

# Existence and Multiplicity of Solutions for a Biharmonic Kirchhoff Equation in $\mathbb{R}^{5*}$

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**Abstract** We consider the biharmonic equation  $\Delta^2 u - (a + b \int_{\mathbb{R}^5} |\nabla u|^2 dx) \Delta u + V(x)u = f(u)$ , where  $V(x)$  and  $f(u)$  are continuous functions. By using a perturbation approach and the symmetric mountain pass theorem, the existence and multiplicity of solutions for this equation are obtained, and the power-type case  $f(u) = |u|^{p-2}u$  is extended to  $p \in (2, 10)$ , where it was assumed  $p \in (4, 10)$  in many papers.

**Keywords** Biharmonic equation, multiplicity of solutions, variational method

**MSC(2010)** 35J85, 47J30, 49J52

## 1. Introduction

We consider the existence and multiplicity of solutions for the following biharmonic equation

$$\begin{cases} \Delta^2 u - \left( a + b \int_{\mathbb{R}^5} |\nabla u|^2 dx \right) \Delta u + V(x)u = f(u), \\ u(x) = u(|x|) \in H^2(\mathbb{R}^5), \end{cases} \quad (1.1)$$

where  $V \in C(\mathbb{R}^5, \mathbb{R})$ ,  $f \in C(\mathbb{R}, \mathbb{R})$ . Biharmonic equations appear in many areas, for example, some of these problems arise from different areas of applied mathematics and physics such as surface diffusion on solids, Micro Electro-Mechanical systems, and flow in Hele-Shaw cells (see [7]). Also, this kind of equations can describe the static deflection of an elastic plate in a fluid and the study of traveling waves in suspension bridges [6, 15]. These equations have been discussed by many authors. Indeed, if we replace  $f(u)$  by  $f(x, u)$  and set  $V(x) = 0$ , and a domain  $\Omega \subset \mathbb{R}^3$ , problem (1.1) becomes the following biharmonic elliptic equation of Kirchhoff type

$$\begin{cases} \Delta^2 u - \left( a + b \int_{\mathbb{R}^5} |\nabla u|^2 dx \right) \Delta u = f(x, u) \text{ in } \Omega, \\ u = \nabla u = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.2)$$

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which is related to the general form of the following stationary analogue of the equation

$$u_{tt} + \Delta^2 u - \left( a + b \int_{\mathbb{R}^5} |\nabla u|^2 dx \right) \Delta u = f(x, u), \quad x \in \Omega. \quad (1.3)$$

Equation (1.3) is used to describe some phenomena appearing in different engineering, physical, and other scientific fields, because it is regarded as a good approximation for describing nonlinear vibrations of beams or plates [2, 4]. For example, on bounded domains, Zhang and Wei [19] used the mountain pass theorem and linking theorem to obtain the existence and multiplicity of results for the following problem

$$\begin{cases} \Delta^2 u + a \Delta u = \lambda |u|^{q-2} u + f(x, u) & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.4)$$

where  $\Omega \subset \mathbb{R}^N$  is a smooth bounded domain,  $a$  is a constant,  $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$  and  $1 < q < 2$ . If  $\lambda = 0$ , An and Liu [1] obtained the existence of solutions of (1.4). By using critical theorems, the multiple results of (1.4) were proved in [9]. Some related results can be found in [8, 10, 16] and the references therein.

As the presence of term  $\int_{\Omega} |\nabla u|^2$ , problem (1.1) is no longer a pointwise identity and therefore, this equation is viewed as an elliptic equation coupled with non-local terms. The competing effect of the non-local term brings some mathematical challenges to the analysis, and also makes the study of such problems particularly interesting. Another difficulty lies in proving the boundedness of PS-sequences, which is very important to use variational methods. In many papers, in order to get the boundedness of PS-sequences, such as in [11], the authors need to assume  $p > 4$  in (H2) and the famous AR-condition. While in our paper, we relax  $p > 2$  and drop the AR-condition.

In order to state our main results, we give the following hypotheses.

- (H0)  $V(x) = V(|x|)$  for any  $x \in \mathbb{R}^5$ , and  $\inf_{x \in \mathbb{R}^5} V(x) := V_0 > 0$ ;
- (H1)  $f \in C(\mathbb{R}, \mathbb{R})$  and  $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$ ;
- (H2)  $\limsup_{|t| \rightarrow \infty} \frac{|f(t)|}{|t|^{p-1}} < \infty$  for some  $p \in (2, 10)$ ;
- (H3) For  $\alpha \in (\frac{1}{3}, \frac{2}{5})$ ,  $t \neq 0$ ,  $f(t)t \geq (2 + 5\alpha)F(t) > 0$ , where  $F(t) = \int_{\mathbb{R}^5} f(t)dt$ ;
- (H4)  $|f(t)| \leq c_1|t| + c_2|t|^{s-1}$ ,  $s \in (2, 10)$ ;
- (H5)  $F(-t) = F(t)$ ,  $\forall t \in \mathbb{R}$ .

Note that if  $b = 0$  in problem (1.1), and it transforms into the following biharmonic equation

$$\Delta^2 u - a \Delta u + V(x)u = f(u), \quad (1.5)$$

which does not depend on the nonlocal term  $\int_{\mathbb{R}^5} |\nabla u|^2$  any more. In contrast to problem (1.5), the nonlocal term makes problem (1.1) more complex in finding sign-changing solutions. The main difficulties are as follows:

- (1) We don't have the following decomposition

$$\hat{I}(u) = \hat{I}(u^+) + \hat{I}(u^-), \quad \langle I'(u), u^\pm \rangle = \langle I'(u^\pm), u^\pm \rangle,$$

where  $\hat{I}$  is the energy functional of (1.5) given by

$$\hat{I}(u) = \frac{1}{2} \int_{\mathbb{R}^5} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) - \int_{\mathbb{R}^5} F(u).$$

Then, we use the method of invariant sets of a descending flow to seek sign-changing solutions of problem (1.1).

- (2) Since the nonlinear term can be written as  $f(u) = |u|^{p-2}u$  with  $p \in (2, 4)$ , it becomes apparent that the associated energy functional lacks a linking structure. This implies that the direct use of the minimax argument is not viable. Hence, we must employ a perturbation method by introducing a higher order term, denoted by  $\mu|u|^{q-2}u$ , to restore the linking structure.
- (3) Without the coercive condition of  $V$  and if  $2 + 5\alpha$  in (H3) is smaller than 4, the method described in [13] is unable to demonstrate the boundedness of PS-sequences. To address this challenge, we propose introducing an additional perturbation term  $\lambda\|u\|_2^{2\alpha}u$  on the left side of the equation.

Our main results are the following.

**Theorem 1.1.** *If (H0) – (H4) hold, then problem (1.1) has at least one radially symmetric ground state sign-changing solution.*

**Theorem 1.2.** *If (H0) – (H5) hold, then problem (1.1) has an unbounded sequence of radially symmetric solutions.*

This paper is organized as follows. In Section 2, we present an auxiliary problem and some necessary preliminary knowledge. We prove our main results in Section 3.

Throughout this paper, we denote by  $c_1, c_2, \dots$  different positive constants in different places.

## 2. Existence

In order to discuss this problem, we define the following Hilbert space

$$E = \left\{ u \in H_r^2(\mathbb{R}^5) : \int_{\mathbb{R}^5} V(x)u^2 < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^5} (\Delta u \Delta v + a \nabla u \nabla v + V(x)uv)$$

and the norm

$$\|u\| = \sqrt{\langle u, u \rangle} = \left( \int_{\mathbb{R}^5} |\Delta u|^2 + a|\nabla u|^2 + V(x)u^2 \right)^{\frac{1}{2}}.$$

The associated energy functional  $I : E \rightarrow \mathbb{R}$  of problem (1.1) is given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^5} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) + \frac{b}{4} \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^5} F(u),$$

from which we derive that  $I$  is a well-defined  $C^1$  functional in  $E$ , and its derivative is

$$\langle I'(u), v \rangle = \langle u, v \rangle + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla u \nabla v - \int_{\mathbb{R}^5} f(u)v, \quad \forall v \in E.$$

We now begin to show the existence of ground state sign-changing solutions to problem (1.1). The lack of the AM-condition makes it is very difficult to prove the boundedness of PS-sequences of problem (1.1). In order to prove Theorem 1.1, we need to introduce a perturbed problem, which is used to overcome this difficulty. Setting  $\alpha \in [\frac{1}{3}, \frac{2}{5})$ ,  $\lambda, \mu \in (0, 1]$  and  $q \in (\max\{p, 6\}, 10)$ , we consider the following modified problem

$$\begin{cases} \Delta^2 u - \left( a + b \int_{\mathbb{R}^5} |\nabla u|^2 dx \right) \Delta u + V(x)u = f_{\lambda, \alpha, \mu}(u), \\ u \in E, \end{cases} \quad (2.1)$$

where  $f_{\lambda, \alpha, \mu}(u) = f(u) + \mu|u|^{q-2}u - \lambda \left( \int_{\mathbb{R}^5} u^2 \right)^\alpha u$ . It is obvious that  $I_{\lambda, \mu}$  is a well defined functional in  $E$ , and its derivative is given by

$$\langle I'_{\lambda, \mu}, v \rangle = I'(u)v + \lambda \left( \int_{\mathbb{R}^5} u^2 \right)^\alpha \int_{\mathbb{R}^5} uv - \mu \int_{\mathbb{R}^5} |u|^{q-2}uv, \quad \forall u, v \in E.$$

From [18] we introduce a Pohožaev identity for problem (2.1), which will be used later.

**Lemma 2.1.** *Let  $u$  be a critical point of  $I_{\lambda, \mu}$  in  $E$  for  $(\lambda, \mu) \in (0, 1] \times (0, 1]$ . Then,*

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^5} |\Delta u|^2 + \frac{3}{2} a \int_{\mathbb{R}^5} |\nabla u|^2 + \frac{5}{2} \int_{\mathbb{R}^5} V(x)|u|^2 + \frac{1}{2} \int_{\mathbb{R}^5} \langle \nabla V(x), x \rangle |u|^2 \\ & + \frac{3}{2} b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^{1+\alpha} - 5 \int_{\mathbb{R}^5} \left( F(u) + \frac{\mu}{q} |u|^q \right) = 0. \end{aligned}$$

For each  $u \in E$ , we can see that the following equation

$$\Delta^2 v - \left( a + b \int_{\mathbb{R}^5} |\nabla u|^2 \right) \Delta v + V(x)v + \lambda \left( \int_{\mathbb{R}^5} u^2 \right)^\alpha v = f(u) + \mu|u|^{q-2}u \quad (2.2)$$

has a unique weak solution. In order to construct the descending flow for  $I_{\lambda, \mu}$ , we give an auxiliary operator  $H_{\lambda, \mu} : u \in E$ , where  $v = H_{\lambda, \mu}(u)$  is the unique weak solution of problem (2.1). It is evident that demonstrating a solution  $u$  for problem (2.1) is tantamount to proving  $u$  as a fixed point of  $H_{\lambda, \mu}$ .

**Lemma 2.2.** *The operator  $H_{\lambda, \mu}$  is well defined and continuous.*

**Proof** Suppose that  $\{u_n\}$  is a sequence in  $E$  converging to  $u$  as  $n \rightarrow \infty$ . Let us define  $v = H_{\lambda, \mu}(u)$  and  $v_n = H_{\lambda, \mu}(u_n)$ . Then

$$\begin{aligned} & \int_{\mathbb{R}^5} (\Delta v_n \Delta w + a \nabla v_n \nabla w + V(x)v_n w) + b \int_{\mathbb{R}^5} |\nabla u_n|^2 \int_{\mathbb{R}^5} \nabla v_n \nabla w \\ & + \lambda \left( \int_{\mathbb{R}^5} u_n^2 \right)^\alpha \int_{\mathbb{R}^5} v_n w = \int_{\mathbb{R}^5} f(u_n)w + \mu \int_{\mathbb{R}^5} |u_n|^{q-2}u_n w, \quad \forall w \in E, \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^5} (\Delta v \Delta w + a \nabla v \nabla w + V(x)vw) + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla v \nabla w + \lambda \left( \int_{\mathbb{R}^5} u^2 \right)^\alpha \int_{\mathbb{R}^5} vw \\ & = \int_{\mathbb{R}^5} f(u)w + \mu \int_{\mathbb{R}^5} |u|^{q-2}uw, \quad \forall w \in E. \end{aligned} \quad (2.4)$$

We need to prove that  $\|v_n - v\| \rightarrow 0$  as  $n \rightarrow \infty$ . (H1) and (H2) deduce that for any  $\epsilon > 0$ , there is  $c_\epsilon > 0$  such that

$$|f(t)| \leq \epsilon|t| + c_\epsilon|t|^{p-1}. \quad (2.5)$$

Putting  $w = v_n$  in (2.3), one has

$$\begin{aligned} & \|v_n\|^2 + b \int_{\mathbb{R}^5} |\nabla u_n|^2 \int_{\mathbb{R}^5} |\nabla v_n|^2 + \lambda \|u_n\|_2^{2\alpha} \int_{\mathbb{R}^5} v_n^2 \\ & \leq \int_{\mathbb{R}^5} (\epsilon|u_n| + c_\epsilon|u_n|^{p-1})|v_n| + \mu \int_{\mathbb{R}^5} |u_n|^{q-1}|v_n|, \end{aligned}$$

which means that from Hölder inequality,  $\{v_n\}$  is bounded in  $E$ . Assuming that  $v_n \rightharpoonup v_0$  in  $E$  and  $v_n \rightarrow v_0$  in  $L^r(\mathbb{R}^5)$  for  $r \in (2, 10)$  after extracting a subsequence, by (2.3), we derive

$$\begin{aligned} & \int_{\mathbb{R}^5} (\Delta v_0 \Delta w + a \nabla v_0 \nabla w + V(x)v_0 w) + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla v_0 \nabla w \\ & + \lambda \left( \int_{\mathbb{R}^5} u^2 \right)^\alpha \int_{\mathbb{R}^5} v_0 w = \int_{\mathbb{R}^5} f(u)w + \mu \int_{\mathbb{R}^5} |u|^{q-2} u w, \quad \forall w \in E. \end{aligned} \quad (2.6)$$

Thus,  $v_0$  is a weak solution of equation (2.2), implying that  $v = v_0$  due to its uniqueness. Additionally, by testing with  $w = v_n - v$  in equations (2.3) and (2.4), and then subtracting, we obtain the following expression:

$$\begin{aligned} & \|v_n - v\|^2 + b \int_{\mathbb{R}^5} |\nabla u_n|^2 \int_{\mathbb{R}^5} |\nabla(v_n - v)| + \lambda \|u_n\|_2^{2\alpha} \int_{\mathbb{R}^5} |v_n - v|^2 \\ & = b \int_{\mathbb{R}^5} (|\nabla u_n|^2 - |\nabla u|^2) \int_{\mathbb{R}^5} \nabla v \nabla(v_n - v) + \lambda (\|u_n\|_2^{2\alpha} - \|u\|_2^{2\alpha}) \int_{\mathbb{R}^5} v(v_n - v) \\ & + \int_{\mathbb{R}^5} (f(u_n) - f(u))(v_n - v) + \mu \int_{\mathbb{R}^5} (|u_n|^{q-2} u_n - |u|^{q-2} u)(v_n - v), \end{aligned} \quad (2.7)$$

which follows from Sobolev's embedding inequality that  $v_n \rightarrow v$  in  $E$  as  $n \rightarrow \infty$ . Thus,  $H_{\lambda,\mu}$  is continuous.

**Lemma 2.3.** (i)  $I'_{\lambda,\mu}(u)(u - H_{\lambda,\mu}(u)) \geq \|u - H_{\lambda,\mu}(u)\|^2$  for all  $u \in E$ ,

(ii)  $\|I'_{\lambda,\mu}(u)\| \leq \|u - H_{\lambda,\mu}(u)\|(1 + c_1\|u\|^2 + c_2\|u\|^{2\alpha})$  for all  $u \in E$ , where  $c_1$  and  $c_2$  are two positive constants.

**Proof** Noting that  $H_{\lambda,\mu}$  is a solution of (2.1), we derive

$$\begin{aligned} & \int_{\mathbb{R}^5} (\Delta H_{\lambda,\mu}(u) \nabla(u - H_{\lambda,\mu}(u)) + a \nabla H_{\lambda,\mu}(u) \nabla(u - H_{\lambda,\mu}(u))) \\ & + V(x)H_{\lambda,\mu}(u)(u - H_{\lambda,\mu}(u)) + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla H_{\lambda,\mu}(u) \\ & + \mu \int_{\mathbb{R}^5} |u|^{q-2} u(u - H_{\lambda,\mu}(u)) + \lambda \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} H_{\lambda,\mu}(u)(u - H_{\lambda,\mu}(u)) \\ & = \int_{\mathbb{R}^5} f(u)(u - H_{\lambda,\mu}(u)) + \mu \int_{\mathbb{R}^5} |u|^{q-2} u(u - H_{\lambda,\mu}(u)), \end{aligned}$$

and

$$\begin{aligned}
 I'_{\lambda,\mu}(u)(u - H_{\lambda,\mu}(u)) &= \int_{\mathbb{R}^5} \Delta u \Delta(u - H_{\lambda,\mu}(u)) + a \nabla u \nabla u \nabla(u - H_{\lambda,\mu}(u)) \\
 &\quad + V(x)u(u - H_{\lambda,\mu}(u)) + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla u \nabla(u - H_{\lambda,\mu}(u)) \\
 &\quad + \lambda \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} u(u - H_{\lambda,\mu}(u)) - \int_{\mathbb{R}^5} f(u)(u - H_{\lambda,\mu}(u)) \\
 &\quad - \mu \int_{\mathbb{R}^5} |u|^{q-2} u(u - H_{\lambda,\mu}(u)) \\
 &= \|u - H_{\lambda,\mu}(u)\|^2 + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} |\nabla(u - H_{\lambda,\mu}(u))|^2 \\
 &\quad + \lambda \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} u(u - H_{\lambda,\mu}(u)),
 \end{aligned}$$

which means that  $I'_{\lambda,\mu}(u)(u - H_{\lambda,\mu}(u)) \geq \|u - H_{\lambda,\mu}(u)\|^2$  for all  $u \in E$ .

Notice that for any  $\varphi$

$$\begin{aligned}
 I'_{\lambda,\mu}(u)\varphi &= \int_{\mathbb{R}^5} [\Delta(u - H_{\lambda,\mu}(u))\Delta\varphi + a \nabla(u - H_{\lambda,\mu}(u))\nabla\varphi + V(x)(u - H_{\lambda,\mu}(u))\varphi] \\
 &\quad + b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla(u - H_{\lambda,\mu}(u))\nabla\varphi + \lambda \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} (u - H_{\lambda,\mu}(u))\varphi.
 \end{aligned}$$

Then  $\|I'_{\lambda,\mu}(u)\| \leq \|u - H_{\lambda,\mu}(u)\|(1 + c_1\|u\|^2 + c_2\|u\|^{2\alpha})$ .

**Lemma 2.4.** For fixed  $(\lambda, \mu) \in (0, 1] \times (0, 1]$  and for  $c < d$  and  $\tau > 0$ , there exists  $\delta > 0$  (which depends on  $\lambda$  and  $\mu$ ) such that  $\|u - H_{\lambda,\mu}(u)\| \geq \delta$  if  $u \in E$ ,  $I_{\lambda,\mu} \in [c, d]$  and  $\|I'_{\lambda,\mu}(u)\| \geq \tau$ .

**Proof** Fixing  $\eta \in (4, q)$ , then for  $u \in E$ , we have

$$\begin{aligned}
 I_{\lambda,\mu}(u) - \frac{1}{\eta} \langle u, u - H_{\lambda,\mu}(u) \rangle &= \frac{\eta - 2}{2\eta} \|u\|^2 + \frac{b}{\eta} \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} (\nabla u - \nabla H_{\lambda,\mu}(u)) \nabla u \\
 &\quad + \lambda \frac{\eta - 2(1 + \alpha)}{2\eta(1 + \alpha)} \|u\|_2^{2\alpha} + \frac{\lambda}{\eta} \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} u(u - H_{\lambda,\mu}(u)) \\
 &\quad + \int_{\mathbb{R}^5} \left( \frac{1}{\eta} f(u)u - F(u) \right) + \frac{\eta - 4}{4\eta} b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 + \frac{q - \eta}{q\eta} \int_{\mathbb{R}^5} |u|^q.
 \end{aligned}$$

Noting  $|f(t)| \leq \epsilon|t| + c_\epsilon|t|^{p-1}$ ,

$$\begin{aligned}
 |I_{\lambda,\mu}(u)| + \frac{1}{\eta} \|u\| \|u - H_{\lambda,\mu}(u)\| &\geq \left( \frac{\eta - 2}{2\eta} - \epsilon c \right) \|u\|^2 \\
 &\quad + \frac{b}{\eta} \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} (\nabla u - \nabla H_{\lambda,\mu}(u)) \nabla u \\
 &\quad + \frac{\eta - 4}{4\eta} b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 + \frac{q - \eta}{q\eta} \mu \int_{\mathbb{R}^5} |u|^q \\
 &\quad + \lambda \frac{\eta - 2(1 + \alpha)}{2\eta(1 + \alpha)} \|u\|_2^{2\alpha+2} \\
 &\quad - c_\epsilon \|u\|_p^p + \frac{\lambda}{\eta} \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} u(u - H_{\lambda,\mu}(u)).
 \end{aligned}$$

Then,

$$\begin{aligned} & \|u\|^2 + b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 + \mu \|u\|_q^q + \lambda \|u\|_2^{2\alpha+2} - c_\epsilon \|u\|_p^p \\ & \leq c(|I_{\lambda,\mu}| + \|u\| \|u - H_{\lambda,\mu}(u)\|) + \frac{\lambda}{\eta} \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} |u| |u - H_{\lambda,\mu}(u)| \\ & \quad + \frac{b}{\eta} \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} |\nabla u - \nabla H_{\lambda,\mu}(u)| |\nabla u|. \end{aligned} \quad (2.8)$$

It follows from Sobolev's inequality and Hölder's inequality that

$$\frac{b}{\eta} \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} |\nabla u - \nabla H_{\lambda,\mu}(u)| |\nabla u| \leq c \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right) \|u\| \|u - H_{\lambda,\mu}(u)\|,$$

and

$$\frac{\lambda}{\eta} \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} |u| |u - H_{\lambda,\mu}(u)| \leq c \|u\|_2^{2\alpha} \|u\| \|u - H_{\lambda,\mu}(u)\|. \quad (2.9)$$

From (2.8), (2.9) and Young's inequality, we have

$$\begin{aligned} & \|u\|^2 + b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 + \mu \|u\|_q^q + \lambda \|u\|_2^{2\alpha+2} - c_\epsilon \|u\|_p^p \\ & \leq c(|I_{\lambda,\mu}| + \|u\| \|u - H_{\lambda,\mu}(u)\| + \|u\|^2 (\|u - H_{\lambda,\mu}(u)\|^2 + \|u\|_2^{4\alpha})). \end{aligned} \quad (2.10)$$

Proceeding by contradiction, assume that there exists  $\{u_n\} \subset E$  with  $I_{\lambda,\mu}(u_n) \in [c, d]$  and  $\|I'_{\lambda,\mu}\| \geq \tau$  such that  $\|u - H_{\lambda,\mu}(u)\| \rightarrow 0$  as  $n \rightarrow \infty$ . Then, for sufficiently large  $n$ , we deduce that

$$\|u\|^2 + b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 + \mu \|u\|_q^q + \lambda \|u\|_2^{2\alpha+2} - c_\epsilon \|u\|_p^p \leq c(1 + \|u\|_2^{4\alpha}). \quad (2.11)$$

Now, we assert that  $\{u_n\}$  is a bounded sequence in  $E$ . Otherwise, for  $\|u_n\| \rightarrow \infty$ , from (2.11), one has

$$\|u\|^2 + b \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 + \mu \|u\|_q^q + \lambda \|u\|_2^{2\alpha+2} - c_\epsilon \|u\|_p^p \leq c. \quad (2.12)$$

It should be noted that, for any  $c_1 > 0$ , there exists a corresponding value of  $c_2 > 0$  such that the inequality  $t^{1+\alpha} > c_1 t - c_2$  holds. By applying this inequality to the equation  $t = \|u_n\|_2^2$  in (2.12), we can derive the following result:

$$\|u_n\|^2 + b \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^2 + \int_{\mathbb{R}^5} (\mu |u_n|^q + \lambda c_1 |u_n|^2 - c_\epsilon |u_n|^p) - c_2 \leq c. \quad (2.13)$$

Noting that  $2 < p < q$ , we can select a sufficiently large  $c_1$  such that the inequality  $\lambda c_1 |t|^2 + \mu |t|^q - c_\epsilon |t|^p > 0$  holds for any  $t \in \mathbb{R}$ . Consequently, (2.13) leads to a contradiction. Hence, our claim stands true, indicating that the sequence  $\{u_n\}$  is bounded in  $E$  for any fixed  $(\lambda, \mu) \in (0, 1] \times (0, 1]$ . Combining this claim with Lemma 2.3, we can conclude that  $\|I'_{\lambda,\mu}(u_n)\| \rightarrow 0$  as  $n \rightarrow \infty$ , which presents a contradiction.  $\square$

In order to obtain sign-changing solutions, we begin by defining the positive and negative cones as follows:  $P^+ := \{u \in E : u \geq 0\}$  and  $P^- := \{u \in E : u \leq 0\}$ . Next,

for any  $\epsilon > 0$ , we define  $P_\epsilon^+$  as the set of elements in  $E$  whose distance to  $P^+$  is less than  $\epsilon$ . Similarly,  $P_\epsilon^-$  is the set of elements in  $E$  whose distance to  $P^-$  is less than  $\epsilon$ . Here, the distance between an element  $u$  and a set  $p^\pm$  is given by  $\text{dist}(u, p^\pm) = \inf_{v \in P^\pm} \|u - v\|$ . Importantly, it should be noted that  $P_\epsilon^- = -P_\epsilon^+$ . Let us denote  $W = P_\epsilon^+ \cap P_\epsilon^-$ . We can easily observe that  $W$  is a symmetric and open subset of  $E$ , and  $E \setminus W$  contains only sign-changing functions. Furthermore, we define the critical points of  $I_{\lambda, \mu}$  as  $K = \{u \in E : I'_{\lambda, \mu}(u) = 0\}$ , and we let  $E_0$  denote the set obtained by removing these critical points from  $E$ . For any  $c \in \mathbb{R}$ , we define  $K_c = \{u \in E : I_{\lambda, \mu}(u) = c, I'_{\lambda, \mu}(u) = 0\}$  and  $I_{\lambda, \mu}^c(u) = \{u \in E : I_{\lambda, \mu}(u) \leq c\}$ .

In the following, we aim to show that for sufficiently small  $\epsilon$ , any sign-changing solution of (2.1) lies within the set  $E \setminus W$ .

**Lemma 2.5.** *There exists  $\epsilon_0 > 0$  such that for  $\epsilon \in (0, \epsilon_0)$ ,*

- (i)  $H_{\lambda, \mu}(\partial P_\epsilon^-) \subset P_\epsilon^-$  and every nontrivial solution  $u \in P_\epsilon^-$  is negative;
- (ii)  $H_{\lambda, \mu}(\partial P_\epsilon^+) \subset P_\epsilon^+$  and every nontrivial solution  $u \in P_\epsilon^+$  is positive.

**Proof** We only need to prove that  $H_{\lambda, \mu}(\partial P_\epsilon^-) \subset P_\epsilon^-$ , and the other case is similar. For  $u \in E$ , define  $v := H_{\lambda, \mu}(u)$ . Since  $\text{dist}(v, P^-) \leq \|v^+\|$ , by Sobolev's inequality and (f1) – (f2), for any  $\epsilon > 0$ , there exists  $c_\epsilon > 0$  such that

$$\begin{aligned} \text{dist}(v, P^-) \|v^+\| &\leq \|v^+\|^2 = \langle v, v^+ \rangle \\ &\leq \int_{\mathbb{R}^5} f(u)v^+ - b \int_{\mathbb{R}^5} |\nabla u|^2 \int_{\mathbb{R}^5} \nabla v \nabla v^+ + \int_{\mathbb{R}^5} |u|^{q-2} uv^+ \\ &\quad - \lambda \|u\|_2^{2\alpha} \int_{\mathbb{R}^5} vv^+ \\ &\leq \int_{\mathbb{R}^5} f(u^+)v^+ + \int_{\mathbb{R}^5} |u^+|^{q-2} u^+ v^+ \\ &\leq \int_{\mathbb{R}^5} (\epsilon u^+ v^+ + c_\epsilon |u^+|^{p-1} v^+) + \int_{\mathbb{R}^5} |u^+|^{q-2} u^+ v^+ \\ &\leq c[\epsilon \text{dist}(u, P^-) + c_\epsilon \text{dist}(u, P^-)^{p-1} + \text{dist}(u, P^-)^{q-1}] \|v^+\|, \end{aligned}$$

which means that

$$\text{dist}(v, P^-) \leq c[\epsilon \text{dist}(u, P^-) + c_\epsilon \text{dist}(u, P^-)^{p-1} + \text{dist}(u, P^-)^{q-1}].$$

Remember that there exists  $\epsilon_0 > 0$  such that for  $\epsilon \in (0, \epsilon_0)$ , and then

$$\text{dist}(H_{\lambda, \mu}(u), P^-) = \text{dist}(v, P^-) < \epsilon.$$

Therefore,  $H_{\lambda, \mu}(u) \in P_\epsilon^-$ , for any  $u \in P_\epsilon^-$ .  $\square$

Since  $H_{\lambda, \mu}$  may not be locally Lipschitz continuous, it needs to construct a locally Lipschitz continuous vector field which inherits its properties. Using the proof of Lemma 2.1 in [3], we obtain the following lemma.

**Lemma 2.6.** *There exists a locally Lipschitz continuous operator  $T_{\lambda, \mu} : E \rightarrow E$  such that*

- (i)  $\langle H'_{\lambda, \mu}(u), u - T_{\lambda, \mu} \rangle \geq \frac{1}{2} \|u - H_{\lambda, \mu}\|^2$ ;
- (ii)  $\frac{1}{2} \|u - T_{\lambda, \mu}\|^2 \leq \|u - H_{\lambda, \mu}\|^2 \leq 2 \|u - T_{\lambda, \mu}\|^2$ ;
- (iii)  $H_{\lambda, \mu}(\partial P_\epsilon^\pm) \subset P_\epsilon^\pm, \forall \epsilon \in (0, \epsilon_0)$ ;



(iv) if  $I_{\lambda,\mu}$  is even, then  $T_{\lambda,\mu}$  is odd.

In the following, we claim that the functional  $I_{\lambda,\mu}$  satisfies the (PS)-condition.

**Lemma 2.7.** *Assume that there exist  $\{u_n\} \subset E$  and  $c \in \mathbb{R}$ , such that  $I_{\lambda,\mu}(u_n) \rightarrow 0$  for any fixed  $(\lambda, \mu) \in (0, 1] \times (0, 1]$  as  $n \rightarrow \infty$ . Then, there exists a convergent sequence  $\{u_n\}$ , denoted as  $\{u_n\}$  for simplicity, such that  $u_n \rightarrow u$  in  $E$ , with  $u \in E$ .*

**Proof** For  $\eta \in (4, p)$ , we derive

$$\begin{aligned} & \eta I_{\lambda, \|u_n\|^2 + \mu}(u_n) - \langle I'_{\lambda,\mu}, u_n \rangle \\ &= \frac{\eta - 2}{2} + \frac{b(\eta - 4)}{4} \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^2 + \lambda \frac{\eta - 2(1 + \alpha)}{2(1 + \alpha)} \|u_n\|_2^{2(1+\alpha)} \\ & \quad + \int_{\mathbb{R}^5} (f(u_n)u_n - \eta F(u_n)) + \mu \frac{q - \eta}{\eta} \int_{\mathbb{R}^5} |u_n|^q. \end{aligned}$$

As argued in the proof of Lemma 2.4,  $\{u_n\}$  is bounded in  $E$ . Passing to a subsequence, suppose that there exists  $u \in E$  such that  $u_n \rightharpoonup u$  in  $E$ , and  $u_n \rightarrow u$  strongly in  $L^r(\mathbb{R}^5)$  for  $r \in (2, 10)$ .

Since

$$\begin{aligned} & \langle I'_{\lambda,\mu}(u_n) - I'_{\lambda,\mu}, u_n - u \rangle \\ &= \|u_n - u\|^2 + b \int_{\mathbb{R}^5} |\nabla u_n|^2 \int_{\mathbb{R}^5} |\nabla(u_n - u)|^2 \\ & \quad + b \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 - \int_{\mathbb{R}^5} |\nabla u|^2 \right) \int_{\mathbb{R}^5} \nabla u \nabla(u_n - u) \\ & \quad - \int_{\mathbb{R}^5} (f(u_n) - f(u))(u_n - u) + \lambda \|u_n\|_2^{2\alpha} \int_{\mathbb{R}^5} (u_n - u) \\ & \quad + \lambda (\|u_n\|_2^{2\alpha} - \|u\|_2^{2\alpha}) \int_{\mathbb{R}^5} u(u_n - u) \\ & \quad - \mu \int_{\mathbb{R}^5} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u), \end{aligned}$$

the boundedness of  $\{u_n\}$  in  $E$  deduces that

$$\begin{aligned} & b \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 - \int_{\mathbb{R}^5} |\nabla u|^2 \right) \int_{\mathbb{R}^5} \nabla u \nabla(u_n - u) \rightarrow 0, \\ & \lambda (\|u_n\|_2^{2\alpha} - \|u\|_2^{2\alpha}) \int_{\mathbb{R}^5} u(u_n - u) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Similarly,

$$\mu \int_{\mathbb{R}^5} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

from which  $u_n \rightarrow u$  in  $E$  as  $n \rightarrow \infty$ .

Now, we give a deformation lemma to functional  $I_{\lambda,\mu}$  whose proof is similar to [13, Lemma 3.6].

**Lemma 2.8.** *(Deformation lemma) Let  $S \subset E$  and  $c \in \mathbb{R}$  such that  $\forall u \in I_{\lambda,\mu}^{-1}([c - 2\epsilon_0, c + 2\epsilon_0]) \cap S_{2\delta}$ ,  $\|I'_{\lambda,\mu}(u)\| \geq \epsilon_0$ , where  $\epsilon_0$  is given in Lemma 2.5 and  $S_{2\delta} := \{u \in S, \text{dist}(u, S) < 2\delta\}$ . Then for  $\epsilon_1 \in (0, \epsilon_0)$  there exists  $\gamma \in C([0, 1] \times E, E)$  such that*

$$(i) \quad \gamma(t, u) = u \text{ if } t = 0 \text{ or if } u \notin I_{\lambda,\mu}^{-1}([c - 2\epsilon_1, c + 2\epsilon_1]);$$

- (ii)  $\gamma(1, I_{\lambda, \mu}^{c+\epsilon_1} \cap S) \subset I_{\lambda, \mu}^{c-\epsilon_1}$ ;
- (iii)  $I_{\lambda, \mu}(\gamma(\cdot, u))$  is not increasing for all  $u \in E$ ;
- (iv)  $\gamma(t, \overline{P_\epsilon^+}) \subset \overline{P_\epsilon^+}$ ,  $\gamma(t, \overline{P_\epsilon^-}) \subset \overline{P_\epsilon^-}$ ,  $\forall t \in [0, 1]$ ;
- (v) if  $f$  is odd, then  $\gamma(t, \cdot)$  is odd,  $\forall t \in [0, 1]$ .

In the following we introduce a critical point theorem. Let  $P, Q \subset E$  be open sets,  $M = P \cap Q$ ,  $\Theta = \partial P \cap \partial Q$  and  $W = P \cup Q$ .

**Definition 2.1** ([14])  $\{P, Q\}$  is called an admissible family of invariant sets with respect to  $J$  at level  $c$ , provided that the following deformation property holds: if  $K_c \setminus W = \emptyset$ , then there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$ , there exists  $\eta \in C(E, E)$  satisfying

- (i)  $\gamma(\bar{P}) = \bar{P}$ ,  $\gamma(\bar{Q}) = \bar{Q}$ ;
- (ii)  $\gamma|_{J^{c-2\epsilon}} = id$ ;
- (iii)  $\gamma(J^{c+\epsilon} \setminus W) \subset J^{c-\epsilon}$ .

**Theorem 2.1.** ([14]) Assume that  $\{P, Q\}$  is an admissible family of invariant sets with respect to  $J$  at any level  $c \geq c_* := \inf_{u \in \Theta} J(u)$ , and there exists a map  $\psi_0 : \Delta \rightarrow E$  satisfying

- (i)  $\psi_0(\partial_1 \Delta) \subset P$  and  $\psi_0(\partial_2 \Delta) \subset Q$ ,
- (ii)  $\psi_0(\partial_1 \Delta) \cap M = \emptyset$ ,
- (iii)  $\sup_{u \in \psi_0(\partial \Delta)} J(u) < c_*$ ,

where  $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 > 0, t_1 + t_2 \leq 1\}$ ,  $\partial_1 \Delta = \{0\} \times [0, 1]$ ,  $\partial_2 \Delta = [0, 1] \times \{0\}$  and  $\partial_0 \Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 > 0, t_1 + t_2 = 1\}$ . Define  $c = \inf_{\psi \in \Gamma} \sup_{u \in \psi(\Delta) \setminus W} J(u)$ , where  $\Gamma := \{\psi \in C(\Delta, E) : \psi(\partial_1 \Delta) \subset P, \psi(\partial_2 \Delta) \subset Q, \psi|_{\partial_0 \Delta} = \psi_0|_{\partial_0 \Delta}\}$ . Then  $c \geq c_*$  and  $K_c \setminus W \neq \emptyset$ .

In order to employ Theorem 2.1 to prove the existence of sign-changing solutions to problem (2.1), setting  $P = P_\epsilon^+$ ,  $Q = P_\epsilon^-$  and  $J = I_{\lambda, \mu}$ , we need to show that  $\{P_\epsilon^+, P_\epsilon^-\}$  is an admissible family of invariance sets for the functional  $I_{\lambda, \mu}$  at any level  $c \in \mathbb{R}$ . Since  $K_c \subset E$ , if  $K_c \setminus W = \emptyset$ , the functional  $I_{\lambda, \mu}$  satisfies the (PS)-condition and  $K_c$  is compact, one has  $2\delta := \text{dist}(K_c, \partial W) > 0$ .

**Lemma 2.9.** For any  $r \in [2, 10]$ , there exists  $m > 0$  independent of  $\epsilon$  such that  $\|u\|_q \leq m\epsilon$  for  $u \in M = P_\epsilon^+ \cap P_\epsilon^-$ .

**Lemma 2.10.** If  $\epsilon > 0$  is sufficiently small, then  $I_{\lambda, \mu}(u) \geq \frac{\epsilon^2}{4}$  for all  $u \in \Theta = \partial P_\epsilon^+ \cap \partial P_\epsilon^-$ , i.e.,  $c_* \geq \frac{\epsilon^2}{4}$ .

The proof of the above two lemmas are similar to that in [12]. Here, we omit their proofs.

**Proof of Theorem 1.1.** We use Theorem 2.1 to prove the existence of sign-changing solutions to problem (2.1). Set  $X = E$ ,  $P = P_\epsilon^+$ ,  $Q = P_\epsilon^-$ , and  $J = I_{\lambda, \mu}$ . Choose  $S = E \setminus W$  in Lemma 2.8, then we can obtain that  $\{P_\epsilon^+, P_\epsilon^-\}$  is an admissible family of invariant sets for the functional  $I_{\lambda, \mu}$  at any level  $c \in \mathbb{R}$ . Now, we divide it into three steps.

*Step 1.* Choose  $\varphi_1, \varphi_2 \in C_0^\infty(B_1(0))$  such that  $\text{supp}(\varphi_1) \cap \text{supp}(\varphi_2) = \emptyset$  and  $\varphi_1 < 0, \varphi_2 > 0$ , where  $B_r(0) = \{x \in \mathbb{R}^5 : |x| < r\}$ . For  $(t, s) \in \Delta$ , define  $\tilde{\varphi}(t, s) = R^2[t\varphi_1(R\cdot) + s\varphi_2(R\cdot)]$ , where  $R > 0$  will be determined later. It is easy to see that for  $t, s \in [0, 1]$ ,  $\tilde{\varphi}(0, s)(\cdot) = R^2s\varphi_2(R\cdot) \in P_\epsilon^+$  and  $\tilde{\varphi}(t, 0) = R^2t\varphi_1(R\cdot) \in P_\epsilon^-$ . By virtue of Lemma 2.10, for small  $\epsilon > 0$ ,

$$I_{\lambda, \mu}(u) \geq \frac{\epsilon^2}{4} \text{ for small } u \in \Theta = \partial P_\epsilon^+ \cap \partial P_\epsilon^-, (\lambda, \mu) \in (0, 1] \times (0, 1].$$

Thus,  $c_* = \inf_{u \in \Theta} I_{\lambda, \mu}(u) \geq \frac{\epsilon^2}{4}$  for any  $(\lambda, \mu) \in (0, 1] \times (0, 1]$ . Let  $u_t = \tilde{\varphi}(t, 1-t)$ , for  $t \in [0, 1]$ . Note that

$$\rho = \min\{\|t\varphi_1 + (1-t)\varphi_2\|_2 : 0 \leq t \leq 1\} > 0.$$

Then  $\|u_t\|_2^2 \geq \rho R^{-1}$  for  $u \in \tilde{\varphi}(\partial_0 \Delta)$ . It follows from Lemma 2.10 that  $\tilde{\varphi}(\partial_0 \Delta) \cap P_\epsilon^+ \cap P_\epsilon^- = \emptyset$ . A direct computation shows that

$$\begin{aligned} \int_{\mathbb{R}^5} |\nabla u_t|^2 &= R \int_{\mathbb{R}^5} (t^2 |\nabla \varphi_1|^2 + (1-t)^2 |\nabla \varphi_2|^2) =: RB_1(t), \\ \int_{\mathbb{R}^5} |\Delta u_t|^2 &= R^3 \int_{\mathbb{R}^5} (t^2 |\Delta \varphi_1|^2 + (1-t)^2 |\Delta \varphi_2|^2) =: R^3 B_2(t), \\ \int_{\mathbb{R}^5} V(x) |u_t|^2 &\leq R^{-1} \max_{x \in B_1(0)} V(x) \int_{\mathbb{R}^5} (t^2 |\varphi_1|^2 + (1-t)^2 |\varphi_2|^2) =: R^{-1} B_3(t), \\ \int_{\mathbb{R}^5} |u_t|^q &= R^{2q-5} \int_{\mathbb{R}^5} (t^q |\varphi_1|^q + (1-t)^q |\varphi_2|^q) =: R^{2q-5} B_q(t), \\ \left( \int_{\mathbb{R}^5} |u_t|^2 \right)^{1+\alpha} &= R^{-(1+\alpha)} \left( \int_{\mathbb{R}^5} (t^2 |\varphi_1|^2 + (1-t)^2 |\varphi_2|^2) \right)^{(1+\alpha)} =: R^{(1+\alpha)} B_3^{1+\alpha}(t), \\ \int_{\mathbb{R}^5} |u_t|^{2+5\alpha} &= R^{5\alpha-1} \int_{\mathbb{R}^5} \left( t^{2+5\alpha} |\varphi_1|^{2+5\alpha} + (1-t)^{2+5\alpha} |\varphi_2|^{2+5\alpha} \right) \\ &=: R^{5\alpha-1} B_{2+5\alpha}(t), \end{aligned}$$

where  $q \in (\max\{p, 6\}, 10)$ . Since  $F(t) \geq c_1 |t|^{2+5\alpha} - c_2$  for any  $t \in \mathbb{R}$ ,

$$\begin{aligned} I(u_t) &= \frac{1}{2} \int_{\mathbb{R}^5} (|\Delta u_t|^2 + a |\nabla u_t|^2 + V(x) u_t^2) + \frac{b}{4} \left( \int_{\mathbb{R}^5} |\nabla u_t|^2 \right)^2 - \int_{\mathbb{R}^5} F(u) \\ &\quad + \frac{1}{2(1+\alpha)} \|u_t\|_2^{2(1+\alpha)} - \frac{\mu}{q} \int_{\mathbb{R}^5} |u_t|^q \\ &\leq \frac{R^3}{2} B_2(t) + \frac{aR}{2} B_1(t) + \frac{1}{2R} B_3(t) + \frac{b}{4} R^2 B_2^2(t) + \frac{1}{2R^{(1+\alpha)}(1+\alpha)} B_3^{1+\alpha}(t) \\ &\quad - c_1 R^{5\alpha-1} B_{2+5\alpha}(t) + cc_2 R^5 - \frac{\mu R^{2q-5}}{q} B_q(t) \\ &\rightarrow -\infty \text{ as } R \rightarrow +\infty \end{aligned}$$

for any fixed  $(\alpha, \mu) \in (0, 1] \times (0, 1]$ , we can choose  $R$  large enough such that

$$\sup_{u \in \tilde{\varphi}(\partial_0 \Delta)} I_{\lambda, \mu}(u) < c_* := \inf_{u \in \Theta} I_{\lambda, \mu}(u).$$

Since  $I_{\lambda, \mu}$  satisfies the assumptions of Theorem 2.1, the number

$$c_{\lambda, \mu} = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(\Delta) \setminus W} I_{\lambda, \mu}(u)$$

is a critical value of  $I_{\lambda,\mu}$  satisfying  $c_{\lambda,\mu} \geq c_*$ . Therefore, there exists  $u_{\lambda,\mu} \in \setminus(P_\epsilon^+ \cup P_\epsilon^-)$  such that  $I_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu}$  and  $I'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$  for  $(\lambda, \mu) \in (0, 1] \times (0, 1]$ .

*Step 2.* Passing to the limit as  $\lambda \rightarrow 0$  and  $\mu \rightarrow 0$ , according to the definition of  $c_{\lambda,\mu}$ , we know that for any  $(\lambda, \mu) \in (0, 1] \times (0, 1]$ ,

$$c_{\lambda,\mu} \leq c_R := \sup_{u \in \bar{\varphi}(\Delta)} I_{1,0}(u) < \infty, \quad (2.14)$$

where  $c_R$  is independent of  $(\lambda, \mu) \in (0, 1] \times (0, 1]$ . Without loss of generality, let  $\lambda = \mu$ . Take a sequence  $\{\lambda_n\}$  satisfying  $\lambda_n \rightarrow 0^+$ , then there exists a sequence of sign-changing critical points  $\{u_{\lambda_n}\}$  of  $I_{\lambda_n, \mu_n}$ , which is still denoted by itself, and  $I_{\lambda_n, \mu_n}(u_n) = c_{\lambda_n, \mu_n}$ . Now, we prove that  $\{u_n\}$  is bounded in  $E$ . According to the definition of  $I_{\lambda,\mu}$ , one has

$$\begin{aligned} c_{\lambda_n, \mu_n} &= \frac{1}{2} \int_{\mathbb{R}^5} (|\Delta u_n|^2 + a|\nabla u_n|^2 + V(x)u_n^2) + \frac{b}{4} \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^2 \\ &\quad - \int_{\mathbb{R}^5} F(u_n) + \frac{1}{2(1+\alpha)} \|u_n\|_2^{2(1+\alpha)} - \frac{\mu}{q} \int_{\mathbb{R}^5} |u_n|^q, \end{aligned} \quad (2.15)$$

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^5} |\Delta u_n|^2 + \frac{3}{2}a \int_{\mathbb{R}^5} |\nabla u_n|^2 + \frac{5}{2} \int_{\mathbb{R}^5} V(x)|u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^5} \langle \nabla V(x), x \rangle |u_n|^2 \\ &\quad + \frac{3b}{2}b \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^2 + \frac{5\lambda}{2} \left( \int_{\mathbb{R}^5} u_n^2 \right)^{1+\alpha} - 5 \int_{\mathbb{R}^5} (F(u_n) + \frac{\mu}{q}|u_n|^q) = 0. \end{aligned} \quad (2.16)$$

Multiplying (2.15), (2.14) and (2.16) by 2, -1, and  $\alpha$  respectively and adding them up

$$\begin{aligned} 2c_{\lambda_n, \mu_n} &= \frac{\alpha}{2} \int_{\mathbb{R}^5} |\nabla u_n|^2 + \frac{3\alpha}{2}a \int_{\mathbb{R}^5} |\nabla u_n|^2 + \frac{5\alpha}{2} \int_{\mathbb{R}^5} V(x)|u_n|^2 \\ &\quad + \left( \frac{\lambda}{1+\alpha} + \frac{3}{2}\lambda \right) \|u_n\|_2^{2(1+\alpha)} + \frac{\alpha}{2} \int_{\mathbb{R}^5} \langle \nabla V(x), x \rangle |u_n|^2 \\ &\quad + \left( \frac{3\alpha}{2} - \frac{1}{2} \right) \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^2 + \int_{\mathbb{R}^5} [f(u_n)u_n - (2+5\alpha)F(u_n)] \\ &\quad + \mu \int_{\mathbb{R}^5} \left( 1 - \frac{1}{q} \right) |u|^q. \end{aligned}$$

By condition (H3), we have

$$10c_R > \frac{7}{2} \int_{\mathbb{R}^5} |\Delta u_n|^2 + \frac{5}{2} \int_{\mathbb{R}^5} |\nabla u_n|^2,$$

from which

$$\int_{\mathbb{R}^5} |\Delta u_n|^2 < c, \quad \int_{\mathbb{R}^5} |\nabla u_n|^2 < c. \quad (2.17)$$

On the other hand, from (2.14), (2.15) and hypotheses (H0), (H1) and (H2),

we infer that for all small  $\xi > 0$  such that

$$\begin{aligned} c_R &> \frac{1}{2} \int_{\mathbb{R}^5} |\Delta u_n|^2 + \frac{a}{2} \int_{\mathbb{R}^5} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^5} V(x)|u_n|^2 + \int_{\mathbb{R}^5} (F(u_n) - \frac{\mu}{q}|u_n|^q) \\ &> \frac{1-\xi}{2} \int_{\mathbb{R}^5} V(x)|u_n|^2 - c_\xi \int_{\mathbb{R}^5} u_n^{10} - \frac{1}{q} \int_{\mathbb{R}^5} |u_n|^q \\ &> \frac{1-\xi}{2} \int_{\mathbb{R}^5} V(x)|u_n|^2 - c_\xi S^{2^*} \left( \int_{\mathbb{R}^5} |\Delta u_n|^2 \right)^{\frac{2^*}{2}} - \frac{1}{q} \int_{\mathbb{R}^5} |u_n|^q. \end{aligned} \quad (2.18)$$

From interpolation inequation, Hölder's inequality and Young's inequality, we obtain that for  $\xi > 0$ , there is  $c_\xi > 0$  such that

$$\begin{aligned} \int_{\mathbb{R}^5} |u_n|^q &\leq \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^{-\frac{q}{8} + \frac{5}{4}} \left( \int_{\mathbb{R}^5} |u_n|^{2^*} \right)^{\frac{5}{22^*}(\frac{q}{2}-1)} \\ &\leq \epsilon \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^{-\frac{q}{16} + \frac{5}{8}} + c_\xi \left( \int_{\mathbb{R}^5} |u_n|^{2^*} \right)^{\frac{5}{42^*}(\frac{q}{2}-1)} \\ &\leq \epsilon \left( \int_{\mathbb{R}^5} |\nabla u_n|^2 \right)^{-\frac{q}{16} + \frac{5}{8}} + c_\xi S^{\frac{5}{4}(\frac{q}{2}-1)} \left( \int_{\mathbb{R}^5} |\Delta u_n|^{2^*} \right)^{\frac{5}{8}(\frac{q}{2}-1)}. \end{aligned} \quad (2.19)$$

Combining (2.17), (2.18) and (2.19), we immediately derive that  $\{u_n\}$  is bounded in  $E$ . In view of (2.14) and Lemma 2.10, we infer that

$$\begin{aligned} \lim_{n \rightarrow \infty} I(u_n) &= \lim_{n \rightarrow \infty} \left( I_{\lambda_n, \mu_n}(u_n) - \frac{\lambda_n}{2(1+\alpha)} \|u_n\|_2^{2(1+\alpha)} + \frac{\mu_n}{q} \int_{\mathbb{R}^5} |u_n|^q \right) \\ &= \lim_{n \rightarrow \infty} c_{\lambda_n, \mu_n} = c^* > \frac{\epsilon^2}{4}. \end{aligned}$$

Furthermore, for any  $\psi \in C_0^\infty(\mathbb{R}^5)$ ,

$$\lim_{n \rightarrow \infty} I'(u_n)\psi = \lim_{n \rightarrow \infty} \left( I'_{\lambda_n, \mu_n}(u_n)\psi - \lambda_n \|u_n\|_2^{2\alpha} \int_{\mathbb{R}^5} u_n \psi + \mu_n \int_{\mathbb{R}^5} |u_n|^{q-2} u_n \psi \right) = 0,$$

which means that  $\{u_n\}$  is a bounded  $PS$ -sequence for  $I$  at level  $c^*$ . Thus, there exists  $u^* \in E$  such that  $u_n \rightharpoonup u^*$  in  $L^r(\mathbb{R}^5)$  for  $r \in (2, 2_*)$ . The similar argument of Lemma 2.7 leads to  $I'(u^*) = 0$  and  $u_n \rightharpoonup u^*$  in  $E$  as  $n \rightarrow \infty$ . Hence,  $u_n \in E \setminus (P_\epsilon^+ \cup P_\epsilon^-)$ , and then  $u^*$  is a sign-changing solution of (2.1).

*Step 3.* Define

$$\bar{c} := \inf_{u \in \Omega} I(u), \quad \Omega := \{u \in E \setminus \{0\}, I'(u) = 0, u^\pm \neq 0\}.$$

Based on Step 2, we have  $\Omega \neq \emptyset$  and  $\bar{c} \leq c^*$ , where  $c^*$  is given in Step 2. From the definition of  $\bar{c}$ , there is  $\{u_n\} \subset E$  such that  $I(u_n) \rightarrow \bar{c}$  and  $I'(u_n) = 0$ . Using the earlier arguments, we can obtain that  $\{u_n\}$  is bounded in  $E$ . Arguing as in Lemma 2.7, there exists a nontrivial  $u \in E$  such that  $I(u) = \bar{c}$  and  $I'(u) = 0$ . Furthermore, we deduce from  $\langle I'(u_n), u_n^\pm \rangle = 0$  that for any  $\epsilon > 0$  there exists  $c_\epsilon > 0$  such that

$$\begin{aligned} c \left( \|u_n^\pm\|_p^2 + \int_{\mathbb{R}^5} |u_n^\pm|^2 \right) &\leq \|u_n^\pm\|^2 \int_{\mathbb{R}^5} f(u_n) u_n^\pm = \int_{\mathbb{R}^5} f(u_n^\pm) u_n^\pm \\ &\leq \epsilon \int_{\mathbb{R}^5} |u_n^\pm|^2 + c_\epsilon \int_{\mathbb{R}^5} |u_n^\pm|^p \leq \epsilon \|u_n^\pm\|_2^2 + c_\epsilon \|u_n^\pm\|, \end{aligned}$$

which together with the boundedness of  $\{u_n\}$  in  $E$ , implies that  $\|u_n^\pm\|_p \geq c$ . Hence  $\|u_n^\pm\| \geq c$ , and then  $u$  is a ground state solution to problem (1.1). Thus, the proof is complete.

### 3. Multiplicity

In this section, we show that problem (1.1) has an unbounded sequence of critical values. In order to obtain infinitely many solutions, we introduce the symmetric mountain pass theorem [17].

**Lemma 3.1.** *Let  $E$  be a real Banach space and  $I \in C^1(E, \mathbb{R})$  with  $I$  even. Suppose  $I(0) = 0$  and  $I$  satisfies (PS)-condition and*

- (i) *there exist constants  $\rho, \alpha > 0$  such that  $I|_{\partial B_\rho} \geq \alpha$ ;*
- (ii) *for all finite dimensional subspaces  $\tilde{E} \subset E$ , there exists an  $R = R(\tilde{E})$  such that  $I(u) \leq 0$  for  $u \in \tilde{E} \setminus B_{R(\tilde{E})}$ .*

*Then  $I$  possesses an unbounded sequence of critical values.*

**Proof of Theorem 1.2.** Since  $E$  is a reflexive and separable Banach space, there exist  $e_i \subset E$  and  $e_i^* \subset E^*$  such that

$$E = \overline{\text{span}\{e_i | i = 1, 2, \dots\}}, \quad E^* = \overline{\text{span}\{e_i^* | i = 1, 2, \dots\}},$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For the sake of convenience, we set  $E_i = \text{span}\{e_i\}$ ,  $Y_k = \bigoplus_{i=1}^k E_i$  and  $Z_k = \bigoplus_{i=k}^\infty E_i$ . It is easy to see that  $I_{\lambda, \mu}(0) = 0$ , and  $I_{\lambda, \mu}$  is even. From (H1) and (H4), we derive

$$|F(u)| \leq \frac{\epsilon}{2}|u|^2 + c|u|^s.$$

Hence,

$$\begin{aligned} I_{\lambda, \mu}(u) &\geq \frac{1}{2} \int_{\mathbb{R}^5} |\Delta u|^2 + \frac{a}{2} \int_{\mathbb{R}^5} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^5} V(x)|u|^2 + \frac{\lambda}{2(1+\alpha)} \left( \int_{\mathbb{R}^5} u^2 \right)^{1+\alpha} \\ &\quad + \frac{b}{4} \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 - \frac{\epsilon}{2} \int_{\mathbb{R}^5} |u|^2 - c \int_{\mathbb{R}^5} |u|^s - \frac{\mu}{q} \int_{\mathbb{R}^5} |u|^q \\ &\geq \left( \frac{1}{2} - \frac{\epsilon}{2}c \right) \|u\|^2 - c\|u\|^s - c\|u\|^q. \end{aligned}$$

Since  $2 < s$  and  $2 < q$ , there exists  $\rho_0 > 0$  such that for all  $0 < \rho < \rho_0$  we derive  $\inf\{I_{\lambda, \mu}(u) : \|u\| = \rho\} > 0$ . We now assert that  $I_{\lambda, \mu} \rightarrow -\infty$  as  $\|u\| \rightarrow +\infty$ ,  $\forall u \in Y_k$ . By virtue of hypothesis (H3), we have

$$\begin{aligned} I_{\lambda, \mu}(u) &= \frac{1}{2} \int_{\mathbb{R}^5} |\Delta u|^2 + \frac{a}{2} \int_{\mathbb{R}^5} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^5} V(x)|u|^2 + \frac{\lambda}{2(1+\alpha)} \left( \int_{\mathbb{R}^5} u^2 \right)^{1+\alpha} \\ &\quad + \frac{b}{4} \left( \int_{\mathbb{R}^5} |\nabla u|^2 \right)^2 - \frac{\mu}{q} \int_{\mathbb{R}^5} |u|^q \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\lambda}{2(1+\alpha)} \|u\|_2^{2(1+\alpha)} + \frac{b}{4} \|\nabla u\|_2^4 - c\|u\|_{2+5\alpha}^{2+5\alpha} - \frac{\mu}{q} \|u\|_q^q. \end{aligned}$$

Since  $Y_k$  is a finite-dimensional space, all norms of  $Y_k$  are equivalent. Given  $q > 2 + 5\alpha$  and  $q > 4$ , it follows that  $I_{\lambda,\mu}(u) \rightarrow -\infty$  as  $\|u\| \rightarrow +\infty$ . Notably,  $I_{\lambda,\mu}(0) = 0$  and  $I_{\lambda,\mu}$  is an even function. By considering  $V = Y_k$  ( $\dim Y_k = k$ ) and  $Y = E$  ( $\text{codim } Y = 0$ ), as well as utilizing the symmetric mountain pass theorem and Lemma 2.7, we can deduce that for any fixed  $\mu \in (0, 1]$  and  $j \geq 2$ , there exists a sequence  $\{u_{\lambda,\mu} \subset E\}$  such that  $I_{\lambda,\mu}^j(u_{\lambda,\mu}^j) = c_{\lambda,\mu}^j$ ,  $I'_{\lambda,\mu}(u_{\lambda,\mu}^j) = 0$ , and  $c_{\lambda,\mu}^j \rightarrow \infty$ . Following a similar approach to the proof of Theorem 1.1, for any fixed  $j \geq 2$ , the sequence  $\{u_{\lambda,\mu}^j\}_{\lambda,\mu \in (0,1]}$  is bounded in  $E$ , which implies the existence of a constant  $c > 0$  independent of  $\lambda$  and  $\mu$  such that  $\|u_{\lambda,\mu}^j\| \leq c$ . Without loss of generality, let us assume that  $u_{\lambda,\mu}^j \rightharpoonup u_*^j$  in  $E$  as  $\mu \rightarrow 0^+$ . Since  $I_{\lambda,\mu}(u) \leq I_{1,0}(u)$ , we can further assume that  $c_{\lambda,\mu}^j \rightarrow c_{0,0}^j$  as  $\lambda, \mu \rightarrow 0^+$ . Consequently, we can show that  $u_{\lambda,\mu}^j \rightarrow u_*^j$  in  $E$  as  $\lambda, \mu \rightarrow 0^+$ , where  $u_*^j$  satisfies  $I'(u_*^j) = 0$  and  $I(u_*^j) = c_{0,0}^j$ . We claim that  $c_{0,0}^j \rightarrow \infty$  as  $j \rightarrow \infty$ .

**Claim:** There holds  $c_{0,0}^j \rightarrow \infty$  as  $j \rightarrow \infty$ .

Hypotheses (H1) and (H2) deduce that

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{1}{2} \int_{\mathbb{R}^5} |\Delta u|^2 + \frac{a}{2} \int_{\mathbb{R}^5} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^5} V(x)|u|^2 - \int_{\mathbb{R}^5} F(u) - \frac{1}{q} \int_{\mathbb{R}^5} |u|^q \\ &\geq \frac{1}{2} \int_{\mathbb{R}^5} |\Delta u|^2 + \frac{a}{2} \int_{\mathbb{R}^5} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^5} V(x)|u|^2 - \int_{\mathbb{R}^5} \left( \frac{V_0}{4} u^2 \right. \\ &\quad \left. + \frac{c_{V_0}}{q} |u|^q \right) - \frac{1}{q} \int_{\mathbb{R}^5} |u|^q \\ &\geq \frac{1}{2} \int_{\mathbb{R}^5} (|\Delta u|^2 + a|\nabla u|^2 + \tilde{V}(x)|u|^2) - \frac{c}{q} \int_{\mathbb{R}^5} |u|^q := H(u), \end{aligned}$$

where  $\tilde{V}(x) := V(x) - \frac{V_0}{2}$  and  $c_{V_0}, c > 0$  are constants. Note that the boundedness of  $PS$ -sequence is not hard to verify for energy functionals which satisfy the famous AR-condition. As a result, with some suitable modification, the methods of functional  $I_{\lambda,\mu}$  are still valid for  $H$ . Without any perturbation, this means that the functional  $H$  satisfies all conditions of Lemma 3.1, and has an unbounded sequence of critical values denoted by  $d^j$ , i.e.,  $d^j \rightarrow +\infty$  as  $j \rightarrow \infty$ . Since  $c_{\lambda,\mu}^j > d^j$ , taking  $\lambda, \mu \rightarrow 0^+$ , we immediately obtain  $c_{0,0}^j > d^j \rightarrow +\infty$  as  $j \rightarrow +\infty$ . Thus, problem (1.1) has infinitely many sequence of critical values, which completes the proof.

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## References

- [1] An Y. and Liu R., *Existence of nontrivial solutions of an asymptotically linear fourth-order elliptic equations*, *Nonlinear Anal.* 2008, 68, 3325-3331.
- [2] Ball J.M., *Initial-boundary value problems for an extensible beam*, *J. Math. Anal. Appl.* 1973, 42, 61-90.

- [3] Badiale M., Greco S. and Rolando S., *Radial solutions of a biharmonic equation with vanishing or singular radial potentials*, *Nonlinear Anal.* 2019, 185, 97-122.
- [4] Berger H.M., *A new approach to the analysis of large deflections of plates*, *J. Appl. Mech.* 1955, 22, 465-472.
- [5] Berestycki H. and Lions P., *Nonlinear scalar field equations. I. Existence of ground state*, *Arch. Ration. Mech. Anal.*, 1983, 82, 313-345.
- [6] Chen Y. and McKenna P., *Traveling waves in a nonlinear suspension beam: theoretical results and numerical observations*, *J. Differential Equations* 1997, 135, 325-355.
- [7] Ferrero A. and Warnault G., *On a solutions of second and fourth order elliptic with power type nonlinearities*, *Nonlinear Anal. T.M.A.* 2009, 70, 2889-2902.
- [8] Fu S. and Qiao Z., *Eigenvalues and Eigenfunctions of a Schrödinger Operator Associated with a Finite Combination of Dirac-Delta Functions and CH Peakons*, *Journal of Nonlinear Modeling and Analysis*, 2021, 3(1), 131C144.
- [9] Han X., Shao M. and Zhao L., *Existence and convergence of solutions for nonlinear biharmonic equations on graphs*, *J. Differential Equations* 2020, 268, 3936-3961.
- [10] Hu S. and Wang L., *Existence of nontrivial solutions for fourth-order asymptotically linear elliptic equations*, *Nonlinear Anal.* 2014, 94, 120-132.
- [11] Liu H., Xiao Q., Shi H., Chen H., and Liu Z., *Ground state and nodal solutions for a class of biharmonic equations with singular potentials*, *Journal of Applied Analysis and Computation*, 2019, 9(4), 1393-1406.
- [12] Liu Z., Ouyang Z. and Zhang J., *Existence and multiplicity of sign-changing standing waves for a gauged nonlinear Schrödinger equation in  $R^2$* , *Nonlinearity*, 2019, 32, 3082-3111.
- [13] Liu Z., Wang Z.Q. and Zhang J., *Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system*, *Annali di Matematica Pura ed Applicata*, 2016, 195, 775-794.
- [14] Liu J., Liu X. and Wang Z.Q., *Multiple mixed states of nodal solutions for nonlinear Schrödinger systems*, *Calc. Var. Partial Differential Equations*, 2015, 52, 565-586.
- [15] McKenna P. and Walter W., *Traveling waves in a suspension bridge*, *SIAM J. Appl. Math.* 1990, 50, 703-715.
- [16] Phan T.V., *Blowup for biharmonic Schrödinger equation with critical nonlinearity*, *Z. Angew. Math. Phys.* 2018, 69, 31.
- [17] Rabinowitz P.H., *Minimax methods in critical point theory with applications to differential equations*, Higher Education Press, Beijing, 2016. 12.
- [18] Xu L. and Hai H., *Existence of positive ground solutions for biharmonic equations via pohoževCnehari manifold*, *Topological Methods in Nonlinear Analysis*, 2018, 52, 541-560.



- [19] Zhang J. and Wei Z., *Multiple solutions for a class of biharmonic equations with a nonlinearity concave at the origin*, J. Math. Anal. Appl. 2011, 383, 291-306.