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Existence and Uniqueness of Solutions for Time-Fractional Oldroyd-B Fluid Equations with Generalized Fractional Derivatives

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Abstract In this paper, we study the existence and uniqueness of solutions for time-fractional Oldroyd-B fluid equations with generalized fractional derivatives. We distinguish two cases. Firstly for the linear case, we get regularity results under some hypotheses of the source function and the initial data. Secondly for the nonlinear case, we use the Banach fixed point theorem to obtain the existence and uniqueness of solutions.

Keywords Time-fractional Oldroyd-B fluid equations, generalized fractional derivatives, generalized Laplace transform, regularity, Banach fixed point theorem

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1. Introduction

The subject of fractional calculus has gained considerable popularity and importance over the past three decades, primarity due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Indeed, it does provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. The Oldroyd-B model is a constitutive model used to describe the flow of viscoelastic fluids. This model can be regarded as an extension of the upper-convected Maxwell model and is equivalent to a fluid filled with elastic bead and spring dumbbells. The model is named after its creator Oldroyd [11]. Moreover, it is considered that the generalized fractional Oldroyd-B fluid model is a special case of non-Newtonian fluids that is critical in a wide range of industries and applied sciences. As a result, there are a lot of papers on this subject, with a lot of distinct research directions. Riemann-Liouville, Caputo, Hadamard, Riesz and other definitions for fractional derivatives and fractional integrals are now in use. We can refer the reader to some papers [1, 2, 5, 8–10, 12, 15, 16].

In [14], Tri considered the following initial problem for the time-fractional Ol-

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droyd -B fluid equation

$$\begin{cases} (1+a\partial_t^{\alpha})\, u_t(x,t) = \mu\left(1+b\partial_t^{\beta}\right)\Delta u(x,t) + F(x,t,u(x,t)),\ x\in\mathcal{D},\ 0<t\le T, \\ u(x,t)=0,\ \ (x,t)\in\partial\mathcal{D}\times(0,T), \\ u(x,0)=u_0(x),\ I^{1-\alpha}u_t(x,0)=0,\ x\in\mathcal{D}, \end{cases} \tag{1.1}$$

where ∂_t^{α} is the Riemann-Liouville fractional derivative [17].

$$\partial_t^{\alpha} v(t) := \frac{\partial}{\partial t}\int_0^t \mu_{1-\alpha}(s)\, v(t-s,x)\, ds,\ \ \mu_{\beta}(s) := \frac{1}{\Gamma(\beta)} s^{\beta-1},\ \ (\beta>0). \tag{1.2}$$

Here u_0 is called the initial data and F is the source function. The author has studied the problem (1.1) for two cases. In the first case or the linear case, under some hypotheses of the source function and initial data, he obtained regularity results, and for the second case or the nonlinear case, he used Banach's fixed point theorem to prove the existence and uniqueness of the solution.

In [3], Al-Maskari et al. considered the following initial boundary-value problem for the time-fractional Oldroyd-B fluid equation

$$(1+a\partial_t^{\alpha})\, u_t(x,t) = \mu\left(1+b\partial_t^{\beta}\right)\Delta u(x,t) + f(x,t),\ \text{in } \Omega\times(0,T]$$

with a homogeneous Dirichlet boundary condition

$$u(x,t) = 0 \text{ on } \partial\Omega\times(0,T],$$

and initial conditions

$$u(x,0)=v(x),\ \ \left(I^{1-\alpha}u_t\right)(x,0)=0 \text{ in } \Omega,$$

where f and v are given functions, the parameters $\alpha,\beta\in(0,1)$, μ, a and b are positive constants, and ∂_t^{α} is the Riemann-Liouville fractional derivative given in (1.2), which established regularity results for the exact solution.

Motivated by the above works, in this paper we consider the following problem

$$\begin{cases} \left(1+a\partial_g^{\alpha}\right) u_t(x,t) = \mu\left(1+b\partial_g^{\beta}\right)\Delta u(x,t) + F(x,t,u(x,t)),\ x\in\mathcal{D},\ d<t\le T, \\ u(x,t)=0,\ \ (x,t)\in\partial\mathcal{D}\times(d,T), \\ u(x,d)=u_d(x),\ I_g^{1-\alpha}u_t(x,d)=0,\ x\in\mathcal{D}, \end{cases} \tag{1.3}$$

where $T>0$ is a fixed time, $0<\alpha<\beta<1$, $a,b,d\ge 0$ and $\mu>0$ are given constant parameters, and ∂_g^{α} is the generalized fractional derivative given by

$$\left(\partial_g^{\alpha} f\right)(t) = \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)}{\Gamma(1-\alpha)}\int_d^t (g(t)-g(u))^{-\alpha} f(u) g'(u) du, \tag{1.4}$$

with $g\in C^1([d,T],\mathbb{R})$ such that $g'(t)>0$ for any $t\in[d,T]$. It can be easily noticed that when $g(t)=t$, (1.4) is the classical Riemann-Liouville fractional derivative and when $g(t)=\ln t$, (1.4) is the Hadamarad fractional derivative [7,13], and $\left(I_g^{\alpha} f\right)(t)$ is the generalized fractional integral given by

$$\left(I_g^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_d^t (g(t)-g(u))^{\alpha-1} f(u)\, g'(u)\, du. \tag{1.5}$$

It is obvious that when $g(t) = t$, (1.5) is the classical Riemann-Liouville fractional integral and when $g(t) = \ln t$, (1.5) is the Hadamarad fractional integral [7, 13]. In the present work we improve and generalize the results in [14].

Outline of the paper. The work is divided as follows. In Section 2, we give some notations and materials needed for our work. In Section 3, we consider the problem (1.3) in the linear case, and obtain regularity results under some assumptions of the initial data and the source function. In Section 4, we consider the problem (1.3) in the nonlinear case, and obtain the existence of a unique solution using Banach's fixed point theorem.

2. Preliminaries

In this section, we present the mathematical backgrounds needed later to prove our main results. Firstly we recall the Hilbert scale space, which is defined as follows

$$\mathcal{H}^s(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}),\ \sum_{j=1}^{\infty} \lambda_j^{2s} \langle f, e_j \rangle_{L^2(\mathcal{D})}^2 < \infty \right\},$$

for any $s \geq 0$. We will use the symbol $\langle .,. \rangle_{L^2(\mathcal{D})}$ to denote the inner product in $L^2(\mathcal{D})$. It is well-known that $\mathcal{H}^r(\mathcal{D})$ is a Hilbert space corresponding to the norm $\|f\|_{\mathcal{H}^s(\mathcal{D})} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^{2s} \langle f, e_j \rangle_{L^2(\mathcal{D})}^2}$, $f \in \mathcal{H}^s(\mathcal{D})$. In view of $\mathcal{H}^v(\Omega) \equiv D((-\mathbb{L})^v)$ is a Hilbert space, then $D\left((-\mathbb{L})^{-v}\right)$ is a Hilbert space with the norm

$$\|v\|_{D\left((-\mathbb{L})^{-v}\right)} = \left(\sum_{j=1}^{\infty} |\langle v, e_j \rangle|^2 \lambda_j^{-2v} \right)^{\frac{1}{2}},$$

where $\langle .,. \rangle$ in the latter equality denotes the duality between $D\left((-\mathbb{L})^{-v}\right)$ and $D((-\mathbb{L})^v)$.

We will denote by $*_g$ the generalized convolution

$$(f *_g h)(t) = \int_d^t f(\tau) h(\phi(t,\tau)) g'(\tau) d\tau \text{ with } \phi(t,\tau) = g^{-1}(g(t) + g(d) - g(\tau)).$$

Moreover, the generalized convolution of two functions is commutative.

Now, we give some properties of the generalized Laplace transform [6]

$$\mathcal{L}_g \{f(t)\}(s) = \int_d^{\infty} e^{-s(g(t)-g(d))} f(t) g'(t) dt,$$

$$\mathcal{L}_g \left\{ f^{[n]}(t) \right\}(s) = s^n \mathcal{L}_g \{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} \left(f^{[k]} \right)(d),$$

$$\mathcal{L}_g \{f *_g h\} = \mathcal{L}_g \{f\} \mathcal{L}_g \{h\}.$$

The generalized Laplace transform for the generalized fractional differential operator ∂_g^{α} with $0 < \alpha < 1$ is given by

$$\mathcal{L}_g \left\{ \partial_g^{\alpha} f(t) \right\}(s) = s^{\alpha} \mathcal{L}_g \{f(t)\} - \left(I_g^{1-\alpha} f \right)(d^+).$$

Lemma 2.1. *The following inclusions hold true*

$$\begin{cases} L^p(\Omega) \hookrightarrow D(\mathcal{A}^\sigma), \ \textit{if} \ -\frac{N}{4} < \sigma \le 0, \ p \ge \frac{2N}{N-4\sigma}, \\ D(\mathcal{A}^\sigma) \hookrightarrow L^p(\Omega), \ \textit{if} \ 0 \le \sigma < \frac{N}{4}, \ p \le \frac{2N}{N-4\sigma}. \end{cases} \tag{2.1}$$

3. Linear inhomogeneous source

In this section, we consider the problem (1.3) in the linear case, that is, the source function has the simple form $F = F(x,t)$. Let $\{\lambda_j\}_{j\in\mathbb{N}}$ and $\{e_j(x)\}_{j\in\mathbb{N}}$ be, respectively, the Dirichlet eigenvalues and eigenfunctions of $\mathcal{A} := -\Delta$ on the domain $\mathcal{D}$, with $\{e_j(x)\}_{j\in\mathbb{N}}$ being an orthogonal basis in $L^2(\mathcal{D})$, and let $0 < \lambda_1 < \lambda_2 < ....$ Denote by $\langle .,. \rangle$ the inner product in $L^2(\mathcal{D})$. Applying eigenfunction decomposition, the solution u of problem (1.3) has the form of Fourier series $u(x,t) = \sum_{j=1}^{\infty} u_j(t) e_j(x)$. We will denote $u_j(t) = \langle u(x,t), e_j \rangle$. Then we get the following equation

$$\left(1 + a\partial_g^\alpha\right) \frac{du_j(t)}{dt} = -\lambda_j \mu \left(1 + b\partial_g^\beta\right) u_j(t) + F_j(t), \ u_j(d) = \langle u_d(x), e_j \rangle. \tag{3.1}$$

Our next step is to solve this equation. For this purpose, we apply the generalized Laplace transform [6], and obtain the formal eigen expansion of solution $u_j(t)$ as follows

$$u_j(t) = H_j(t) \langle u_d, e_j \rangle + \int_d^t L_j(\phi(t,\tau)) \langle F(\tau), e_j \rangle g'(\tau) d\tau, \tag{3.2}$$

which allows us to get the explicit formula of the solution u

$$u(x,t) = \sum_{j=1}^{\infty} H_j(t) \langle u_d, e_j \rangle e_j(x) + \sum_{j=1}^{\infty} \left(\int_d^t L_j(\phi(t,\tau)) \langle F(\tau), e_j \rangle g'(\tau) d\tau \right) e_j(x). \tag{3.3}$$

Here the generalized Laplace transform of the following two functions H_j and L_j is given by

$$\begin{aligned} \mathcal{L}_g(H_j)(s) &= \frac{1 + as^\alpha}{s(1 + as^\alpha) + \mu\lambda_j(1 + bs^\beta)}, \\ \mathcal{L}_g(L_j)(s) &= \frac{1}{s(1 + as^\alpha) + \mu\lambda_j(1 + bs^\beta)}. \end{aligned} \tag{3.4}$$

Thanks for the results from the work of Bazhlekova and Bazhlekov [4], we have the following lemma right away.

Lemma 3.1. *Two functions H_j and L_j satisfy the following properties*

$$\begin{aligned} &H_j(d) = 1, \ L_j(d) = 0, \ |H_j(t)| \le C_1, \ t \ge d, \\ &|H_j(t)| \le \frac{C_2\left(t^{\beta-1} + at^{\beta-\alpha-1}\right)}{\lambda_j}, \ \int_d^t |L_j(\phi(t,\tau))| g'(\tau) d\tau \le \frac{C_3}{\lambda_j}, \end{aligned}$$

where the constants C_1, C_2 and C_3 are independent of n and t.

Theorem 3.1. *Let the source function $F \in L^\infty\left(d, T, \mathcal{H}^\theta(\mathcal{D})\right)$.*

a) If $u_d \in \mathcal{H}^s(\mathcal{D})$ then

$$\|u\|^2_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))} \le 2C_1^2 \|u_d\|^2_{\mathcal{H}^s(\mathcal{D})} + 2\overline{CC_2}(s,\theta,N) C_3^2 \|F\|^2_{L^\infty(d,T,\mathcal{H}^\theta(\mathcal{D}))}. \tag{3.5}$$

Here s and θ satisfy the condition $4+4\theta-4s>N$.

b) If $u_d \in \mathcal{H}^{s-1}(\mathcal{D})$ then we obtain

$$\|u(.,t)\|_{\mathcal{H}^s(\mathcal{D})} \le \sqrt{2}C_2\left(t^{\beta-1}+at^{\beta-\alpha-1}\right)\|u_d\|_{\mathcal{H}^{s-1}(\mathcal{D})} + \sqrt{2\overline{CC_2}(s,\theta,N)}C_3\|F\|_{L^\infty(d,T,\mathcal{H}^\theta(\mathcal{D}))}. \tag{3.6}$$

Remark 3.1. We can see from part 2 of the above theorem, that if $u_d \in \mathcal{H}^{s-1}(\mathcal{D})$ then $t^\gamma \|u(.,t)\|_{\mathcal{H}^s(\mathcal{D})}$ belongs to the space $L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))$ with $\gamma \ge 1+\alpha-\theta$.

Remark 3.2. Let us suppose that $u_d \in L^p(\mathcal{D})$ for $1 \le p < 2$. Then using Lemma 2.1, we find that $u_d \in \mathcal{H}^\sigma(\mathcal{D})$ for $-\frac{N}{4} < \sigma \le \frac{(p-2)N}{4p}$. Let us choose $\sigma = \frac{(p-2)N}{4p}$. Then if $F \in L^\infty\left(d,T,\mathcal{H}^\theta(\mathcal{D})\right)$ for $\theta > \frac{1}{4}\left(N-\frac{2N}{p}-3\right)$ from Theorem 3.1, we can deduce that $u \in L^\infty\left(d,T,\mathcal{H}^{\frac{(p-2)N}{4p}}(\mathcal{D})\right)$.

Proof. By using the Parseval equality, we get

$$\begin{aligned}
\|u(.,t)\|^2_{\mathcal{H}^s(\mathcal{D})} &\le 2\left\|\sum_{j=1}^\infty H_j(t)\langle u_d,e_j\rangle e_j(x)\right\|^2_{\mathcal{H}^s(\mathcal{D})} \\
&+ 2\left\|\sum_{j=1}^\infty\left(\int_d^t L_j(\phi(t,\tau))\langle F(\tau),e_j\rangle g'(\tau)d\tau\right)e_j(x)\right\|^2_{\mathcal{H}^s(\mathcal{D})} \\
&\le 2\sum_{j=1}^\infty \lambda_j^{2s}|H_j(t)|^2\langle u_d,e_j\rangle^2 \\
&+ 2\sum_{j=1}^\infty \lambda_j^{2s}\left(\int_d^t L_j(\phi(t,\tau))\langle F(\tau),e_j\rangle g'(\tau)d\tau\right)^2 \\
&= \partial_1+\partial_2.
\end{aligned}$$

For the term ∂_2, by using the Hölder inequality, we obtain

$$\begin{aligned}
&\lambda_j^{2s}\left(\int_d^t L_j(\phi(t,\tau))\langle F(\tau),e_j\rangle g'(\tau)d\tau\right)^2 \\
&\le \lambda_j^{2s}\left(\int_d^t L_j(\phi(t,\tau))g'(\tau)d\tau\right)\left(\int_d^t L_j(\phi(t,\tau))\langle F(\tau),e_j\rangle^2 g'(\tau)d\tau\right) \\
&\le C_3\lambda_j^{2s-1}\left(\int_d^t L_j(\phi(t,\tau))\langle F(\tau),e_j\rangle^2 g'(\tau)d\tau\right).
\end{aligned} \tag{3.7}$$

It is obvious that

$$\begin{aligned}
&\lambda_j^{2s-1}\left(\int_d^t L_j(\phi(t,\tau))\langle F(\tau),e_j\rangle^2 g'(\tau)d\tau\right) \\
&= \lambda_j^{2s-2-2\theta}\left(\int_d^t \lambda_j L_j(\phi(t,\tau))\lambda_j^{2\theta}\langle F(\tau),e_j\rangle^2 g'(\tau)d\tau\right).
\end{aligned} \tag{3.8}$$

We may deduce from the definition of the space $L^{\infty}\left(d,T,\mathcal{H}^{s-1}\left(\mathcal{D}\right)\right)$, that the function F satisfies the following inequality

$$\|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))} = \underset{d\leq\tau\leq T}{sup}\|F(\tau)\|^2_{\mathcal{H}^{\theta}(\mathcal{D})} \geq \lambda_j^{2\theta}\left\langle F(\tau), e_j\right\rangle^2, \tag{3.9}$$

which allows us to get that

$$\begin{aligned}&\left(\int_d^t \lambda_j L_j(\phi(t,\tau))\lambda_j^{2\theta}\left\langle F(\tau), e_j\right\rangle^2 g'(\tau)d\tau\right)\\ &\leq \|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))}\left(\int_d^t \lambda_j L_j(\phi(t,\tau))g'(\tau)d\tau\right)\\ &\leq C_3\|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))}.\end{aligned} \tag{3.10}$$

Combining (3.7), (3.8), and (3.10), we obtain that

$$\partial_2 \leq 2C_3^2\|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))}\sum_{j=1}^{\infty}\lambda_j^{2s-2-2\theta}. \tag{3.11}$$

It is well-known that $\lambda_j \leq \overline{C_1}j^{2/N}$ with N is the dimensional number of the domain $\mathcal{D}$. As a result, we arrive at the following conclusion $\sum_{j=1}^{\infty}\lambda_j^{2s-2-2\theta} \leq \overline{C}\sum_{j=1}^{\infty}j^{\frac{4s-4-4\theta}{N}}$. It is clear that under the following condition $4+4\theta-4s > N$, this infinite series $\sum_{j=1}^{\infty}j^{\frac{4s-4-4\theta}{N}}$ is convergent. Let us suppose that $\sum_{j=1}^{\infty}j^{\frac{4s-4-4\theta}{N}} = \overline{C_2}(s,\theta,N)$ then we follows from (3.11) that

$$\partial_2 \leq 2\overline{CC_2}(s,\theta,N)C_3^2\|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))}. \tag{3.12}$$

We distinguish two cases for considering the first term ∂_1.

Case 1. Let us suppose that $u_d \in \mathcal{H}^s(\mathcal{D})$. In this case, the term ∂_1 we can bound is as follows

$$\partial_1 = 2\sum_{j=1}^{\infty}\lambda_j^{2s}|H_j(t)|^2\left\langle u_d, e_j\right\rangle^2 \leq 2C_1^2\sum_{j=1}^{\infty}\lambda_j^{2s}\left\langle u_d, e_j\right\rangle^2 = 2C_1^2\|u_d\|^2_{\mathcal{H}^s(\mathcal{D})}. \tag{3.13}$$

Combining (3.12) and (3.13), we find

$$\|u(.,t)\|^2_{\mathcal{H}^s(\mathcal{D})} \leq \partial_1+\partial_2 \leq 2C_1^2\|u_d\|^2_{\mathcal{H}^s(\mathcal{D})} + 2\overline{CC_2}(s,\theta,N)C_3^2\|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))}. \tag{3.14}$$

The right hand side of the above expression is independent of t. As a result, we conclude that $u \in L^{\infty}(d,T,\mathcal{H}^s(\mathcal{D}))$. We also give the following regularity result

$$\|u\|^2_{L^{\infty}(d,T,\mathcal{H}^s(\mathcal{D}))} \leq \partial_1+\partial_2 \leq 2C_1^2\|u_d\|^2_{\mathcal{H}^s(\mathcal{D})} + 2\overline{CC_2}(s,\theta,N)C_3^2\|F\|^2_{L^{\infty}(d,T,\mathcal{H}^{\theta}(\mathcal{D}))}. \tag{3.15}$$

Case 2. Let us suppose that $u_d \in \mathcal{H}^{s-1}(\mathcal{D})$. In this case, we can give for the term ∂_1 the following estimation

$$\partial_1 = 2\sum_{j=1}^{\infty}\lambda_j^{2s}|H_j(t)|^2\left\langle u_d, e_j\right\rangle^2$$

$$\leq 2C_2^2\left(t^{\beta-1}+at^{\beta-\alpha-1}\right)^2\sum_{j=1}^{\infty}\lambda_j^{2s-2}\left\langle u_d,e_j\right\rangle^2$$
$$=2C_2^2\left(t^{\beta-1}+at^{\beta-\alpha-1}\right)^2\left\|u_d\right\|_{\mathcal{H}^{s-1}(\mathcal{D})}^2. \tag{3.16}$$

Combining (3.12) and (3.16), we get

$$\begin{aligned}\left\|u\left(.,t\right)\right\|_{\mathcal{H}^s(\mathcal{D})} &\leq \sqrt{\partial_1}+\sqrt{\partial_2}\\ &\leq \sqrt{2}C_2\left(t^{\beta-1}+at^{\beta-\alpha-1}\right)\left\|u_d\right\|_{\mathcal{H}^{s-1}(\mathcal{D})}\\ &+\sqrt{2\overline{CC_2}\left(s,\theta,N\right)}C_3\left\|F\right\|_{L^\infty(d,T,\mathcal{H}^\theta(\mathcal{D}))}.\end{aligned} \tag{3.17}$$

□

4. Nonlinear time-fractional Oldroyd-B fluid equation

In this section, we consider the following nonlinear problem

$$\begin{cases}\left(1+a\partial_g^\alpha\right)u_t\left(x,t\right)=\mu\left(1+b\partial_g^\beta\right)\Delta u\left(x,t\right)+F\left(u\left(x,t\right)\right),\ x\in\mathcal{D},\ d<t\leq T,\\ u=0,\ \ \left(x,t\right)\in\partial\mathcal{D}\times\left(d,T\right),\\ u\left(x,d\right)=u_d\left(x\right),\ I_g^{1-\alpha}u_t\left(x,d\right)=0,\ x\in\mathcal{D}.\end{cases} \tag{4.1}$$

We can deduce the following result by using a similar technique as in the previous section

$$\begin{aligned}u\left(x,t\right)&=\sum_{j=1}^{\infty}H_j\left(t\right)\left\langle u_d,e_j\right\rangle e_j\left(x\right)\\ &+\sum_{j=1}^{\infty}\left(\int_d^t L_j(\phi\left(t,\tau\right))\left\langle F\left(u\left(\tau\right)\right),e_j\right\rangle g'(\tau)d\tau\right)e_j\left(x\right).\end{aligned} \tag{4.2}$$

Theorem 4.1. *Let the initial datum $u_d\in\mathcal{H}^s\left(\mathcal{D}\right)$. Let F satisfy that $F(0)=0$ and*

$$\left\|F\left(w_1\right)-F\left(w_2\right)\right\|_{\mathcal{H}^\theta(\mathcal{D})}\leq K_f\left\|w_1-w_2\right\|_{\mathcal{H}^s(\mathcal{D})}, \tag{4.3}$$

for K_f is a positive constant. Then if K_f is enough small then problem (4.1) has a unique solution $u\in L^\infty\left(d,T,\mathcal{H}^s\left(\mathcal{D}\right)\right)$.

Proof. Set the following mapping

$$\begin{aligned}\left(Qw\right)\left(t\right)&=\sum_{j=1}^{\infty}H_j\left(t\right)\left\langle u_d,e_j\right\rangle e_j\left(x\right)\\ &+\sum_{j=1}^{\infty}\left(\int_d^t L_j(\phi\left(t,\tau\right))\left\langle F\left(w\left(\tau\right)\right),e_j\right\rangle g'(\tau)d\tau\right)e_j\left(x\right).\end{aligned} \tag{4.4}$$

If $w=0$ then under this condition $F(0)=0$, it is obvious that

$$(Qw)(t)=\sum_{j=1}^{\infty}H_j(t)\langle u_d,e_j\rangle e_j(x).$$

Since according to Lemma 3.1 $|H_j(t)|\le C_1$ and the initial datum $u_d\in\mathcal{H}^s(\mathcal{D})$, it is easily seen that $Qw\in L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))$.

Take any functions $w_1,w_2\in L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))$. And by exploiting (4.4), we find

$$\begin{aligned}&(Qw_1)(t)-(Qw_2)(t)\\&=\sum_{j=1}^{\infty}\left(\int_d^t L_j(\phi(t,\tau))\langle F(w_1(\tau))-F(w_2(\tau)),e_j\rangle g'(\tau)d\tau\right)e_j(x).\end{aligned}\tag{4.5}$$

By performing some calculations on the above expression and using Parseval's equality and Hölder inequality, we arrive at the following result

$$\begin{aligned}&\|(Qw_1)(t)-(Qw_2)(t)\|^2_{\mathcal{H}^s(\mathcal{D})}\\&=\sum_{j=1}^{\infty}\lambda_j^{2s}\left(\int_d^t L_j(\phi(t,\tau))\langle F(w_1(\tau))-F(w_2(\tau)),e_j\rangle g'(\tau)d\tau\right)^2\\&\le\sum_{j=1}^{\infty}\lambda_j^{2s}\left(\int_d^t|L_j(\phi(t,\tau))|g'(\tau)d\tau\right)\\&\times\left(\int_d^t|L_j(\phi(t,\tau))|\langle F(w_1(\tau))-F(w_2(\tau)),e_j\rangle^2 g'(\tau)d\tau\right)\\&\le C_3\sum_{j=1}^{\infty}\lambda_j^{2s-1}\left(\int_d^t|L_j(\phi(t,\tau))|\langle F(w_1(\tau))-F(w_2(\tau)),e_j\rangle^2 g'(\tau)d\tau\right).\end{aligned}\tag{4.6}$$

It is easy to see that

$$\begin{aligned}&\|Qw_1(t)-Qw_2(t)\|^2_{\mathcal{H}^s(\mathcal{D})}\\&\le C_3\sum_{j=1}^{\infty}\lambda_j^{2s-2-2\theta}\left(\int_d^t\lambda_j|L_j(\phi(t,\tau))|\lambda_j^{2\theta}\langle F(w_1(\tau))-F(w_2(\tau)),e_j\rangle^2 g'(\tau)d\tau\right).\end{aligned}\tag{4.7}$$

Let us continue to deal with the integral term on the right hand side of the above expression. By looking at the globally Lipschitz condition of F as in (4.3), we infer that

$$\begin{aligned}&\lambda_j^{2\theta}\langle F(w_1(\tau))-F(w_2(\tau)),e_j\rangle^2\\&\le\|F(w_1(\tau))-F(w_2(\tau))\|^2_{\mathcal{H}^\theta(\mathcal{D})}\\&\le K_f^2\|w_1(\tau)-w_2(\tau)\|^2_{\mathcal{H}^s(\mathcal{D})}\\&\le K_f^2\sup_{d\le\tau\le T}\|w_1(\tau)-w_2(\tau)\|^2_{\mathcal{H}^s(\mathcal{D})}\\&\le K_f^2\|w_1-w_2\|^2_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))}.\end{aligned}\tag{4.8}$$

We may obtain the result of the upper bound of the integral on the right hand side of (4.7), by exploiting the two evaluations (4.7) and (4.8)

$$\begin{aligned}&\int_d^t \lambda_j \left|L_j(\phi(t,\tau))\right| \lambda_j^{2\theta} \left\langle F(w_1(\tau)) - F(w_2(\tau)), e_j\right\rangle^2 g'(\tau) d\tau \\ &\leq K_f^2 \|w_1 - w_2\|^2_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))} \left(\int_d^t \lambda_j \left|L_j(\phi(t,\tau))\right| g'(\tau) d\tau\right). \end{aligned} \tag{4.9}$$

According to some above observations, we can deduce the following

$$\begin{aligned}&\|(Qw_1)(t) - (Qw_2)(t)\|^2_{\mathcal{H}^s(\mathcal{D})} \\ \leq& K_f^2 C_3 \|w_1 - w_2\|^2_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))} \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\theta} \left(\int_d^t \lambda_j \left|L_j(\phi(t,\tau))\right| g'(\tau) d\tau\right) \\ \leq& K_f^2 C_3^2 \|w_1 - w_2\|^2_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))} \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\theta} \\ \leq& K_f^2 C_4 \|w_1 - w_2\|^2_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))}, \end{aligned} \tag{4.10}$$

where we observe that the infinite series $\sum_{j=1}^{\infty} \lambda_j^{2s-2-2\theta}$ is convergent. According to the latter inequality we can get the following result

$$\|Qw_1 - Qw_2\|_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))} \leq \sqrt{K_f^2 C_4} \|w_1 - w_2\|_{L^\infty(d,T,\mathcal{H}^s(\mathcal{D}))}. \tag{4.11}$$

By using Banach fixed point theorem with the following notation $K_f^2 C_4 < 1$, if K_f^2 is small enough, we deduce that Q has a fixed point $u \in L^\infty(d, T, \mathcal{H}^s(\mathcal{D}))$. □

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