{3.12}$$

holds for all $x,y \in \widetilde{X}$ with $x \neq y$, then $\widetilde{S}$ has a stationary point in $\widetilde{X}$.

Proof. Obviously, condition (3.10) can be followed from condition (3.11) by taking $\lambda = \sum_{i=1}^{l} \lambda_i$. Furthermore, let $x, y \in \widetilde{X}$ with $x \neq y$. Then, from (3.12), we have

$$\begin{aligned}\widetilde{H}([\widetilde{S}(x,\cdots,x)]_{\widetilde{\alpha}_L}, [\widetilde{S}(y,\cdots,y)]_{\widetilde{\alpha}_L}) &\leq \widetilde{H}([\widetilde{S}(x,\cdots,x)]_{\widetilde{\alpha}_L}, [\widetilde{S}(y,\cdots,x,y)]_{\widetilde{\alpha}_L}) \\ &\quad + \widetilde{H}([\widetilde{S}(x,\cdots,x,y)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x,\cdots,x,y,y)]_{\widetilde{\alpha}_L}) \\ &\quad + \cdots + \widetilde{H}([\widetilde{S}(x,y,\cdots,y)]_{\widetilde{\alpha}_L}, [\widetilde{S}(y,\cdots,y)]_{\widetilde{\alpha}_L}) \\ &\leq \sum_{i=1}^{l} \lambda_i \sigma(x,y).\end{aligned}$$

Thus, all the assertions of Corollary 3.1 are satisfied with $\lambda = \sum_{i=1}^{l} \lambda_i$, and that completes the proof. □

The following theorem is a Presic-type generalization of the main result of Heilpern [14] using the concept of σ_L^{∞} distance for L-fuzzy sets.

Theorem 3.2. *Let $(\widetilde{X}, \sigma)$ be a complete metric space, l a positive integer and $\widetilde{S} : \widetilde{X}^l \longrightarrow W(\widetilde{X})$ an L-fuzzy set-valued map. Assume that there exists a lower semi-continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varphi(t) = 0$, if and only if $t = 0$ such that*

$$\begin{aligned}\sigma_L^{\infty}(\widetilde{S}(x_1,\cdots,x_l), \widetilde{S}(x_2,\cdots,x_{l+1})) \leq\ & \max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\} \\ & - \varphi(\max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\}),\end{aligned} \tag{3.13}$$

for all $(x_1,\cdots,x_{l+1}) \in \widetilde{X}^{l+1}$. Then, for each arbitrary point $x_1,\cdots,x_l \in \widetilde{X}$, the sequence $\{x_{n+l}\}_{n\geq 1}$ defined by

$$x_{n+l} \in \widetilde{S}(x_n,\cdots,x_{n+l-1}),\ n \in \mathbb{N}$$

converges to $u \in \widetilde{X}$ and $\{u\} \subset \widetilde{S}(u,\cdots,u)$.

Proof. Let $x_1,\cdots,x_l \in \widetilde{X}$ and $\widetilde{\alpha}_L \in L \setminus \{0_L\}$. Then, according to the hypothesis, $\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}} \in \mathcal{K}(\widetilde{X})$. Now, by definitions of $D_{\widetilde{\alpha}_L}$ and σ_L^{∞}-metric for L-fuzzy sets, for all $x_1,\cdots,x_{l+1} \in \widetilde{X}^{l+1}$, we have

$$\begin{aligned}D_{\widetilde{\alpha}_L}(\widetilde{S}(x_1,\cdots,x_l), \widetilde{S}(x_2,\cdots,x_{l+1})) &= \widetilde{H}([\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L}) \\ &\leq \sup_{\widetilde{\alpha}_L} \widetilde{H}([\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L}) \\ &= \sigma_L^{\infty}(\widetilde{S}(x_1,\cdots,x_l), \widetilde{S}(x_2,\cdots,x_{l+1})) \\ &\leq \max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\} \\ &\quad -\varphi(\max\{\sigma(x_i,x_{i+1}) : 1 \leq i \leq l\}).\end{aligned}$$

Thus, Theorem 3.1 can be applied to find $u \in \widetilde{X}$ such that $\{u\} \subset \widetilde{S}(u,\cdots,u)$. □

Remark 3.1. If we take $L = [0,1]$, $l = 1$ and $\varphi(t) = (1-\lambda)t$ for all $t \in \mathbb{R}_+$ and $\lambda \in (0,1)$, then Theorem 3.2 reduces to the main result of Heilpern [14].

Definition 3.2. [14] Let $(\widetilde{X}, \sigma)$ be a metric space. A fuzzy set-valued map $\widetilde{S} : \widetilde{X} \longrightarrow W(\widetilde{X})$ is called fuzzy λ-contraction, if there exists a constant $\lambda \in (0,1)$ such that for all $x, y \in \widetilde{X}$,

$$\sigma_{\infty}(\widetilde{S}(x), \widetilde{S}(y)) \leq \lambda \sigma(x,y).$$

In [14], it has been shown that every fuzzy λ-contraction on a complete metric space has a fuzzy fixed point. Following this idea, we inaugurate the next definition in order to establish a significant consequence of Theorem 3.2.

Definition 3.3. Let $(\widetilde{X}, \sigma)$ be a metric space. An L-fuzzy set-valued map $\widetilde{S} : \widetilde{X} \longrightarrow W(\widetilde{X})$ is called an L-fuzzy weak contraction, if there exists a lower semi-continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varphi(t) = 0$, if and only if $t = 0$ such that for all $x, y \in \widetilde{X}$ with $x \neq y$,

$$\sigma_L^\infty(\widetilde{S}(x), \widetilde{S}(y)) \leq \sigma(x, y) - \varphi(\sigma(x, y)).$$

Corollary 3.3. *Let $(\widetilde{X}, \sigma)$ be a complete metric space and $\widetilde{S} : \widetilde{X} \longrightarrow W(\widetilde{X})$ be an L-fuzzy weak contraction on $\widetilde{X}$. Then, $\widetilde{S}$ has at least one L-fuzzy fixed point in $\widetilde{X}$.*

Proof. It is enough to take $l = 1$ in Theorem 3.2. □

Remark 3.2. If we put $L = [0, 1]$, $\varphi(t) = (1 - \lambda)t$ for all $t \geq 0$ and $\lambda \in (0, 1)$, Corollary 3.3 reduces to the main result of Heilpern [14, Theorem 3.1].

4. Further consequences

Here, we apply the results from Section 3 to discuss some new fixed point results of fuzzy, multi-valued and single-valued mappings. To this end, recall that a point $u \in \widetilde{X}$ is called a fixed point of a multi-valued (single-valued) mapping A on $\widetilde{X}$, if $u \in Au$ ($u = Au$). A point $u \in \widetilde{X}$ is said to be a stationary point of a multi-valued mapping A, if $Au = \{u\}$.

Theorem 4.1. *Let $(\widetilde{X}, \sigma)$ be a complete metric space, l a positive integer and $\widetilde{S}(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow [0, 1]$ a fuzzy set-valued map. Assume that the following conditions hold*

(i) there exists an $\widetilde{\alpha} \in (0, 1]$ such that $[\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}}$ is a nonempty compact subset of $\widetilde{X}$;

(ii) there exists a lower semi-continuous function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying $\varphi(t) = 0$, if and only if $t = 0$ such that

$$\widetilde{H}([\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}}, [\widetilde{S}(x_2, \cdots, x_{l+1})]_{\widetilde{\alpha}}) \leq \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\} - \varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\}),$$

for all $(x_1, \cdots, x_{l+1}) \in \widetilde{X}^{l+1}$. Then, for any arbitrary point $x_1, \cdots, x_l \in \widetilde{X}$, the sequence $\{x_{n+l}\}_{n \geq 1}$ defined by

$$x_{n+l} \in [\widetilde{S}(x_n, \cdots, x_{n+l-1})]_{\widetilde{\alpha}},\ n \in \mathbb{N}$$

converges to $u \in \widetilde{X}$ and $u \in [\widetilde{S}(u, \cdots, u)]_{\widetilde{\alpha}}$. Moreover, if

$$\widetilde{H}([\widetilde{S}(x_1, \cdots, x)]_{\widetilde{\alpha}}, [\widetilde{S}(y, \cdots, y)]_{\widetilde{\alpha}}) \leq \sigma(x, y) - \varphi(\sigma(x, y))$$

holds for all $x, y \in \widetilde{X}$ with $x \neq y$, then $\widetilde{S}$ has a stationary point in $\widetilde{X}$.

Proof. Put $L = [0, 1]$ in Theorem 3.1. □

Theorem 4.2. *Let $(\widetilde{X},\sigma)$ be a complete metric space, l be a positive integer and $A:\widetilde{X}^l\longrightarrow\mathcal{K}(\widetilde{X})$ be a multi-valued mapping. Assume that*

$$\begin{aligned}\widetilde{H}(A(x_1,\cdots,x_l),A(x_2,\cdots,x_{l+1}))&\leq \max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\}\\ &= -\varphi(\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\})\end{aligned}$$

holds for all $(x_1,\cdots,x_{l+1})\in\widetilde{X}^{l+1}$, where $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a lower semi-continuous function with $\varphi(t)=0$, if and only if $t=0$. Then, for any arbitrary point $x_1,\cdots,x_l\in\widetilde{X}$, the sequence $\{x_{n+l}\}_{n\geq 1}$ defined by $x_{n+l}\in A(x_n,\cdots,x_{n+l-1})$, $n\in\mathbb{N}$ converges to $u\in\widetilde{X}$ and $u\in A(u,\cdots,u)$. Moreover, if

$$\widetilde{H}(A(x,\cdots,x),A(y,\cdots,y))\leq\sigma(x,y)-\varphi(\sigma(x,y))$$

holds for all $x,y\in\widetilde{X}$ with $x\neq y$, then A has a stationary point in $\widetilde{X}$.

Proof. Let $L=\{\xi,\varpi,\varsigma,\tau\}$ with $\xi\preceq_L\varpi\preceq_L\tau$, $\xi\preceq_L\varsigma\preceq_L\tau$, ϖ and ς are not comparable, then $(L,\preceq_L)$ is a complete distributive lattice. Let $\widetilde{\alpha}_L:\widetilde{X}^l\longrightarrow L\backslash\{0_L\}$ be a mapping, and consider an L-fuzzy set-valued map $\widetilde{S}(x_1,\cdots,x_l):\widetilde{X}^l\longrightarrow L$ defined by

$$\widetilde{S}(x_1,\cdots,x_l)(t_1,\cdots,t_l)=\begin{cases}\widetilde{\alpha}_L(x_1,\cdots,x_l), & \text{if } (t_1,\cdots,t_l)\in A(x_1,\cdots,x_l);\\ 0_L, & \text{otherwise.}\end{cases}$$

If we take $\widetilde{\alpha}_L:=\widetilde{\alpha}_L(x_1,\cdots,x_l)=\tau$, then, for all $(x_1,\cdots,x_l)\in\widetilde{X}$, there exists $\widetilde{\alpha}_L\in L\setminus\{0_L\}$ such that

$$\begin{aligned}[\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}&=\{(t_1,\cdots,t_l)\in\widetilde{X}:\tau\preceq_L\widetilde{S}(x_1,\cdots,x_l)(t_1,\cdots,t_l)\}\\ &=A(x_1,\cdots,x_l).\end{aligned}$$

From this point, Theorem 3.1 can be applied to find $u\in\widetilde{X}$ such that $u\in A(u,\cdots,u)$ and $\{u\}=A(u,\cdots,u)$. □

Theorem 4.3. *(see [1, Theorem 2.1]) Let $(\widetilde{X},\sigma)$ be a complete metric space, l a positive integer and $\vartheta:\widetilde{X}^l\longrightarrow\widetilde{X}$ be a single-valued mapping. Assume that there exists a lower semi-continuous function $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ with $\varphi(t)=0$, if and only if $t=0$ such that*

$$\begin{aligned}\sigma(\vartheta(x_1,\cdots,x_l),\vartheta(x_2,\cdots,x_{l+1}))\leq\ &\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\}\\ &-\varphi(\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\})\end{aligned}$$

holds for all $(x_1,\cdots,x_{l+1})\in\widetilde{X}^{l+1}$. Then, for any arbitrary point $x_1,\cdots,x_l\in\widetilde{X}$, the sequence $\{x_{n+l}\}_{n\in\mathbb{N}}$ defined by $x_{n+l}=\vartheta(x_n,\cdots,x_{n+l-1})$, $n\in\mathbb{N}$ converges to $u\in\widetilde{X}$ and $u=\vartheta(u,\cdots,u)$. Moreover, if

$$\sigma(\vartheta(x,\cdots,x),\vartheta(y,\cdots,y))\leq\sigma(x,y)-\varphi(\sigma(x,y))$$

holds for all $x,y\in\widetilde{X}$ with $x\neq y$, then $u\in\widetilde{X}$ is the unique fixed point of ϑ.

Proof. Let $L=\{\xi,\varpi,\varsigma,\tau\}$ be as defined in the proof of Theorem 4.2. Then $(L,\preceq_L)$ is a complete distributive lattice. Let $\widetilde{\alpha}_{L_\vartheta}:\widetilde{X}^l\longrightarrow L\backslash\{0_L\}$ be an arbitrary mapping and define an L-fuzzy set-valued map $\widetilde{S}(x_1,\cdots,x_l):\widetilde{X}^l\longrightarrow L$ as

$$\widetilde{S}(x_1,\cdots,x_l)(t_1,\cdots,t_l)=\begin{cases}\widetilde{\alpha}_{L_\vartheta}(x_1,\cdots,x_l), & \text{if } (t_1,\cdots,t_l)=\vartheta(x_1,\cdots,x_l);\\ 0_L, & \text{if } (t_1,\cdots,t_l)\neq\vartheta(x_1,\cdots,x_l).\end{cases}$$

By taking $\widetilde{\alpha}_L:=\widetilde{\alpha}_{L_\vartheta}(x_1,\cdots,x_l)$, we have

$$\begin{aligned}[\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}&=\{(t_1,\cdots,t_l)\in\widetilde{X}:\widetilde{\alpha}_{L_\vartheta}(x_1,\cdots,x_l)\preceq_L\widetilde{S}(x_1,\cdots,x_l)(t_1,\cdots,t_l)\}\\&=\{\vartheta(x_1,\cdots,x_l)\}.\end{aligned}$$

Obviously, $\{\vartheta(x_1,\cdots,x_l)\}\in\mathcal{K}(\widetilde{X})$. Notice that in this case,

$$\widetilde{H}([\widetilde{S}(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L},[\widetilde{S}(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L})=\sigma(\vartheta(x_1,\cdots,x_l),\vartheta(x_2,\cdots,x_{l+1})).$$

Consequently, Theorem 3.1 can be applied to find $u\in\widetilde{X}$ such that $u\in[\widetilde{S}(u,\cdots,u)]_{\widetilde{\alpha}_L}=\{\vartheta(u,\cdots,u)\}$, which further implies that $\vartheta(u,\cdots,u)=u$. □

Remark 4.1.

(i) Theorems 3.1 and 4.2 are L-fuzzy set-valued and multi-valued extensions of the result of Abbas et al., [1, Theorem 2.1].

(ii) Theorem 3.1 is an L-fuzzy generalization of the results of Ciric [10] and Presic [25].

(iii) If $l=1$, Theorem 3.1 is an L-fuzzy improvement of the result of Rhoades [28].

(iv) By setting $\varphi(t)=(1-\lambda)t$, where $\lambda\in(0,1)$ and $t\geq 0$, we can deduce the Banach contraction theorem from Theorem 3.1 by employing the method of proving Theorem 4.3.

5. Stability of L-fuzzy mappings

In this section, the study of stability of Presic type L-fuzzy fixed point problems is initiated. We start with the following result.

Theorem 5.1. *Let $(\widetilde{X},\sigma)$ be a complete metric space, l a positive integer and $\widetilde{S}_i(x_1,\cdots,x_l):\widetilde{X}^l\longrightarrow L$ be a sequence of L-fuzzy set-valued maps for $i=1,2$. Assume that the following assertions hold:*

(i) there exists $\widetilde{\alpha}_L\in L\setminus\{0_L\}$ such that $[\widetilde{S}_i(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L}$ is a nonempty compact subset of $\widetilde{X}$;

(ii) there exists a lower semi-continuous function $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ satisfying $\varphi(t)=0$, if and only if $t=0$ such that

$$\begin{aligned}&\widetilde{H}([\widetilde{S}_i(x_1,\cdots,x_l)]_{\widetilde{\alpha}_L},[\widetilde{S}_i(x_2,\cdots,x_{l+1})]_{\widetilde{\alpha}_L})\\ \leq\ &\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\}\\&-\varphi(\max\{\sigma(x_i,x_{i+1}):1\leq i\leq l\}),\end{aligned}\tag{5.1}$$

for all $(x_1,\cdots,x_{l+1})\in\widetilde{X}^{l+1}$ and

$$\widetilde{H}([\widetilde{S}_i(x,\cdots,x)]_{\widetilde{\alpha}},[\widetilde{S}_i(y,\cdots,y)]_{\widetilde{\alpha}})\leq\ \sigma(x,y)-\varphi(\sigma(x,y))\tag{5.2}$$

holds for all $x, y \in \widetilde{X}$. *Then,*

$$\varphi(\widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_1), \mathcal{F}_{ix}(\widetilde{S}_2))) \leq \delta,$$

where

$$\delta = \sup_{x \in \widetilde{X}} \widetilde{H}([\widetilde{S}_1(x, \cdots, x)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(x, \cdots, x)]_{\widetilde{\alpha}_L}).$$

Proof. Following Theorem 3.1, we have that $\mathcal{F}_{ix}(\widetilde{S}_i)$ is nonempty. Let $\theta_0 \in [\widetilde{S}_1(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}$. Then, by Lemma 2.1, there exists $\theta_1 \in [\widetilde{S}_2(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}$ such that

$$\sigma(\theta_0, \theta_1) \leq \widetilde{H}([\widetilde{S}_1(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}).$$

Since $[\widetilde{S}_1(\theta_1, \cdots, \theta_1)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L} \in \mathcal{K}(\widetilde{X})$ and $\theta_1 \in [\widetilde{S}_2(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}$, then by Lemma 2.1, there exists $\theta_2 \in [\widetilde{S}_1(\theta_1, \cdots, \theta_1)]_{\widetilde{\alpha}_L}$ such that

$$\sigma(\theta_1, \theta_2) \leq \widetilde{H}([\widetilde{S}_2(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}, [\widetilde{S}_1(\theta_1, \cdots, \theta_1)]_{\widetilde{\alpha}_L}).$$

Continuing in this way, we generate a sequence $\{\theta_n\}_{n\geq 1}$ in $\widetilde{X}$ with $\theta_n \in [\widetilde{S}_2(\theta_{n-1}, \cdots, \theta_{n-1})]_{\widetilde{\alpha}_L}$, $\theta_{n+1} \in [\widetilde{S}_1(\theta_n, \cdots, \theta_n)]_{\widetilde{\alpha}_L}$ such that

$$\begin{aligned}\sigma(\theta_{l+1}, \theta_{l+1}) &\leq \widetilde{H}([\widetilde{S}_1(\theta_l, \cdots, \theta_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(\theta_{l+1}, \cdots, \theta_{l+1})]_{\widetilde{\alpha}_L}) \\ &\leq \sigma(\theta_l, \theta_{l+1}) - \varphi(\sigma(\theta_l, \theta_{l+1})).\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma(\theta_l, \theta_{l+1}) &\leq \widetilde{H}([\widetilde{S}_1(\theta_{l-1}, \cdots, \theta_{l-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(\theta_l, \cdots, \theta_l)]_{\widetilde{\alpha}_L}) \\ &\leq \sigma(\theta_{l-1}, \theta_l) - \varphi(\sigma(\theta_{l-1}, \theta_l)).\end{aligned}$$

Therefore, for all $l \geq n$, we have

$$\begin{aligned}\sigma(\theta_{l-n}, \theta_{l-n+1}) &\leq \widetilde{H}([\widetilde{S}_1(\theta_{l-n-1}, \cdots, \theta_{l-n-1})]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(\theta_{l-n}, \cdots, \theta_{l-n})]_{\widetilde{\alpha}_L}) \quad (5.3) \\ &\leq \sigma(\theta_{l-n-1}, \theta_{l-n}) - \varphi(\sigma(\theta_{l-n-1}, \theta_{l-n})). \quad (5.4)\end{aligned}$$

Continuing as in Theorem 3.1, it follows that $\{\theta_n\}_{n\geq 1}$ is a Cauchy sequence in $\widetilde{X}$, and the completeness of this space implies that there exists $u \in \widetilde{X}$ such that $\theta_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, let $u \in [\widetilde{S}_2(u, \cdots, u)]_{\widetilde{\alpha}_L}$. Then, by assumption, we have

$$\begin{aligned}\sigma(\theta_0, \theta_1) &\leq \widetilde{H}([\widetilde{S}_1(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}) \\ &\leq \sup_{x \in \widetilde{X}} \widetilde{H}([\widetilde{S}_1(x, \cdots, x)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(x, \cdots, x)]_{\widetilde{\alpha}_L}) = \delta.\end{aligned}$$

Thus, by triangle inequality, we get

$$\begin{aligned}\sigma(\theta_0, u) &\leq \sigma(\theta_0, \theta_1) + \sigma(\theta_1, u) \\ &\leq \sigma(\theta_0, \theta_1) + \widetilde{H}([\widetilde{S}_1(\theta_0, \cdots, \theta_0)]_{\widetilde{\alpha}_L}, [\widetilde{S}_2(u, \cdots, u)]_{\widetilde{\alpha}_L}) \\ &\leq \delta + \sigma(\theta_0, u) - \varphi(\sigma(\theta_0, u)),\end{aligned}$$

which implies that $\varphi(\sigma(\theta_0, u)) \leq \delta$. It follows that for any arbitrary point $\theta_0 \in \mathcal{F}_{ix}(\widetilde{S}_1)$, there exists $u \in \mathcal{F}_{ix}(\widetilde{S}_2)$ such that $\varphi(\sigma(\theta_0, u)) \leq \delta$. On similar steps, for any point $\xi_0 \in \mathcal{F}_{ix}(\widetilde{S}_2)$, we can find an element $\gamma \in \mathcal{F}_{ix}(\widetilde{S}_1)$ such that $\varphi(\sigma(\xi_0, \gamma)) \leq \delta$. Consequently, it follows that $\widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_1), \mathcal{F}_{ix}(\widetilde{S}_2)) \leq \delta$. □

For the next results, we introduce the following concept of uniform convergence of sequence of L-fuzzy set-valued maps.

Definition 5.1. Let $(\widetilde{X}, \sigma)$ be a metric space. A sequence of L-fuzzy set-valued maps $\{\widetilde{S}_n(x) : \widetilde{X} \longrightarrow L,\ n \in \mathbb{N}\}$ is said to converge uniformly to an L-fuzzy set-valued map $\widetilde{S}(x) : \widetilde{X} \longrightarrow L$, if for every $\epsilon > 0$ and for all $x \in \widetilde{X}$, there exists $\widetilde{\alpha}_L(x) := \widetilde{\alpha}_L \in L \setminus \{0_L\}$ and $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$,

$$\widetilde{H}([\widetilde{S}_n x]_{\widetilde{\alpha}_L}, [Tx]_{\widetilde{\alpha}_L}) < \epsilon. \tag{5.5}$$

If (5.5) holds, then we write

$$\lim_{n \longrightarrow \infty} \widetilde{H}([\widetilde{S}_n x]_{\widetilde{\alpha}_L}, [Tx]_{\widetilde{\alpha}_L}) = 0,$$

where $[Tx]_{\widetilde{\alpha}_L}$ is called the limiting cut set, and is given by

$$[Tx]_{\widetilde{\alpha}_L} = \lim_{n \longrightarrow \infty} [\widetilde{S}_n x]_{\widetilde{\alpha}_L}.$$

.

Example 5.1. Take $\widetilde{X} = [0, 5]$ and define $\sigma : \widetilde{X} \times \widetilde{X} \longrightarrow \mathbb{R}$ as $\sigma(x, y) = |x - y|$ for all $x, y \in \widetilde{X}$. Let $L = \{\xi, \varpi, \varsigma, \tau\}$ be as defined in the proof of Theorem 4.2, then $(L, \preceq_L)$ is a complete distributive lattice. Consider a mapping $\widetilde{\alpha}_L : \widetilde{X} \longrightarrow L \setminus \{0_L\}$ and a sequence of L-fuzzy set-valued maps $\{\widetilde{S}_n\}_{n \geq 1}$ defined by

$$\widetilde{S}_n(x)(t) = \begin{cases} \widetilde{\alpha}_L(x), & \text{if } 0 \leq t \leq \frac{1}{(n+x)}; \\ 0_L, & \text{if } \frac{1}{(n+x)} < t \leq 5. \end{cases}$$

Assume that $\widetilde{\alpha}_L := \widetilde{\alpha}_L(x)$ for all $x \in \widetilde{X}$, then

$$[\widetilde{S}_n x]_{\widetilde{\alpha}_L} = \left[0, \frac{1}{(n+x)}\right].$$

Given $\epsilon > 0$, we get

$$\widetilde{H}([\widetilde{S}_n x]_{\widetilde{\alpha}_L}, [Tx]_{\widetilde{\alpha}_L}) = \frac{1}{(n+x)} < \epsilon.$$

Notice that $n \geq \frac{1}{\epsilon} - x$ decreases with x and the maximum value is $\frac{1}{\epsilon}$. Thus, choose $n_\epsilon \geq \frac{1}{\epsilon}$, so that for $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, $\widetilde{H}([\widetilde{S}_n x]_{\widetilde{\alpha}_L}, [Tx]_{\widetilde{\alpha}_L}) < \epsilon$. Hence, $\{\widetilde{S}_n\}_{n \geq 1}$ converges uniformly to $\widetilde{S}$ on $\widetilde{X}$.

We recall that the fixed point sets $\mathcal{F}_{ix}(\widetilde{S}_n)$ of a sequence of multi-valued mappings $\widetilde{S}_n : \widetilde{X} \longrightarrow \mathcal{K}(\widetilde{X})$ are stable, if $\widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_n), \mathcal{F}_{ix}(\widetilde{S})) \longrightarrow 0$, as $n \longrightarrow \infty$, where $\widetilde{S} = \lim_{n \longrightarrow \infty} \widetilde{S}_n$. Similar to the concept of stability of fixed points in [9,16,18], we propose the following definition of stability of fixed point sets of sequence of L-fuzzy set-valued maps.

Definition 5.2. Let $\{\widetilde{S}_n(x) : \widetilde{X} \longrightarrow L,\ x \in \widetilde{X}, n \in \mathbb{N}\}$ be a sequence of L-fuzzy set-valued maps that converges uniformly to an L-fuzzy set-valued map $\widetilde{S}(x) : \widetilde{X} \longrightarrow L$. Suppose that $\{\mathcal{F}_{ix}(\widetilde{S}_n)\}_{n \geq 1}$ is the sequence of fixed point sets of the sequence $\{\widetilde{S}_n\}_{n \geq 1}$ and $\{\mathcal{F}_{ix}(\widetilde{S})\}$ is the fixed point set of $\widetilde{S}$. Then, we say that the L-fuzzy fixed point sets of $\{\widetilde{S}_n\}_{n \geq 1}$ are stable, if

$$\lim_{n \longrightarrow \infty} \widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_n), \mathcal{F}_{ix}(\widetilde{S})) = 0.$$

Lemma 5.1. *Let $(\widetilde{X}, \sigma)$ be a complete metric space, l a positive integer and $\{\widetilde{S}_n(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow L,\ n \in \mathbb{N}\}$ be a sequence of L-fuzzy set-valued maps uniformly convergent to $\widetilde{S}(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow L$. If $\{\widetilde{S}_n(x_1, \cdots, x_l)\}_{n \geq 1}$ satisfies* (5.1) *and* (5.2) *for each $n \in \mathbb{N}$, then $\widetilde{S}$ also satisfies* (5.1) *and* (5.2).

Proof. Since $\widetilde{S}_n$ satisfies (5.1) and (5.2) for each $n \in \mathbb{N}$, then for all $(x_1, \cdots, x_{l+1}) \in \widetilde{X}$, we have

$$\widetilde{H}([\widetilde{S}_n(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}_n(x_2, \cdots, x_{l+1})]_{\widetilde{\alpha}_L}) \leq \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\} - \varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\}), \tag{5.6}$$

and

$$\widetilde{H}([\widetilde{S}_n(x, \cdots, x)]_{\widetilde{\alpha}_L}, [\widetilde{S}_n(y, \cdots, y)]_{\widetilde{\alpha}_L}) \leq \sigma(x, y) - \varphi(\sigma(x, y)). \tag{5.7}$$

As $\widetilde{S}_n$ converges to $\widetilde{S}$ uniformly and φ is lower semi-continuous, taking upper limit in (5.6) and (5.7) yields

$$\widetilde{H}([\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}, [\widetilde{S}(x_2, \cdots, x_{l+1})]_{\widetilde{\alpha}_L}) \leq \max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\} - \varphi(\max\{\sigma(x_i, x_{i+1}) : 1 \leq i \leq l\}),$$

and

$$\widetilde{H}([\widetilde{S}(x, \cdots, x)]_{\widetilde{\alpha}_L}, [\widetilde{S}(y, \cdots, y)]_{\widetilde{\alpha}_L}) \leq \sigma(x, y) - \varphi(\sigma(x, y)).$$

□

In what follows, we apply Theorem 5.1 and Lemma 5.1 to establish a stability result for the sequence of L-fuzzy set-valued maps.

Theorem 5.2. *Let $(\widetilde{X}, \sigma)$ be a complete metric space and $\{\widetilde{S}_n(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow L,\ n \in \mathbb{N}\}$ be a sequence of L-fuzzy set-valued maps, uniformly convergent to $\widetilde{S}(x_1, \cdots, x_l) : \widetilde{X}^l \longrightarrow L$. Assume that the following conditions are satisfied*

(i) there exists $\widetilde{\alpha}_L \in L \setminus \{0_L\}$ such that $[\widetilde{S}_n(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}$ and $[\widetilde{S}(x_1, \cdots, x_l)]_{\widetilde{\alpha}_L}$ are nonempty compact subsets of $\widetilde{X}$;

(ii) $\widetilde{S}_n$ satisfies (5.1) *and* (5.2) *for each $n \in \mathbb{N}$.*

Then,

$$\lim_{n \longrightarrow \infty} \widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_n), \mathcal{F}_{ix}(\widetilde{S})) = 0.$$

That is, the set of all L-fuzzy fixed points of $\widetilde{S}_n$ are stable.

Proof. By Lemma 5.1, $\widetilde{S}$ satisfies (5.1) and (5.2).
Let $\delta_n = \sup_{x \in \widetilde{X}} \widetilde{H}([\widetilde{S}_n(x)]_{\widetilde{\alpha}_L}, [\widetilde{S}x]_{\widetilde{\alpha}_L})$. Since $\widetilde{S}_n$ converges to $\widetilde{S}$ uniformly on $\widetilde{X}$, we have

$$\lim_{n \longrightarrow \infty} \delta_n = \lim_{n \longrightarrow \infty} \sup_{x \in \widetilde{X}} \widetilde{H}([\widetilde{S}_n x]_{\widetilde{\alpha}_L}, [\widetilde{S}x]_{\widetilde{\alpha}_L}) = 0.$$

Applying Theorem 5.1 yields $\varphi(\widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_n), \mathcal{F}_{ix}(\widetilde{S}))) \leq \delta_n$ for all $n \in \mathbb{N}$. Given that φ is lower semi-continuous, we have

$$\lim_{n \longrightarrow \infty} \inf \varphi(\widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_n), \mathcal{F}_{ix}(\widetilde{S}))) \leq \lim_{n \longrightarrow \infty} \delta_n = 0,$$

from which it follows that

$$\lim_{n \longrightarrow \infty} \widetilde{H}(\mathcal{F}_{ix}(\widetilde{S}_n), \mathcal{F}_{ix}(\widetilde{S})) = 0.$$

□

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