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Local Existence of Strong Solutions to the Generalized MHD Equations*

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Abstract This paper devotes to consider the local existence of the strong solutions to the generalized MHD system with fractional dissipative terms $\Lambda^{2\alpha}u$ for the velocity field and $\Lambda^{2\alpha}b$ for the magnetic field, respectively. We construct the approximate solutions by the Fourier truncation method, and use energy method to obtain the local existence of strong solutions in $H^s(\mathbb{R}^n)$ $(s > \max\left\{\frac{n}{2} + 1 - 2\alpha, 0\right\})$ for any $\alpha \geq 0$.

Keywords Generalized MHD system, local existence, Fourier truncation

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1. Introduction

In this paper, we consider the Cauchy problem of the following $n-$dimensional $(n \geq 2)$ generalized MHD (GMHD) equations:

$$u_t + u \cdot \nabla u + \nabla \pi + \Lambda^{2\alpha} u - b \cdot \nabla b = 0, \, (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.1}$$

$$b_t + u \cdot \nabla b + \Lambda^{2\beta} b - b \cdot \nabla u = 0, \, (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.2}$$

$$\mathrm{div} u = 0, \mathrm{div} b = 0, \, (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1.3}$$

$$(u,b)(x,0) = (u_0, b_0), \qquad x \in \mathbb{R}^n, \tag{1.4}$$

where $u = u(x,t) \in \mathbb{R}^n, b = b(x,t) \in \mathbb{R}^n$ and $\pi = \pi(x,t) \in \mathbb{R}$ denote the velocity field, magnetic field and scalar pressure respectively. $\alpha \geq 0$ is a real parameter. The fractional Laplacian operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ is defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

where the Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx.$$

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For the local existence results related to our equations, when $\alpha = 1,\ \beta = 0$, Fefferman et al. [1] established the local-in-time existence and uniqueness of strong solutions in $H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}\ (n = 2,\ 3)$. When $\alpha \geq 0,\ \beta > 0$, Wu [2] proved that the system has a unique local solution in $H^s(\mathbb{R}^n),\ s > max\{2, \beta\} + \frac{n}{2}$. When $\alpha = \beta \in (0,\ \frac{3}{2})$, Yuan [3] obtained the local existence of solution in $H^s(\mathbb{R}^3),\ s > \frac{5}{2} - 2\alpha$. For generalized $\alpha,\ \beta \geq 0$, Jiang and Zhou [4] proved the local existence results in $H^s(\mathbb{R}^n)$ with $s > max\{\frac{n}{2} + 1 - \alpha,\ 1\}$.

For the global existence results related to our equations, when $\alpha \geq \frac{1}{2} + \frac{n}{4},\ \alpha + \beta \geq 1 + \frac{n}{2}$, Zhou [5] established the global existence of solutions. When $\frac{1}{2} < \alpha,\ \beta \leq 1$, Ye [6] proved the global existence of mild solutions and small solutions in Fourier-Herz space. When $0 < \alpha < \frac{1}{2},\ \beta > 1,\ 3\alpha + 2\beta > 3$, Cheng [7] showed the existence of global regular solutions for logarithmically supercritical $2-$dimensional GMHD equations. Zhao [8] established the decay results for $\alpha,\ \beta \in (0, 2]$ when $u_0,\ b_0 \in L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)\ (p > 1)$. Some regularity criteria were studied for the GMHD system in [9–11].

Now, we introduce some notations that will be used in our paper. $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^n)$ norm. $\|u\|_{H^s(\mathbb{R}^n)}$ and $\|u\|_{\dot{H}^s(\mathbb{R}^n)}$ denote the norm of u in the non-homogeneous Sobolev spaces $H^s(\mathbb{R}^n)$ and homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ respectively. C denotes a generic positive constant which may vary from line to line.

In this paper, we will consider the case $\alpha = \beta$ of GMHD system and establish the local existence of strong solutions. The main result of our paper is given by the following Theorem.

Theorem 1.1. *For any $\alpha \geq 0$, if $u_0,\ b_0 \in H^s(\mathbb{R}^n)$ with $s > \max\left\{\frac{n}{2} + 1 - 2\alpha,\ 0\right\}$, $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$, then there exists a positive time T_* and a unique solution $(u,\ b)$ to equations (1.1)-(1.4) on $[0,\ T_*]$ such that*

$$u, b \in L^\infty([0, T_*];\ H^s(\mathbb{R}^n)) \cap L^2([0, T_*];\ H^{s+\alpha}(\mathbb{R}^n)).$$

Moreovre, we could obtain $u, b \in C_w([0, T_*];\ H^s(\mathbb{R}^n))$.

This paper is organized as follows. In Section 2, we introduce some preliminary lemmas that will be used frequently in the proof of our main results, and establish the energy estimates for the solutions of (1.1)-(1.4). In Section 3, we construct the approximate solutions, and give the proof of Theorem 1.1.

2. Preliminaries and energy estimates

In this section, we recall some elementary lemmas and give energy estimates for smooth solutions of (1.1)-(1.4), which are crucial in the proof of Theorem 1.1.

2.1. Preliminaries

Lemma 2.1. *[1] Define the Fourier truncation S_R as follows:*

$$\widehat{S_R f}(\xi) = 1_{B_R(\xi)}\hat{f}(\xi) = \begin{cases} \hat{f}(\xi), & |\xi| \leq R, \\ 0, & |\xi| > R, \end{cases}$$

which satisfies

$$\|S_R f - f\|_{H^s} \le C\frac{1}{R^k}\|f\|_{H^{s+k}}, \tag{2.1}$$

$$\|S_R f - S_{R'} f\|_{H^s} \le C\max\{\frac{1}{R^k}, \frac{1}{R'^k}\}\|f\|_{H^{s+k}}. \tag{2.2}$$

The following inequalities are used frequently in our estimates.

Lemma 2.2. *[12, 13] (Gagliardo-Nirenberg inequality). Let $u \in L^q(\mathbb{R}^n)$ and its derivatives of order m, $D^m u \in L^r$, $1 \le q$ and $r \le \infty$. For the derivatives $D^j u$, $0 \le j < m$, the following inequalities hold,*

$$\|D^j u\|_p \le C\|D^m u\|_r^{\alpha}\|u\|_q^{1-\alpha}, \tag{2.3}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q},$$

for all α in the interval $\frac{j}{m} \le \alpha \le 1$.

Lemma 2.3. *[14, 15] (Kato-Ponce inequality) Let $s > 0$, $p \in (1, \infty)$. Assume that $f \in W^{1,p_1} \cap W^{s,q_2}$ and $g \in L^{p_2} \cap W^{s,q_1}$. Then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_p \le C\left(\|\nabla f\|_{p_1}\left\|\Lambda^{s-1}g\right\|_{q_1} + \|g\|_{p_2}\|\Lambda^s f\|_{q_2}\right), \tag{2.4}$$

and

$$\|\Lambda^s(fg)\|_p \le C\left(\|f\|_{p_1}\|\Lambda^s g\|_{q_1} + \|g\|_{p_2}\|\Lambda^s f\|_{q_2}\right), \tag{2.5}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

2.2. Energy estimates

Proposition 2.1. *For any $\alpha \ge 0$, if the initial data $u_0, b_0 \in H^s(\mathbb{R}^n)$ with $s > \max\left\{\frac{n}{2} + 1 - 2\alpha,\ 0\right\}$, $\mathrm{div} u_0 = \mathrm{div} b_0 = 0$, then there exists a time $T_* = T_*(\|u_0\|_{H^s}$, $\|b_0\|_{H^s}) > 0$ such that*

$$\sup_{t\in[0,T_*]}\left[\|u\|_{H^s(\mathbb{R}^n)}^2(t) + \|b\|_{H^s(\mathbb{R}^n)}^2(t)\right] \le C(\alpha,\ T_*,\ \|u_0\|_{H^s},\ \|b_0\|_{H^s}),$$

and

$$\int_0^{T_*}\left[\|u\|_{H^{s+\alpha}(\mathbb{R}^n)}^2(t) + \|b\|_{H^{s+\alpha}(\mathbb{R}^n)}^2(t)\right]dt \le C(\alpha,\ T_*,\ \|u_0\|_{H^s},\ \|b_0\|_{H^s}).$$

Proof. Multiplying (1.1) and (1.2) by u and b, respectively, after integrating by parts and taking the divergence-free property into account, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_2^2 + \|b\|_2^2\right) + \|\Lambda^\alpha u\|_2^2 + \|\Lambda^\alpha b\|_2^2 = 0. \tag{2.6}$$

Taking the inner product of (1.1) and (1.2) with $\Lambda^{2s}u$ and $\Lambda^{2s}b$, respectively, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2+\|b\|_{H^s}^2\right)+\left(\|u\|_{\dot H^{s+\alpha}}^2+\|b\|_{\dot H^{s+\alpha}}^2+\|u\|_{\dot H^{\alpha}}^2+\|b\|_{\dot H^{\alpha}}^2\right)\\
&=\langle\Lambda^s\left[(b\cdot\nabla)\,b\right],\Lambda^s u\rangle+\langle\Lambda^s\left[(b\cdot\nabla)\,u\right],\Lambda^s b\rangle\\
&\quad-\langle\Lambda^s\left[(u\cdot\nabla)\,u\right],\Lambda^s u\rangle-\langle\Lambda^s\left[(u\cdot\nabla)\,b\right],\Lambda^s b\rangle\\
&\triangleq I_1+I_2+I_3+I_4.
\end{aligned}
\tag{2.7}
$$

The estimates of $I_i(i=1,2,3,4)$ are split up into the following two cases:
Case 1. When $\alpha<\frac{n}{4}+\frac{1}{2}$, using the Lemma 2.2, Lemma 2.3, Hölder and Young inequalities, we have

$$
\begin{aligned}
|I_1|+|I_2|&\le\langle\Lambda^s\left(b\cdot\nabla b\right)-b\cdot\nabla\Lambda^s b,\Lambda^s u\rangle+\langle\Lambda^s\left(b\cdot\nabla u\right)-b\cdot\nabla\Lambda^s u,\Lambda^s b\rangle\\
&\le\|\Lambda^s\left(b\cdot\nabla b\right)-b\cdot\nabla\Lambda^s b\|_2\|\Lambda^s u\|_2+\|\Lambda^s\left(b\cdot\nabla u\right)-b\cdot\nabla\Lambda^s u\|_2\|\Lambda^s b\|_2\\
&\le C\|\nabla b\|_{\frac{n}{\alpha}}\|\Lambda^s b\|_{\frac{2n}{n-2\alpha}}\|\Lambda^s u\|_2+C\|\nabla b\|_{\frac{n}{\alpha}}\|\Lambda^s u\|_{\frac{2n}{n-2\alpha}}\|\Lambda^s b\|_2\\
&\quad+C\|\nabla u\|_{\frac{n}{\alpha}}\|\Lambda^s b\|_{\frac{2n}{n-2\alpha}}\|\Lambda^s b\|_2\\
&\le C\|b\|_2^{1-\theta}\|b\|_{H^{s+\alpha}}^{1+\theta}\|u\|_{H^s}+C\|b\|_2^{1-\theta}\|b\|_{H^{s+\alpha}}^{\theta}\|u\|_{H^{s+\alpha}}\|b\|_{H^s}\\
&\quad+C\|u\|_2^{1-\theta}\|u\|_{H^{s+\alpha}}^{\theta}\|b\|_{H^{s+\alpha}}\|b\|_{H^s}\\
&\le\frac{1}{6}\|u\|_{H^{s+\alpha}}^2+\frac{1}{4}\|b\|_{H^{s+\alpha}}^2+C\|u\|_{H^s}^{\frac{2}{1-\theta}}+C\|b\|_{H^s}^{\frac{2}{1-\theta}},
\end{aligned}
$$

where $\theta=\frac{2(1-\alpha)+n}{2(s+\alpha)}$, which implies $s>\frac{n}{2}+1-2\alpha$.
Case 2. When $\alpha\ge\frac{n}{4}+\frac{1}{2}$,

$$
\begin{aligned}
|I_1|+|I_2|&\le\|\Lambda^s\left(b\cdot\nabla b\right)-b\cdot\nabla\Lambda^s b\|_2\|\Lambda^s u\|_2+\|\Lambda^s\left(b\cdot\nabla u\right)-b\cdot\nabla\Lambda^s u\|_2\|\Lambda^s b\|_2\\
&\le C\|\nabla b\|_{\frac{4n}{n+2}}\|\Lambda^s b\|_{\frac{4n}{n-2}}\|\Lambda^s u\|_2+C\|\nabla b\|_{\frac{4n}{n+2}}\|\Lambda^s u\|_{\frac{4n}{n-2}}\|\Lambda^s b\|_2\\
&\quad+C\|\nabla u\|_{\frac{4n}{n+2}}\|\Lambda^s b\|_{\frac{4n}{n-2}}\|\Lambda^s b\|_2\\
&\le C\|b\|_2^{1-\theta}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}b\|_2^{1+\theta}\|\Lambda^s u\|_2\\
&\quad+C\|b\|_2^{1-\theta}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}b\|_2^{\theta}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}u\|_2\|\Lambda^s b\|_2\\
&\quad+C\|u\|_2^{1-\theta}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}u\|_2^{\theta}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}b\|_2\|\Lambda^s b\|_2\\
&\le\frac{1}{6}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}u\|_2^2+\frac{1}{4}\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}b\|_2^2+C\|\Lambda^s u\|_2^{\frac{2}{1-\theta}}+C\|\Lambda^s b\|_2^{\frac{2}{1-\theta}}\\
&\le\frac{1}{6}\|u\|_{H^{s+\alpha}}^2+\frac{1}{4}\|b\|_{H^{s+\alpha}}^2+C\|u\|_{H^s}^{\frac{2}{1-\theta}}+C\|b\|_{H^s}^{\frac{2}{1-\theta}},
\end{aligned}
$$

where $\theta=\frac{n+2}{4(s+\frac{n}{4}+\frac{1}{2})}$.

Similarly, we obtain

$$
\begin{aligned}
|I_3|&\le|\langle\Lambda^s\left(u\cdot\nabla u\right)-u\cdot\nabla\Lambda^s u,\Lambda^s u\rangle|\\
&\le\|\Lambda^s\left(u\cdot\nabla u\right)-u\cdot\nabla\Lambda^s u\|_2\|\Lambda^s u\|_2\\
&\le C\|u\|_2^{1-\theta}\|u\|_{H^{s+\alpha}}^{1+\theta}\|u\|_{H^s}\\
&\le\frac{1}{6}\|u\|_{H^{s+\alpha}}^2+C\|u\|_{H^s}^{\frac{2}{1-\theta}}.\\
|I_4|&\le|\langle\Lambda^s\left(u\cdot\nabla b\right)-u\cdot\nabla\Lambda^s b,\Lambda^s b\rangle|
\end{aligned}
$$

$$\begin{aligned}
&\leq \|\Lambda^s (u\cdot\nabla b) - u\cdot\nabla\Lambda^s b\|_2 \|\Lambda^s b\|_2 \\
&\leq C\|u\|_2^{1-\theta}\|u\|_{H^{s+\alpha}}^{\theta}\|b\|_{H^{s+\alpha}}\|b\|_{H^s} + C\|b\|_2^{1-\theta}\|b\|_{H^{s+\alpha}}^{\theta}\|u\|_{H^{s+\alpha}}\|b\|_{H^s} \\
&\leq \frac{1}{6}\|u\|_{H^{s+\alpha}}^2 + \frac{1}{4}\|b\|_{H^{s+\alpha}}^2 + C\|b\|_{H^s}^{\frac{2}{1-\theta}}.
\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2 + 1\right) + \left(\|u\|_{H^{s+\alpha}}^2 + \|b\|_{H^{s+\alpha}}^2\right) \\
\leq\ & C\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2 + 1\right)^N,
\end{aligned}$$

where $N = \frac{1}{1-\theta} > 1$.

By using Gronwall inequality, we obtain

$$\sup_{t\in[0,T_*]}(\|u\|_{H^s}^2 + \|b\|_{H^s}^2) \leq \frac{\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 + 1}{\{1 - C(N-1)[\|u_0\|_{H^s}^2 + \|b_0\|_{H^s}^2 + 1]^{N-1}t\}^{\frac{1}{N-1}}} - 1.$$

□

3. Local existence and uniqueness

Now we consider the following truncated MHD equations:

$$u_t^R = -\Lambda^{2\alpha}u^R - \nabla\pi^R - S_R\left[\left(u^R\cdot\nabla\right)u^R\right] + S_R\left[\left(b^R\cdot\nabla\right)b^R\right], \tag{3.1}$$

$$b_t^R = -\Lambda^{2\beta}b^R - S_R\left[\left(u^R\cdot\nabla\right)b^R\right] + S_R\left[\left(b^R\cdot\nabla\right)u^R\right], \tag{3.2}$$

$$\mathrm{div}u^R = 0, \mathrm{div}b^R = 0, \tag{3.3}$$

$$\left(u^R, b^R\right)(x,0) = \left(S_R u_0, S_R b_0\right). \tag{3.4}$$

Let $X^R = (u^R, b^R)^T$, $X_1^R = (u_1^R, b_1^R)^T$ and $X_2^R = (u_2^R, b_2^R)^T$, the truncated MHD equations (3.1)-(3.4) could be reformulated as

$$\begin{aligned}
&\frac{dX^R}{dt} = F(X^R), \\
&\left(u^R, b^R\right)(x,0) = \left(S_R u_0, S_R b_0\right) = X_0^T,
\end{aligned}$$

where

$$F(X^R) = \begin{pmatrix} -\Lambda^{2\alpha}u^R - PS_R\left[\left(u^R\cdot\nabla\right)u^R\right] + PS_R\left[\left(b^R\cdot\nabla\right)b^R\right] \\ -\Lambda^{2\alpha}b^R - PS_R\left[\left(u^R\cdot\nabla\right)b^R\right] + PS_R\left[\left(b^R,\nabla\right)u^R\right] \end{pmatrix}, \tag{3.5}$$

and P is the Leray projection operator which projects functions onto the space of divergence-free functions. Taking $V^s = \left\{f \in H^s(\mathbb{R}^n) : \mathrm{div}f = 0,\ \mathrm{supp}\hat{f} \subset B_R\right\}$, by using (2.1), (2.2) and the fact of $\|S_R f\|_{H^s} \leq CR^s\|f\|_2$, we have

$$\left\|F\left(X_1^R\right) - F\left(X_2^R\right)\right\|_{H^s} \leq C\left(\|X_0\|_2,\ R,\ n\right)\left\|X_1^R - X_2^R\right\|_{H^s}. \tag{3.6}$$

Thus, F is locally Lipschitz continuous on any open set $V^s \times V^s$. So, the Picard's theorem implies that, given any initial condition $X_0^R \in V^s \times V^s$, there exists a unique solution $X^R \in C^1([0,T_R);\ V^s) \times C^1([0,T_R);\ V^s)$ for some $T_R > 0$.

By using the similar method that is used in the proof of Proposition 2.1 and the fact $\|(u_0^R,\ b_0^R)\|_{H^s} \le \|(u_0,\ b_0)\|_{H^s}$, we obtain the following uniform estimates

$$\left\|u^R\right\|^2_{H^s(\mathbb{R}^n)}(t) + \left\|b^R\right\|^2_{H^s(\mathbb{R}^n)}(t) \le C(\alpha,\ T_*,\ \|u_0\|_{H^s},\ \|b_0\|_{H^s}), \tag{3.7}$$

and

$$\int_0^{T_*}\left[\left\|u^R\right\|^2_{H^{s+\alpha}(\mathbb{R}^n)} + \left\|b^R\right\|^2_{H^{s+\alpha}(\mathbb{R}^n)}\right](t)dt \le C\left(\alpha,\ T_*,\ \|u_0\|_{H^s},\ \|b_0\|_{H^s}\right). \tag{3.8}$$

In order to establish the continuity of u, b on the interval $[0,T_*]$ with values in the weak topology of $H^s(\mathbb{R}^n)$), we need the strong convergence of the sequence $\left(u^R,b^R\right)$ in $L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)\times L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)$.

Proposition 3.1. *For any $\alpha \ge 0$, $s > \max\left\{\frac{n}{2}+1-2\alpha,0\right\}$, if the initial data $u_0,\ b_0 \in H^s\left(\mathbb{R}^n\right)$ with $\mathrm{div}u_0 = \mathrm{div}b_0 = 0$, and $\left(u^R,b^R\right)$ is the solution of* (3.1)-(3.4)*, then it is a Cauchy sequence (as $R\to\infty$) in $L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)\times L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)$.*

Proof. Without loss of generality, we assume that $R' > R \ge 1$. Taking the difference between the equations (3.1)-(3.2) for R and R', we have

$$\begin{aligned}\left(u^R-u^{R'}\right)_t =& -\Lambda^{2\alpha}\left(u^R-u^{R'}\right)-\nabla\left(\pi^R-\pi^{R'}\right)-S_R\left[\left(u^R\cdot\nabla\right)u^R-\left(b^R\cdot\nabla\right)b^R\right]\\ &+S_{R'}\left[\left(u^{R'}\cdot\nabla\right)u^{R'}-\left(b^{R'}\cdot\nabla\right)b^{R'}\right].\end{aligned} \tag{3.9}$$

$$\begin{aligned}\left(b^R-b^{R'}\right)_t =& -\Lambda^{2\beta}\left(b^R-b^{R'}\right)-S_R\left[\left(u^R\cdot\nabla\right)b^R-\left(b^R\cdot\nabla\right)u^R\right]\\ &+S_{R'}\left[\left(u^{R'}\cdot\nabla\right)b^{R'}-\left(b^{R'}\cdot\nabla\right)u^{R'}\right].\end{aligned} \tag{3.10}$$

Taking the inner product of (3.9)- (3.10) with $u^R-u^{R'}$, $b^R-b^{R'}$ respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|u^R-u^{R'}\right\|_2^2+\left\|b^R-b^{R'}\right\|_2^2\right)+\left\|\Lambda^\alpha\left(u^R-u^{R'}\right)\right\|_2^2+\left\|\Lambda^\alpha\left(b^R-b^{R'}\right)\right\|_2^2\\ =&\left\langle S_R\left[\left(b^R\cdot\nabla\right)b^R\right]-S_{R'}\left[\left(b^{R'}\cdot\nabla\right)b^{R'}\right],u^R-u^{R'}\right\rangle\\ &-\left\langle S_R\left[\left(u^R\cdot\nabla\right)u^R\right]-S_{R'}\left[\left(u^{R'}\cdot\nabla\right)u^{R'}\right],u^R-u^{R'}\right\rangle\\ &+\left\langle S_R\left[\left(b^R\cdot\nabla\right)u^R\right]-S_{R'}\left[\left(b^{R'}\cdot\nabla\right)u^{R'}\right],b^R-b^{R'}\right\rangle\\ &-\left\langle S_R\left[\left(u^R\cdot\nabla\right)b^R\right]-S_{R'}\left[\left(u^{R'}\cdot\nabla\right)b^{R'}\right],b^R-b^{R'}\right\rangle\\ \triangleq&\sum_{j=1}^4 I_j.\end{aligned}$$

For I_4, we have

$$\begin{aligned}I_4 &= -\left\langle S_R\left[\left(u^R\cdot\nabla\right)b^R\right]-S_{R'}\left[\left(u^{R'}\cdot\nabla\right)b^{R'}\right],b^R-b^{R'}\right\rangle\\ &= -\left\langle\left(S_R-S_{R'}\right)\left[\left(u^R\cdot\nabla\right)b^R\right],b^R-b^{R'}\right\rangle\end{aligned}$$

$$
\begin{aligned}
&-\left\langle S_{R'}\left[\left(\left(u^R-u^{R'}\right)\cdot\nabla\right)b^R\right],b^R-b^{R'}\right\rangle\\
&-\left\langle S_{R'}\left[\left(u^{R'}\cdot\nabla\right)\left(b^R-b^{R'}\right)\right],b^R-b^{R'}\right\rangle\\
\triangleq&\sum_{j=1}^{3}I_{4j}.
\end{aligned}
$$

For the first term I_{41},

$$
\begin{aligned}
|I_{41}|&=\left\langle\left(S_R-S_{R'}\right)\left[\left(u^R\cdot\nabla\right)b^R\right],b^R-b^{R'}\right\rangle\\
&\le\frac{1}{R^s}\left\|b^R-b^{R'}\right\|_{H^s}\left\|\left(u^R\cdot\nabla\right)b^R\right\|_2\\
&\le\frac{1}{R^s}\left(\|b^R\|_{H^s}+\|b^{R'}\|_{H^s}\right)\left\|\left(u^R\cdot\nabla\right)b^R\right\|_2\\
&\le\frac{C}{R^s}\left(\|u^R\|_{H^\alpha}^2+\|b^R\|_{H^{s+\alpha}}^2\right).
\end{aligned}
$$

The estimate of $\left\|\left(u^R\cdot\nabla\right)b^R\right\|_2$ is split up into the following two cases:
When $\alpha<\frac{n}{4}+\frac{1}{2}$, using (2.3),

$$
\begin{aligned}
\left\|\left(u^R\cdot\nabla\right)b^R\right\|_2&\le\left\|u^R\right\|_{\frac{2n}{n-2\alpha}}\left\|\nabla b^R\right\|_{\frac{n}{\alpha}}\\
&\le\left\|\Lambda^\alpha u^R\right\|_2\left\|b^R\right\|_2^{1-\theta}\left\|\Lambda^{s+\alpha}b^R\right\|_2^{\theta},
\end{aligned}
$$

where $\theta=\frac{n+2(1-\alpha)}{2(s+\alpha)}$ implies $s>\frac{n}{2}+1-2\alpha$.
When $\alpha\ge\frac{n}{4}+\frac{1}{2}$,

$$
\begin{aligned}
\left\|\left(u^R\cdot\nabla\right)b^R\right\|_2&\le\left\|u^R\right\|_{\frac{4n}{n-2}}\left\|\nabla b^R\right\|_{\frac{4n}{n+2}}\\
&\le\left\|\Lambda^{\frac{n}{4}+\frac{1}{2}}u^R\right\|_2\left\|b^R\right\|_2^{1-\theta}\left\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}b^R\right\|_2^{\theta}\\
&\le\left\|\Lambda^\alpha u^R\right\|_2\left\|b^R\right\|_2^{1-\theta}\left\|\Lambda^{s+\alpha}b^R\right\|_2^{\theta},
\end{aligned}
$$

where $\theta=\frac{n+2}{4(s+\frac{n}{4}+\frac{1}{2})}$.

For the second term I_{42},

$$
\begin{aligned}
|I_{42}|&=\left\langle S_{R'}\left[\left(\left(u^R-u^{R'}\right)\cdot\nabla\right)b^R\right],b^R-b^{R'}\right\rangle\\
&\le\left\|\left(\left(u^R-u^{R'}\right)\cdot\nabla\right)b^R\right\|_2\left\|b^R-b^{R'}\right\|_2\\
&\le\frac{1}{4}\left\|u^R-u^{R'}\right\|_{H^\alpha}^2+C\left\|b^R\right\|_{H^{s+\alpha}}^2\left\|b^R-b^{R'}\right\|_2^2.
\end{aligned}
$$

The estimate of $\left\|\left(\left(u^R-u^{R'}\right)\cdot\nabla\right)b^R\right\|_2$ is as follows:
When $\alpha<\frac{n}{4}+\frac{1}{2}$, using (2.3),

$$
\begin{aligned}
\left\|\left(\left(u^R-u^{R'}\right)\cdot\nabla\right)b^R\right\|_2&\le\left\|u^R-u^{R'}\right\|_{\frac{2n}{n-2\alpha}}\left\|\nabla b^R\right\|_{\frac{n}{\alpha}}\\
&\le\left\|u^R-u^{R'}\right\|_{H^\alpha}\left\|b^R\right\|_2^{1-\theta}\left\|\Lambda^{s+\alpha}b^R\right\|_2^{\theta},
\end{aligned}
$$

where $\theta=\frac{n-2\alpha+2}{2(s+\alpha)}$, which implies $s>\frac{n}{2}+1-2\alpha$.
When $\alpha\geq\frac{n}{4}+\frac{1}{2}$,

$$\begin{aligned}\left\|\left(\left(u^R-u^{R'}\right)\cdot\nabla\right)b^R\right\|_2&\leq\left\|u^R-u^{R'}\right\|_{\frac{4n}{n-2}}\left\|\nabla b^R\right\|_{\frac{4n}{n+2}}\\&\leq\left\|\Lambda^{\frac{n}{4}+\frac{1}{2}}\left(u^R-u^{R'}\right)\right\|_2\left\|b^R\right\|_2^{1-\theta}\left\|\Lambda^{s+\frac{n}{4}+\frac{1}{2}}b^R\right\|_2^{\theta}\\&\leq C\left\|u^R-u^{R'}\right\|_{H^{\frac{n}{4}+\frac{1}{2}}}\left\|b^R\right\|^{\theta}_{H^{s+\frac{n}{4}+\frac{1}{2}}}\\&\leq C\left\|u^R-u^{R'}\right\|_{H^\alpha}\left\|b^R\right\|^{\theta}_{H^{s+\alpha}},\end{aligned}$$

where $\theta=\frac{n+2}{4(s+\frac{n}{4}+\frac{1}{2})}$, which implies $s>0$.

Integrating by parts and using the divergence-free condition, we have $I_{43}=0$.

Similarly, we have an estimate for I_2, which is

$$I_2\leq\frac{C}{R^s}\|u^R\|^2_{H^\alpha}+\frac{1}{4}\left\|u^R-u^{R'}\right\|^2_{H^\alpha}+C\|u^R\|^2_{H^{s+\alpha}}\left\|u^R-u^{R'}\right\|_2^2.$$

Now we estimate the other terms

$$\begin{aligned}&I_1+I_3\\=&\left\langle S_R\left[\left(b^R\cdot\nabla\right)b^R\right]-S_{R'}\left[\left(b^{R'}\cdot\nabla\right)b^{R'}\right],u^R-u^{R'}\right\rangle\\+&\left\langle S_R\left[\left(b^R\cdot\nabla\right)u^R\right]-S_{R'}\left[\left(b^{R'}\cdot\nabla\right)u^{R'}\right],b^R-b^{R'}\right\rangle\\=&\left\langle\left(S_R-S_{R'}\right)\left[\left(b^R\cdot\nabla\right)b^R\right],u^R-u^{R'}\right\rangle+\left\langle S_{R'}\left[\left(\left(b^R-b^{R'}\right)\cdot\nabla\right)b^R\right],u^R-u^{R'}\right\rangle\\+&\left\langle\left(S_R-S_{R'}\right)\left[\left(b^R\cdot\nabla\right)u^R\right],b^R-b^{R'}\right\rangle+\left\langle S_{R'}\left[\left(\left(b^R-b^{R'}\right)\cdot\nabla\right)u^R\right],b^R-b^{R'}\right\rangle\\\triangleq&I_{11}+I_{12}+I_{31}+I_{32}.\end{aligned}$$

Using the same method as is used in the estimates of I_{41}, I_{11} and I_{31} can be estimated as follows.

$$\begin{aligned}I_{11}&=\left\langle\left(S_R-S_{R'}\right)\left[\left(b^R\cdot\nabla\right)b^R\right],u^R-u^{R'}\right\rangle\\&\leq\frac{1}{R^s}\left\|u^R-u^{R'}\right\|_{H^s}\left\|\left(b^R\cdot\nabla\right)b^R\right\|_2\\&\leq\frac{1}{R^s}\left(\|u^R\|_{H^s}+\|u^{R'}\|_{H^s}\right)\left\|\left(b^R\cdot\nabla\right)b^R\right\|_2\\&\leq\frac{C}{R^s}\left(\|b^R\|^2_{H^\alpha}+\|b^R\|^2_{H^{s+\alpha}}\right).\end{aligned}$$

$$\begin{aligned}I_{31}&=\left\langle\left(S_R-S_{R'}\right)\left[\left(b^R\cdot\nabla\right)u^R\right],b^R-b^{R'}\right\rangle\\&\leq\frac{1}{R^s}\left\|b^R-b^{R'}\right\|_{H^s}\left\|\left(b^R\cdot\nabla\right)u^R\right\|_2\\&\leq\frac{1}{R^s}\left(\|b^R\|_{H^s}+\|b^{R'}\|_{H^s}\right)\left\|\left(b^R\cdot\nabla\right)u^R\right\|_2\\&\leq\frac{C}{R^s}\left(\|b^R\|^2_{H^\alpha}+\|u^R\|^2_{H^{s+\alpha}}\right).\end{aligned}$$

Using the same method as is used in the estimates of I_{42}, we have

$$
\begin{aligned}
I_{12} &= \left\langle S_{R'}\left[\left((b^R - b^{R'})\cdot\nabla\right) b^R\right], u^R - u^{R'}\right\rangle \\
&\leq \left\|\left(\left(b^R - b^{R'}\right)\cdot\nabla\right) b^R\right\|_2 \left\|u^R - u^{R'}\right\|_2 \\
&\leq \frac{1}{4}\left\|b^R - b^{R'}\right\|_{H^\alpha}^2 + C\left\|b^R\right\|_{H^{s+\alpha}}^2 \left\|u^R - u^{R'}\right\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
I_{32} &= \left\langle S_{R'}\left[\left(\left(b^R - b^{R'}\right)\cdot\nabla\right) u^R\right], b^R - b^{R'}\right\rangle \\
&\leq \left\|\left(\left(b^R - b^{R'}\right)\cdot\nabla\right) u^R\right\|_2 \left\|b^R - b^{R'}\right\|_2 \\
&\leq \frac{1}{4}\left\|b^R - b^{R'}\right\|_{H^\alpha}^2 + C\left\|u^R\right\|_{H^{s+\alpha}}^2 \left\|b^R - b^{R'}\right\|_2^2.
\end{aligned}
$$

Summing up the above estimates and setting $Y(t) = \left\|u^R - u^{R'}\right\|_2^2 + \left\|b^R - b^{R'}\right\|_2^2$, we have

$$
\begin{aligned}
&\frac{dY(t)}{dt} + \left\|u^R - u^{R'}\right\|_{H^\alpha}^2 + \left\|b^R - b^{R'}\right\|_{H^\alpha}^2 \\
\leq &\frac{C}{R^s}(\left\|u^R\right\|_{H^\alpha}^2 + \left\|b^R\right\|_{H^\alpha}^2 + \left\|u^R\right\|_{H^{s+\alpha}}^2 + \left\|b^R\right\|_{H^{s+\alpha}}^2) \\
+ &C\left(\left\|u^R\right\|_{H^{s+\alpha}}^2 + \left\|b^R\right\|_{H^{s+\alpha}}^2 + 1\right) Y(t).
\end{aligned}
$$

By using the uniform estimates (3.7), (3.8) and the Gronwall's inequality, we have

$$
\sup_{t\in[0,T_*]} Y(t) \leq \frac{C}{R^s},
$$

where C depends on the parameter α and the time T_*. □

Now, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Using the Banach-Alaoglu theorem, we can extract a sequence that converges weakly to $u,\ b \in L^2(0,T_*;H^s(\mathbb{R}^n))$. Moreover, for each $t\in[0,T_*]$, the subsequence is uniformly bounded in $H^s(\mathbb{R}^n)$, so it also has a subsequence that converges weakly in $L^\infty(0,T_*;H^s(\mathbb{R}^n))$. Hence, the limit $u,\ b \in L^\infty(0,T_*;H^s(\mathbb{R}^n))$.

From Propositon 2.1 and 3.1, $u^R,\ b^R$ converge strongly in $L^\infty(0,T_*;\ L^2(\mathbb{R}^n))$. By interpolation, we obtain $u^R,\ b^R \to u,\ b$ strongly in $L^\infty(0,T_*;H^{s'}(\mathbb{R}^n))$ for any $0<s'<s$. By using the standard argument in Majda and Bertozzi [16](proof of Theorem 3.4), we have $u,b\in C_w([0,T_*];H^s(\mathbb{R}^n))$. □

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