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Optimal Decay Rates for the Highest-Order Derivatives of Solutions for the Compressible MHD Equations with Coulomb Force*

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Abstract For the Cauchy problem of the 3D compressible MHD equations with Coulomb force, the large time behavior of this model is further investigated in this article. Compared to the previous related works in Tan-Tong-Wang [*J. Math. Anal. Appl.* 427 (2015) 600–617], the main novelty of this paper is that we prove the optimal decay rates for the highest-order spatial derivatives of the solutions to the compressible MHD equations with Coulomb force, which are the same as those of the heat equation.

Keywords MHD equations, highest-order derivatives, optimal decay rates

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1. Introduction

Considering the initial value problem of 3D isentropic MHD equations with Coulomb force for viscous compressible fluid:

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ (\rho v)_t + \operatorname{div}(\rho v \otimes v) + \nabla P = curl H \times H + \mu \Delta v + (\lambda + \mu) \nabla \operatorname{div} v + \rho \nabla \Phi, \\ H_t - curl(v \times H) - \nu \Delta H = 0, \quad \operatorname{div} H = 0, \\ \Delta \Phi = \rho - \bar{\rho}, \end{cases} \tag{1.1}$$

with the initial data as follows:

$$(\rho, v, H, \nabla\Phi)(t,x)\big|_{t=0} = (\rho_0, v_0, H_0, \nabla\Phi_0)(x), \tag{1.2}$$

the far field behavior of solutions, we assume:

$$(\rho_0, v_0, H_0, \nabla\Phi_0)(x) \to (\bar{\rho}, 0, 0, 0) \quad \text{as } |x| \to \infty. \tag{1.3}$$

Here the unknown functions $\rho = \rho(t,x) \geq 0$, $v = v(t,x) \in \mathbb{R}^3$, $H = H(t,x) \in \mathbb{R}^3$ and $\Phi = \Phi(t,x) \in \mathbb{R}^3$ are density, velocity, magnetic field and electric potential

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respectively. The pressure $P = P(\rho)$ is a smooth function with $P^{'}(\rho) > 0$ for $\rho > 0$. The constants μ, λ are the viscosity coefficients of the flow, and they satisfy the physical restrictions $\mu > 0$ and $2\mu + 3\lambda \geq 0$. The constant $\nu > 0$ represents the magnetic diffusivity. $\bar{\rho}$ represents the positive constant.

1.1. History of the problem

Let us give some explanations about the above model. When the magnetic field and the Coulomb force are taken into account, the compressible Navier-Stokes-Poisson equations are transformed into the compressible viscous magnetohydrodynamic(MHD) equations with Coulomb force, which is of hyperbolic-parabolic-elliptic mixed type. Owing to the physical importance and mathematical challenges, there is a huge literature on the investigations of well-posedness of smooth or weak solutions to the compressible MHD equations with Coulomb force, cf. [4,6,8,10,12]. To get straight to the point of this article, let's just give a brief overview of the results for this model. When time goes to infinity, Tan-Wang [8] showed the vanishing vacuum phenomena of the finite energy weak solutions via the weak convergence method. Wang [9] proved that the weak solutions decay exponentially to the equilibrium state in L^2 norm. What'more, Zheng-Tan [10] obtained the optimal time decay estimate of the solutions by spectral analysis and energy methods. And Tan-Tong-Wang [11] obtained the global existence and time decay rates of the solutions through a general energy method. More precisely, under the assumptions that the initial data $\|(\rho_0 - \bar{\rho}, v_0, H_0, \nabla\Phi_0)(x)\|_{H^3}$ is sufficiently small, one has

$$\begin{aligned}&\|(\rho-\bar{\rho},v,H,\nabla\Phi)(t)\|_{H^N}^2+\int_0^t\|(\rho-\bar{\rho},\nabla v,\nabla H,\nabla\nabla\Phi)(s)\|_{H^N}^2\mathrm{d}s\\ \leq &C\|(\rho_0-\bar{\rho},v_0,H_0,\nabla\Phi_0)\|_{H^N}^2.\end{aligned} \tag{1.4}$$

Furthermore, if $(\rho_0 - \bar{\rho}, v_0, H_0, \nabla\Phi_0)(x) \in H^N \cap L^1$, it holds that

$$\|\nabla^\ell(\rho-\bar{\rho},v,H,\nabla\Phi)(t)\|_{H^{N-l}} \lesssim (1+t)^{-\frac{3}{4}-\frac{\ell}{2}} \quad \ell = 0,1,...,N-1, \tag{1.5}$$

and

$$\|\nabla^\ell(\rho-\bar{\rho})(t)\|_{L^2} \lesssim (1+t)^{-(\frac{5}{4}+\frac{l}{2})} \quad \ell = 0,1,...,N-2. \tag{1.6}$$

However, when taking $\ell = N-1$ in (1.5), we find that the L^2–decay rate of the highest-order (i.e. N–order) spatial derivative of the solution $(\rho-\bar{\rho}, v, H, \nabla\Phi)$ is the same as that of its lower order, which is $(1+t)^{-\frac{3}{4}-\frac{N-1}{2}}$ and is slower than L^2-rate $(1+t)^{-\frac{3}{4}-\frac{N}{2}}$. Comparing the related results with those of the heat equation, it does not appear to be optimal. Therefore, to improve the time decay rate in (1.5) is an interesting and meaningful problem.

1.2. Notation

Throughout this paper, we denote $L^p(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$ as usual Lebesgue space and Sobolev spaces with norm $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$ respectively. In order to construct the low–high frequency decomposition, we will introduce some symbols in the frequency space. Let $\phi_j \in C_0^\infty(\mathbb{R}_\xi^3)(j = 1,\infty)$ be the following cut–off functions:

$$\phi_1(\xi) = \begin{cases} 1 & |\xi| \leq a_0, \\ 0 & |\xi| \geq A_0, \end{cases} \tag{1.7}$$

and $\phi_\infty(\xi) := 1 - \phi_1(\xi)$. Here, the two positive numbers satisfy the following relationship: $0 < a_0 < A_0$. Then, we can define the following operators:

$$\mathcal{Q}_j f = \mathfrak{F}^{-1}[\phi_j(\xi)\widehat{f}], \qquad j = 1, \infty.$$

For the sake of simplicity, denote the low and high frequencies respectively as

$$f^l = \mathcal{Q}_1 f, \text{ and } f^h = \mathcal{Q}_\infty f.$$

In addition, we utilize the symbol $a \lesssim b$ to represent $a \le Cb$ for a universal positive constant C. We use C_0 to denote the positive constant depending additionally on the initial data, and we use ∇^k with an integer $k \ge 0$ to represent the usual arbitrary spatial derivative of order k. Besides, we adopt the following simplified notation: $\|(f,g)\|_X := \|f\|_X + \|g\|_X$.

Similar to the approach developed by [13], we establish the optimal decay-in-time of the highest-order derivatives of the solutions in the following theorem.

1.3. Main results

Theorem 1.1. *Suppose that the initial data* $(\rho_0 - \bar\rho, v_0, H_0, \nabla\Phi_0)(x) \in H^N \cap L^1$ *for any integer* $N \ge 3$ *and there exists a constant* $\delta_0 > 0$ *such that if*

$$\|(\rho_0 - \bar\rho, v_0, H_0, \nabla\Phi_0)\|_{H^3} \le \delta_0, \tag{1.8}$$

then the Cauchy problem (1.1)-(1.3) *has a unique global solution* $(\rho(t), v(t), H(t), \nabla\Phi(t))$ *such that for any* $t \in [0,\infty)$*, we have the following optimal time decay rates*

$$\|\nabla^N(\rho - \bar\rho, v, H, \nabla\Phi)(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{N}{2}}, \tag{1.9}$$

and

$$\|\nabla^{N-1}(\rho - \bar\rho)(t)\|_{L^2} \lesssim (1+t)^{-\frac{5}{4}-\frac{N-1}{2}}. \tag{1.10}$$

Remark 1.1. Let's do a comparison between the Theorem 1.1 and the main results (1.5)-(1.6) in [11]. In this paper, we establish the optimal decay rates of the highest-order spatial derivatives of the solution, which are the same as those of the heat equation. It is totally new and meaningful.

2. Proof of Theorem 1.1

For the Cauchy problem (1.1)-(1.3) in this section, we will rewrite the system in its perturbation form which makes it easier to study. Thus, let

$$\sigma = \rho - \bar\rho, z = \frac{\bar\rho}{\gamma} v, \gamma = \sqrt{P'(\bar\rho)}, \mu_1 = \frac{\mu}{\bar\rho}, \mu_2 = \frac{\lambda + \mu}{\bar\rho}.$$

Then the systems (1.1)-(1.3) can be rewritten as

$$\begin{cases} \sigma_t + \gamma \operatorname{div} z = G_1, \\ z_t - \mu_1 \Delta z - \mu_2 \nabla \operatorname{div} z + \gamma \nabla\sigma - \frac{\bar\rho}{\gamma}\nabla\Phi = G_2, \\ H_t - \nu\Delta H = G_3, \quad \operatorname{div} H = 0, \\ \Delta\Phi = \sigma, \end{cases} \tag{2.1}$$

where the nonhomogeneous source terms $G_i (i = 1, 2, 3)$ are defined as

$$\begin{cases} G_1 = -\frac{\gamma}{\bar{\rho}}(\sigma \mathrm{div} z + z \cdot \nabla \sigma), \\ G_2 = -\frac{\gamma}{\bar{\rho}} z \cdot \nabla z + \left(\frac{\mu}{\sigma + \bar{\rho}} - \mu_1\right) \Delta u + \left(\frac{\lambda + \mu}{\sigma + \bar{\rho}} - \mu_2\right) \nabla \mathrm{div} z \\ \qquad + \frac{\bar{\rho}}{\gamma}\left(\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\sigma + \bar{\rho})}{(\sigma + \bar{\rho})}\right) \nabla \sigma + \frac{\bar{\rho}}{\gamma(\sigma + \bar{\rho})} H \cdot \nabla H - \frac{\bar{\rho}}{2\gamma(\sigma + \bar{\rho})} \nabla(|H|^2), \\ G_3 = \frac{\gamma}{\bar{\rho}}(H \cdot \nabla z - z \cdot \nabla H - H \mathrm{div} z). \end{cases} \tag{2.2}$$

The associated initial data are given by

$$(\sigma, z, H, \nabla\Phi)(x, 0) = (\rho_0 - \bar{\rho}, z_0, H_0, \nabla\Phi_0)(x) \to (0, 0, 0, 0), \quad \text{as} \quad |x| \to \infty. \tag{2.3}$$

Before our proofs, we need to construct the following time-weighted energy functional

$$\mathfrak{M}(t) = \sup_{0 \le s \le t} (1 + s)^{\frac{3}{2} + N} \|\nabla^N(\sigma, z, H, \nabla\Phi)(s)\|_{L^2}^2. \tag{2.4}$$

Due to the benefits of (2.4), we only prove the following proposition to obtain the decay rate (1.9) in Theorem 1.1:

Proposition 2.1. *Let the hypothesis of Theorem 1.1 hold. Then we have the following conclusion:*

$$\mathfrak{M}(t) \le C(C_0 + \|U_0\|_{L^1}^2). \tag{2.5}$$

Then, we will divide the proof of proposition 2.1 into the following steps: deriving the optimal decay rates on the low-frequent parts and the high-frequent parts of the highest order derivatives of the solution. To this end, we first give a useful lemma to deal with the integration over time, the proof of which can be found in [1]:

Lemma 2.1. *Let $l_1, l_2 > 0$, and it holds that*

$$\int_0^{\frac{t}{2}} (1 + t - s)^{-l_1} (1 + s)^{-l_2} \mathrm{d}s \le \begin{cases} C(1 + t)^{-l_1}, & l_2 > 1, \\ C(1 + t)^{-(l_1 - \epsilon)}, & l_2 = 1, \\ C(1 + t)^{-(l_1 + l_2 - 1)}, & l_2 < 1, \end{cases}$$

and

$$\int_{\frac{t}{2}}^{t} (1 + t - s)^{-l_1} (1 + s)^{-l_2} \mathrm{d}s \le \begin{cases} C(1 + t)^{-l_2}, & l_1 > 1, \\ C(1 + t)^{-(l_2 - \epsilon)}, & l_1 = 1, \\ C(1 + t)^{-(l_1 + l_2 - 1)}, & l_1 < 1, \end{cases}$$

where $\epsilon > 0$ is a small but fixed constant.

Next, we can state the optimal decay rate on linearized system (2.1), whose proof can be seen in [3,14]:

Lemma 2.2. *Let $(\bar{\sigma}, \bar{z}, \bar{H}, \nabla\bar{\Phi})$ be the global solution of the linearized system of* (2.1) *with the initial data $(\bar{\sigma}_0, \bar{z}_0, \bar{H}, \nabla\bar{\Phi}_0) \in H^\ell(\mathbb{R}^3)$. Then if there exists a sufficiently small constant $\delta_0 > 0$, such that*

$$K_0 = \|(\bar{\sigma}_0, \bar{z}_0, \bar{H}, \nabla\bar{\Phi}_0)\|_{L^1} \le \delta_0, \tag{2.6}$$

then for all $0 \le k \le N$ *(k be an integer), it holds that*

$$\|\nabla^k(\bar{\sigma}^l, \bar{z}^l, \bar{H}, \nabla\bar{\Phi}^l)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}\|(\bar{\sigma}_0, \bar{z}_0, \bar{H}, \nabla\bar{\Phi}_0)\|_{L^1}, \tag{2.7}$$

for all $t \ge 0$.

For simplicity, we define $U = (\sigma, z, H, \nabla\Phi)$ to be the global smooth solution of the Cauchy problem (2.1)–(2.3). Next, we can deduce the L^2–decay estimate for the low frequency part of the solution for the perturbation system (2.1):

Lemma 2.3. *Under the assumptions of Theorem 1.1, we have the following estimates:*

$$\|\nabla^N(\sigma^l, z^l, H^l, \nabla\Phi^l)(t)\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{N}{2}}(\|U_0\|_{L^1} + \delta\sqrt{\mathfrak{M}(t)}), \tag{2.8}$$

Proof. Defining $G = (G_1, G_2, G_3)^T$, by (2.1)-(2.2), Lemma 2.1, Lemma 2.2, Plancherel theorem and the Hausdorff-Young's inequality, we obtain from the Duham el's principle that

$$\begin{aligned}
&\|\nabla^N(\sigma^l, z^l, H^l, \nabla\Phi^l)(t)\|_{L^2} \\
\lesssim&(1+t)^{-\frac{3}{4}-\frac{N}{2}}\|U_0\|_{L^1} + \int_0^{\frac{t}{2}}(1+t-s)^{-\frac{3}{4}-\frac{N}{2}}\|G^l(s)\|_{L^1}\mathrm{d}s \\
&+ \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{5}{4}}\big\||\xi|^{N-1}\widehat{G}^l(s)\big\|_{L^\infty}\mathrm{d}s.
\end{aligned} \tag{2.9}$$

To derive the decay on $\nabla^N(\sigma^L, z^l, H^L, \nabla\Phi^L)$, by taking advantage of low and high frequency decomposition, we can estimate the nonlinear terms in (2.2) like Lemma 3.1. What's more, by the estimates (1.5) and Lemma 3.2-Lemma 3.5, we have

$$\begin{aligned}
\|G^l(s)\|_{L^1} \lesssim& \Big\|\frac{\gamma}{\bar{\rho}}(\sigma\mathrm{div}z + z\cdot\nabla\sigma)(s)\Big\|_{L^1} + \Big\|\frac{\gamma}{\bar{\rho}}z\cdot\nabla z(s)\Big\|_{L^1} \\
&+ \Big\|\Big[\Big(\frac{\mu}{\sigma+\bar{\rho}} - \mu_1\Big)\Delta z + \Big(\frac{\lambda+\mu}{\sigma+\bar{\rho}} - \mu_2\Big)\nabla\mathrm{div}z\Big](s)\Big\|_{L^1} \\
&+ \Big\|\frac{\bar{\rho}}{\gamma}\Big(\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\sigma+\bar{\rho})}{(\sigma+\bar{\rho})}\Big)\nabla\sigma(s)\Big\|_{L^1} \\
&+ \Big\|\Big[\frac{\bar{\rho}}{\gamma(\sigma+\bar{\rho})}H\cdot\nabla H - \frac{\bar{\rho}}{2\gamma(\sigma+\bar{\rho})}\nabla(|H|^2)\Big](s)\Big\|_{L^1} \\
&+ \Big\|\frac{\gamma}{\bar{\rho}}(H\cdot\nabla z - z\cdot\nabla H - H\mathrm{div}z)(s)\Big\|_{L^1} \\
\lesssim& \|(\sigma,z)(s)\|_{L^2}\|\nabla(\sigma,z)(s)\|_{L^2} \\
&+ \Big\|\Big[\Big(\frac{\mu}{\sigma+\bar{\rho}} - \mu_1\Big), \Big(\frac{\lambda+\mu}{\sigma+\bar{\rho}} - \mu_2\Big)\Big](s)\Big\|_{L^2}\|\Delta z(s)\|_{L^2} \\
&+ \Big\|\Big(\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\sigma+\bar{\rho})}{(\sigma+\bar{\rho})}\Big)(s)\Big\|_{L^2}\|\nabla\sigma(s)\|_{L^2} \\
&+ \Big\|\frac{\bar{\rho}}{(\sigma+\bar{\rho})}(s)\Big\|_{L^\infty}\|H(s)\|_{L^2}\|\nabla H(s)\|_{L^2} \\
&+ \|(z,H)(s)\|_{L^2}\|\nabla(z,H)(s)\|_{L^2} \\
\lesssim& \|(\sigma,z)(s)\|_{L^2}\|\nabla(\sigma,z)(s)\|_{L^2} + \|\sigma(s)\|_{L^2}\|\Delta z(s)\|_{L^2} \\
&+ \|(z,H)(s)\|_{L^2}\|\nabla(z,H)(s)\|_{L^2} \\
\lesssim& (1+s)^{-2}.
\end{aligned} \tag{2.10}$$

Moreover, with the help of decay rate (1.5), the definition (2.4) of $\mathfrak{M}(t)$, Lemma 3.6, remark 3.1 and the Hölder's and Hausdorff-Young's inequality, we can estimate the derivatives of the nonlinear term (2.2) as:

$$\begin{aligned}
&\big\||\xi|^{N-1}\widehat{G}^l(s)\big\|_{L^\infty} \\
\lesssim\, & \|\nabla^{N-1}G^l(s)\|_{L^1} \\
\lesssim\, & \|\nabla^{N-1}(\sigma\mathrm{div}z+z\cdot\nabla\sigma)(s)\|_{L^1}+\|\nabla^{N-1}(z\cdot\nabla z)(s)\|_{L^1} \\
&+\Big\|\nabla^{N-2}\Big[\Big(\frac{\mu}{\sigma+\bar\rho}-\mu_1\Big)\Delta z+\Big(\frac{\lambda+\mu}{\sigma+\bar\rho}-\mu_2\Big)\nabla\mathrm{div}z\Big](s)\Big\|_{L^1} \\
&+\Big\|\nabla^{N-1}\Big[\Big(\frac{P'(\bar\rho)}{\bar\rho}-\frac{P'(\sigma+\bar\rho)}{(\sigma+\bar\rho)}\Big)\nabla\sigma\Big](s)\Big\|_{L^1} \\
&+\Big\|\nabla^{N-1}\Big[\frac{\bar\rho}{(\sigma+\bar\rho)}(H\cdot\nabla H-\nabla(|H|^2))\Big](s)\Big\|_{L^1} \\
&+\Big\|\nabla^{N-1}(H\cdot\nabla z-z\cdot\nabla H-H\mathrm{div}z)(s)\Big\|_{L^1} \\
\lesssim\, & \|(\sigma,z)(s)\|_{L^2}\|\nabla^N(\sigma,z)(s)\|_{L^2}+\|\nabla(\sigma,z)(s)\|_{L^2}\|\nabla^{N-1}(\sigma,z)(s)\|_{L^2} \\
&+\|\nabla^{N-2}\sigma(s)\|_{L^2}\|\Delta z(s)\|_{L^2}+\|\sigma(s)\|_{L^2}\|\nabla^N z(s)\|_{L^2} \\
&+\|\nabla^{N-1}\sigma(s)\|_{L^2}\|\nabla\sigma(s)\|_{L^2}+\|\nabla H(s)\|_{L^2}\|\nabla^{N-1}H(s)\|_{L^2} \\
&+\|(z,H)(s)\|_{L^2}\|\nabla^N(z,H)(s)\|_{L^2} \\
\lesssim\, & \delta(1+s)^{-\frac34-\frac N2}\sqrt{\mathfrak{M}(s)}+(1+s)^{-1-\frac N2}.
\end{aligned} \tag{2.11}$$

Inserting estimates (2.10) and (2.11) into (2.9), then using the monotonicity of $\mathfrak{M}(t)$ and Lemma 2.1, we can obtain Lemma 2.3.

□

Next, we will work on establishing the energy estimate at high frequency, i.e. the N-order spatial derivative of the solution. Our results are illustrated in the following Lemma:

Lemma 2.4. *Under the conditions of Theorem 1.1, the estimate*

$$\|\nabla^N(\sigma^h,z^h,H^h,\nabla\Phi^h)(t)\|_{L^2}\le C(C_0+\delta\sqrt{\mathfrak{M}(t)})(1+t)^{-\frac34-\frac N2} \tag{2.12}$$

holds.

Proof. By the ∇^N energy estimate, for $\mathcal{Q}_\infty((2.1)_1)$, $\mathcal{Q}_\infty((2.1)_2)$, $\mathcal{Q}_\infty((2.1)_3)$ on σ^h, z^h, H^h respectively, we can see that

$$\begin{cases}
\dfrac12\dfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\sigma^h\|_{L^2}^2+\gamma\langle\nabla^N\mathrm{div}z^h,\nabla^N\sigma^h\rangle=\langle\nabla^N G_1^h,\nabla^N\sigma^h\rangle, \\
\dfrac12\dfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N z^h\|_{L^2}^2+\gamma\langle\nabla^N\nabla\sigma^h,\nabla^N z^h\rangle-\mu_1\langle\nabla^N\Delta z^h,\nabla^N z^h\rangle \\
\qquad-\mu_2\langle\nabla^N\nabla\mathrm{div}z^h,\nabla^N z^h\rangle-\dfrac{\bar\rho}{\gamma}\langle\nabla^N\nabla\Phi^h,\nabla^N z^h\rangle=\langle\nabla^N G_2^h,\nabla^N z^h\rangle, \\
\dfrac12\dfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N H^h\|_{L^2}^2-\nu\langle\nabla^N\Delta H^h,\nabla^N H^h\rangle=\langle\nabla^N G_3^h,\nabla^N H^h\rangle.
\end{cases} \tag{2.13}$$

And note that

$$
\begin{aligned}
-\frac{\bar\rho}{\gamma}\langle\nabla^N\nabla\Phi^h,\nabla^N z^h\rangle &= \frac{\bar\rho}{\gamma}\langle\nabla^N\Phi^h,\nabla^N\mathrm{div}z^h\rangle\\
&= -\frac{\bar\rho}{\gamma}\langle\nabla^N\Phi^h,\nabla^N\Big[\frac{1}{\bar\rho}\mathrm{div}(\sigma z)+\frac{1}{\gamma}\sigma_t\Big]^h\rangle\\
&= -\frac{1}{\gamma}\langle\nabla^N\Phi^h,\nabla^N\mathrm{div}(\sigma z)^h\rangle-\frac{\bar\rho}{\gamma^2}\langle\nabla^N\Phi^h,\nabla^N\Delta\Phi_t^h\rangle\\
&= \frac{1}{\gamma}\langle\nabla^N\nabla\Phi^h,\nabla^N(\sigma z)^h\rangle+\frac{\bar\rho}{\gamma^2}\langle\nabla^N\nabla\Phi^h,\nabla^N\nabla\Phi_t^h\rangle\\
&= \frac{\bar\rho}{2\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\nabla\Phi^h\|_{L^2}^2+\frac{1}{\gamma}\langle\nabla^N\nabla\Phi^h,\nabla^N(\sigma z)^h\rangle.
\end{aligned}\tag{2.14}
$$

Thus by (2.13), (2.14) and integration by parts, we infer that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N(\sigma^h,z,H)\|_{L^2}^2+\frac{\bar\rho}{2\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\nabla\Phi^h\|_{L^2}^2+(\mu_1+\mu_2)\|\nabla^{N+1}z^h\|_{L^2}^2\\
&\quad+\nu\|\nabla^{N+1}H^h\|_{L^2}^2=\langle\nabla^N G_1^h,\nabla^N\sigma^h\rangle+\langle\nabla^N G_2^h,\nabla^N z^h\rangle+\langle\nabla^N G_3^h,\nabla^N H^h\rangle\\
&\quad-\frac{1}{\gamma}\langle\nabla^N\nabla\Phi^h,\nabla^N(\sigma z)^h\rangle.
\end{aligned}\tag{2.15}
$$

Now, let us estimate the terms in the righthand side of (2.15) one by one. For the first term, we know that:

$$
\begin{aligned}
\langle\nabla^N G_1^h,\nabla^N\sigma^h\rangle &= -\frac{\gamma}{\bar\rho}\langle\nabla^N(\sigma\mathrm{div}z+z\cdot\nabla\sigma)^h,\nabla^N\sigma^h\rangle\\
&= -\frac{\gamma}{\bar\rho}\langle\nabla^N(\sigma\mathrm{div}z)^h,\nabla^N\sigma^h\rangle-\frac{\gamma}{\bar\rho}\langle\nabla^N(z\cdot\nabla\sigma)^h,\nabla^N\sigma^h\rangle\\
&:= I_1+I_2.
\end{aligned}\tag{2.16}
$$

For I_1, by (1.8) and Lemma 3.1–Lemma 3.5, we get

$$
\begin{aligned}
|I_1| &\lesssim \|\nabla^N(\sigma\mathrm{div}z)^h\|_{L^2}\|\nabla^N\sigma^h\|_{L^2}\\
&\lesssim \|\nabla^N(\sigma\mathrm{div}z)\|_{L^2}\|\nabla^N\sigma^h\|_{L^2}\\
&\lesssim (\|\sigma\|_{L^\infty}\|\nabla^N\mathrm{div}z\|_{L^2}+\|\nabla^N\sigma\|_{L^2}\|\mathrm{div}z\|_{L^\infty})\|\nabla^N\sigma\|_{L^2}\\
&\lesssim (\|\nabla\sigma\|_{H^1}\|\nabla^N\mathrm{div}z\|_{L^2}+\|\nabla^N\sigma\|_{L^2}\|\nabla\mathrm{div}z\|_{H^1})\|\nabla^N\sigma\|_{L^2}\\
&\lesssim \delta(\|\nabla^N\sigma\|_{L^2}^2+\|\nabla^{N+1}z\|_{L^2}^2),
\end{aligned}\tag{2.17}
$$

where we have used Young's and Hölder's inequalities. For the term I_2, we obtain from $f=f^h+f^l$ that

$$
\begin{aligned}
I_2 &= -\frac{\gamma}{\bar\rho}\langle\nabla^N(z\cdot\nabla\sigma)^h,\nabla^N\sigma^h\rangle\\
&= -\frac{\gamma}{\bar\rho}\langle\nabla^N(z\cdot\nabla\sigma)-\nabla^N(z\cdot\nabla\sigma)^l,\nabla^N\sigma^h\rangle\\
&= -\frac{\gamma}{\bar\rho}\langle\nabla^N(z\cdot\nabla\sigma^h)+\nabla^N(z\cdot\nabla\sigma^l)-\nabla^N(z\cdot\nabla\sigma)^l,\nabla^N\sigma^h\rangle\\
&:= I_{2,1}+I_{2,2}+I_{2,3}.
\end{aligned}\tag{2.18}
$$

In terms of the commutator notation of Lemma 3.4, one can obtain

$$-\frac{\gamma}{\bar{\rho}}\langle\nabla^N(z\cdot\nabla\sigma^h),\nabla^N\sigma^h\rangle=-\frac{\gamma}{\bar{\rho}}\langle z\cdot\nabla\nabla^N\sigma^h,\nabla^N\sigma^h\rangle-\frac{\gamma}{\bar{\rho}}\langle[\nabla^N,z]\nabla\sigma^h,\nabla^N\sigma^h\rangle.$$

We can employ integrating by parts to imply

$$\begin{aligned}|\langle z\cdot\nabla\nabla^N\sigma^h,\nabla^N\sigma^h\rangle|=\left|\frac{1}{2}\langle\mathrm{div}z,|\nabla^N\sigma^h|^2\rangle\right|&\lesssim\|\nabla z\|_{L^\infty}\|\nabla^N\sigma^h\|_{L^2}^2\\&\lesssim\delta\|\nabla^N\sigma^h\|_{L^2}^2.\end{aligned}\tag{2.19}$$

On the other hand, based on the estimates (3.5) of Lemma 3.4, we can derive

$$\begin{aligned}|\langle[\nabla^N,z]\nabla\sigma^h,\nabla^N\sigma^h\rangle|&\lesssim\|[\nabla^N,z]\nabla\sigma^h\|_{L^2}\|\nabla^N\sigma^h\|_{L^2}\\&\lesssim(\|\nabla z\|_{L^\infty}\|\nabla^N\sigma^h\|_{L^2}+\|\nabla^N z\|_{L^6}\|\nabla\sigma^h\|_{L^3})\|\nabla^N\sigma^h\|_{L^2}\\&\lesssim(\|\nabla^2 z\|_{H^1}\|\nabla^N\sigma^h\|_{L^2}+\|\nabla^{N+1}z\|_{L^2}\|\nabla\sigma^h\|_{H^1})\|\nabla^N\sigma^h\|_{L^2}\\&\lesssim\delta(\|\nabla^N\sigma^h\|_{L^2}^2+\|\nabla^{N+1}z\|_{L^2}^2).\end{aligned}\tag{2.20}$$

Using (2.19)-(2.20), we have

$$|I_{2,1}|\lesssim\delta(\|\nabla^N\sigma^h\|_{L^2}^2+\|\nabla^{N+1}z\|_{L^2}^2).\tag{2.21}$$

For the term $I_{2,2}$, by the fact that $\|\nabla^k f^l\|_{L^2}\le\|\nabla^{k-1}f^l\|_{L^2}(k\ge1)$, we have:

$$\begin{aligned}|I_{2,2}|&\lesssim\|\nabla^N(z\cdot\nabla\sigma^l)\|_{L^2}\|\nabla^N\sigma^h\|_{L^2}\\&\lesssim(\|z\|_{L^\infty}\|\nabla^{N+1}\sigma^l\|_{L^2}+\|\nabla^N z\|_{L^6}\|\nabla\sigma^l\|_{L^3})\|\nabla^N\sigma\|_{L^2}\\&\lesssim(\|\nabla z\|_{H^1}\|\nabla^N\sigma\|_{L^2}+\|\nabla^{N+1}z\|_{L^2}\|\nabla\sigma\|_{H^1})\|\nabla^N\sigma\|_{L^2}\\&\lesssim\delta(\|\nabla^N\sigma\|_{L^2}^2+\|\nabla^{N+1}z\|_{L^2}^2).\end{aligned}\tag{2.22}$$

Similar to (2.22), we have

$$\begin{aligned}|I_{2,3}|&\lesssim\|\nabla^N(z\cdot\nabla\sigma)^l\|_{L^2}\|\nabla^N\sigma^h\|_{L^2}\\&\lesssim\|\nabla^{N-1}(z\cdot\nabla\sigma)\|_{L^2}\|\nabla^N\sigma\|_{L^2}\\&\lesssim(\|z\|_{L^\infty}\|\nabla^N\sigma\|_{L^2}+\|\nabla^{N-1}z\|_{L^6}\|\nabla\sigma\|_{L^3})\|\nabla^N\sigma\|_{L^2}\\&\lesssim(\|\nabla z\|_{H^1}\|\nabla^N\sigma\|_{L^2}+\|\nabla^N z\|_{L^2}\|\nabla\sigma\|_{H^1})\|\nabla^N\sigma\|_{L^2}\\&\lesssim\delta(\|\nabla^N\sigma\|_{L^2}^2+\|\nabla^N z\|_{L^2}^2).\end{aligned}\tag{2.23}$$

Inserting estimates (2.21)-(2.23) into (2.18), it is easy to obtain

$$|I_2|\lesssim\delta\|(\nabla^N\sigma,\nabla^N z,\nabla^{N+1}z)\|_{L^2}^2.\tag{2.24}$$

Therefore, summing up estimates (2.17) and (2.24), we have

$$|\langle\nabla^N G_1^h,\nabla^N\sigma^h\rangle|\lesssim\delta\|(\nabla^N\sigma,\nabla^N z,\nabla^{N+1}z)\|_{L^2}^2.\tag{2.25}$$

For the second term in the right-hand side of (2.15), we obtain from (2.2) that

$$\begin{aligned}\langle\nabla^N G_2^h,\nabla^N z^h\rangle =& -\frac{\gamma}{\bar\rho}\langle\nabla^N(z\cdot\nabla z)^h,\nabla^N z^h\rangle\\ &+\Big\langle\nabla^N\Big[\Big(\frac{\mu}{\sigma+\bar\rho}-\mu_1\Big)\Delta z+\Big(\frac{\lambda+\mu}{\sigma+\bar\rho}-\mu_2\Big)\nabla\mathrm{div}z\Big]^h,\nabla^N z^h\Big\rangle\\ &+\frac{\bar\rho}{\gamma}\Big\langle\nabla^N\Big[\Big(\frac{P'(\bar\rho)}{\bar\rho}-\frac{P'(\sigma+\bar\rho)}{(\sigma+\bar\rho)}\Big)\nabla\sigma\Big]^h,\nabla^N z^h\Big\rangle\\ &+\Big\langle\nabla^N\Big[\frac{\bar\rho}{\gamma(\sigma+\bar\rho)}H\cdot\nabla H\Big]^h,\nabla^N z^h\Big\rangle\\ &-\Big\langle\nabla^N\Big[\frac{\bar\rho}{2\gamma(\sigma+\bar\rho)}\nabla(|H|^2)\Big]^h,\nabla^N z^h\Big\rangle:=\sum_{i=3}^{7}I_i.\end{aligned}\tag{2.26}$$

Next, we devote ourselves to estimating the five terms on the right-hand side of (2.26). For the term I_3, I_6 and I_7, similar to the proof of (2.17), we can obtain

$$|I_3|+|I_6|+|I_7|\lesssim\delta\|\nabla^{N+1}z\|_{L^2}^2+\|\nabla^{N+1}H\|_{L^2}^2.\tag{2.27}$$

I_4 is estimated by

$$\begin{aligned}|I_4| &=\Big|\Big\langle\ \nabla^N\Big[\Big(\frac{\mu}{\sigma+\bar\rho}-\mu_1\Big)\Delta z+\Big(\frac{\lambda+\mu}{\sigma+\bar\rho}-\mu_2\Big)\nabla z\Big]^h,\nabla^N z^h\Big\rangle\Big|\\ &=\Big|\Big\langle\ \nabla^{N-1}\Big[\Big(\frac{\mu}{\sigma+\bar\rho}-\mu_1\Big)\Delta z+\Big(\frac{\lambda+\mu}{\sigma+\bar\rho}-\mu_2\Big)\nabla z\Big]^h,\nabla^N\mathrm{div}z^h\Big\rangle\Big|\\ &\lesssim\Big\|\nabla^{N-1}\Big[\Big(\frac{\mu}{\sigma+\bar\rho}-\mu_1\Big)\Delta z+\Big(\frac{\lambda+\mu}{\sigma+\bar\rho}-\mu_2\Big)\nabla z\Big]^h\Big\|_{L^2}\|\nabla^N\mathrm{div}z^h\|_{L^2}\\ &\lesssim\Big\|\nabla^{N-1}\Big[\Big(\frac{\mu}{\sigma+\bar\rho}-\mu_1\Big)\Delta z+\Big(\frac{\lambda+\mu}{\sigma+\bar\rho}-\mu_2\Big)\nabla z\Big]\Big\|_{L^2}\|\nabla^{N+1}z^h\|_{L^2}\\ &\lesssim(\|\sigma\|_{L^\infty}\|\nabla^{N+1}z\|_{L^2}+\|\nabla^{N-1}\sigma\|_{L^6}\|\Delta z\|_{L^3})\|\nabla^{N+1}z\|_{L^2}\\ &\lesssim(\|\nabla\sigma\|_{H^1}\|\nabla^{N+1}z\|_{L^2}+\|\nabla^{N}\sigma\|_{L^2}\|\Delta z\|_{H^1})\|\nabla^{N+1}z\|_{L^2}\\ &\lesssim\delta(\|\nabla^{N+1}z\|_{L^2}^2+\|\nabla^N\sigma\|_{L^2}^2).\end{aligned}\tag{2.28}$$

Similarly, we also have

$$|I_5|\lesssim\delta(\|\nabla^{N+1}z\|_{L^2}^2+\|\nabla^N\sigma\|_{L^2}^2).\tag{2.29}$$

Plugging (2.27)-(2.29) into (2.26) yields that

$$|\langle\nabla^N G_2^h,\nabla^N z^h\rangle|\lesssim\delta\|(\nabla^N\sigma,\nabla^{N+1}H,\nabla^{N+1}z)\|_{L^2}^2.\tag{2.30}$$

We also need to estimate the last term $\langle\nabla^N G_3^h,\nabla^N z^h\rangle$ which is equivalent to the following:

$$\begin{aligned}|\langle\nabla^N G_3^h,\nabla^N H^h\rangle| &= \Big|\frac{\gamma}{\bar\rho}\langle\nabla^N(H\cdot\nabla z - z\cdot\nabla H - H\mathrm{div}z)^h,\nabla^N H^h\rangle\Big| \\ &\lesssim \|\nabla^N(H\cdot\nabla z - z\cdot\nabla H - H\mathrm{div}z)^h\|_{L^2}\|\nabla^N H^h\|_{L^2} \\ &\lesssim \|\nabla^N(H\cdot\nabla z - z\cdot\nabla H - H\mathrm{div}z)\|_{L^2}\|\nabla^{N+1} H\|_{L^2} \\ &\lesssim (\|(z,H)\|_{L^\infty}\|\nabla^{N+1}(z,H)\|_{L^2} \\ &\quad + \|\nabla(z,H)\|_{L^\infty}\|\nabla^N(z,H)\|_{L^2})\|\nabla^{N+1}H\|_{L^2} \\ &\lesssim \delta(\|\nabla^N(z,H)\|_{L^2}^2 + \|\nabla^{N+1}(z,H)\|_{L^2}^2).\end{aligned} \tag{2.31}$$

The same as before

$$\begin{aligned}\Big| -\frac{1}{\gamma}\langle\nabla^N\nabla\Phi^h,\nabla^N(\sigma z)^h\rangle\Big| &\lesssim \|\nabla^N(\sigma z)^h\|_{L^2}\|\nabla^N\nabla\Phi^h\|_{L^2} \\ &\lesssim \|\nabla^N(\sigma z)\|_{L^2}\|\nabla^{N-1}\sigma^h\|_{L^2} \\ &\lesssim \|(\sigma,z)\|_{L^\infty}\|\nabla^N(\sigma,z)\|_{L^2}\|\nabla^N\sigma\|_{L^2} \\ &\lesssim \delta\|\nabla^N(\sigma,z)\|_{L^2}^2.\end{aligned} \tag{2.32}$$

Thus, with the aid of the estimates (2.25), (2.30)-(2.32), we obtain from (2.15) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N(\sigma^h,z,H)\|_{L^2}^2 + \frac{\bar\rho}{2\gamma^2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\nabla\Phi^h\|_{L^2}^2 + (\mu_1+\mu_2)\|\nabla^{N+1}z^h\|_{L^2}^2 \\ &\quad + \nu\|\nabla^{N+1}H^h\|_{L^2}^2 \lesssim \delta(\|\nabla^N(\sigma,z,H)\|_{L^2}^2 + \|\nabla^{N+1}(z,H)\|_{L^2}^2).\end{aligned} \tag{2.33}$$

Next, we will derive the dissipation estimate for σ^h to close the estimate. To end this, taking $\mathcal{Q}_\infty(\nabla^{N-1}(2.1)_2)$ and multiplying $\nabla^N\sigma^h$, integrating over $\mathbb{R}^3$ and using equation $(2.1)_1$ and $(2.1)_4$ we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^{N-1}z^h\nabla^N\sigma^h\mathrm{d}x + \gamma\|\nabla^N\sigma^h\|_{L^2}^2 + \frac{\bar\rho}{\gamma}\|\nabla^N(\nabla\Phi)^h\|_{L^2}^2 \\ =&\gamma\|\nabla^N z^h\|_{L^2}^2 + \mu_1\langle\nabla^{N+1}z^h,\nabla^N\sigma^h\rangle + \mu_2\langle\nabla^N\mathrm{div}z^h,\nabla^N\sigma^h\rangle \\ &+\langle\nabla^N G_1^h,\nabla^{N-1}z^h\rangle + \langle\nabla^{N-1}G_2^h,\nabla^N\sigma^h\rangle.\end{aligned} \tag{2.34}$$

The Young inequality implies

$$|\mu_1\langle\nabla^{N+1}z^h,\nabla^N\sigma^h\rangle| \lesssim \|\nabla^{N+1}z^h\|_{L^2}^2 + \frac{\gamma}{4}\|\nabla^N\sigma^h\|_{L^2}^2, \tag{2.35}$$

and

$$|\mu_2\langle\nabla^N\mathrm{div}z^h,\nabla^N\sigma^h\rangle| \lesssim \|\nabla^N\mathrm{div}z^h\|_{L^2}^2 + \frac{\gamma}{4}\|\nabla^N\sigma^h\|_{L^2}^2. \tag{2.36}$$

Then, we should estimate the term $\langle\nabla^N G_1^h,\nabla^{N-1}z^h\rangle$. From $(2.2)_1$, we can get

$$\begin{aligned}\langle\nabla^N G_1^h,\nabla^{N-1}z^h\rangle &= -\frac{\gamma}{\bar\sigma}\langle\nabla^N(\sigma\mathrm{div}z)^h,\nabla^{N-1}z^h\rangle - \frac{\gamma}{\bar\sigma}\langle\nabla^N(z\cdot\nabla\sigma)^h,\nabla^{N-1}z^h\rangle \\ &:= J_1 + J_2.\end{aligned} \tag{2.37}$$

Following the idea of (2.17) and (2.28), we can get

$$|J_1| + |J_2| \lesssim \delta(\|\nabla^N\sigma\|_{L^2}^2 + \|\nabla^N z\|_{L^2}^2 + \|\nabla^{N+1}z\|_{L^2}^2). \tag{2.38}$$

For the last term $\langle \nabla^{N-1} G_2, \nabla^N \sigma^h \rangle$, it is easy to check that

$$
\begin{aligned}
\langle \nabla^{N-1} G_2, \nabla^N \sigma^h \rangle = & -\frac{\gamma}{\bar\rho} \langle \nabla^{N-1} (z \cdot \nabla z)^h, \nabla^N \sigma^h \rangle \\
& + \Big\langle \nabla^{N-1} \Big[\Big(\frac{\mu}{\sigma + \bar\rho} - \mu_1 \Big) \Delta z + \Big(\frac{\lambda + \mu}{\sigma + \bar\rho} - \mu_2 \Big) \nabla \mathrm{div} z \Big]^h, \nabla^N \sigma^h \Big\rangle \\
& + \frac{\bar\rho}{\gamma} \Big\langle \nabla^{N-1} \Big[\Big(\frac{P^{'}(\bar\rho)}{\bar\rho} - \frac{P^{'}(\sigma + \bar\rho)}{(\sigma + \bar\rho)} \Big) \nabla \sigma \Big]^h, \nabla^N \sigma^h \Big\rangle \\
& + \Big\langle \nabla^{N-1} \Big[\frac{\bar\rho}{\gamma(\sigma + \bar\rho)} H \cdot \nabla H \Big]^h, \nabla^N \sigma^h \Big\rangle \\
& - \Big\langle \nabla^{N-1} \Big[\frac{\bar\rho}{2\gamma(\sigma + \bar\rho)} \nabla (|H|^2) \Big]^h, \nabla^N \sigma^h \Big\rangle \\
:= & \sum_{j=3}^{7} J_j.
\end{aligned} \tag{2.39}
$$

For the term J_3, we have

$$
\begin{aligned}
|J_3| & \lesssim \| \nabla^{N-1} (z \cdot \nabla z)^h \|_{L^2} \| \nabla^N \sigma^h \|_{L^2} \\
& \lesssim \| \nabla^N (z \cdot \nabla z) \|_{L^2} \| \nabla^N \sigma^h \|_{L^2} \\
& \lesssim (\|z\|_{L^\infty} \| \nabla^{N+1} z \|_{L^2} + \| \nabla^N z \|_{L^6} \| \nabla z \|_{L^3}) \| \nabla^N \sigma \|_{L^2} \\
& \lesssim \delta (\| \nabla^N \sigma \|_{L^2}^2 + \| \nabla^{N+1} z \|_{L^2}^2).
\end{aligned} \tag{2.40}
$$

Similarly, we have

$$
|J_4| + |J_5| \lesssim \delta (\| \nabla^N \sigma \|_{L^2}^2 + \| \nabla^{N+1} z \|_{L^2}^2). \tag{2.41}
$$

For the terms J_6 and J_7, we can get

$$
\begin{aligned}
|J_6| + |J_7| & \lesssim \Big\| \nabla^{N-1} \Big[\frac{\bar\rho}{\gamma(\sigma + \bar\rho)} H \cdot \nabla H \Big]^h \Big\|_{L^2} \| \nabla^N \sigma^h \|_{L^2} \\
& \lesssim \Big\| \nabla^{N-1} \Big[\frac{\bar\rho}{\gamma(\sigma + \bar\rho)} H \cdot \nabla H \Big] \Big\|_{L^2} \| \nabla^N \sigma^h \|_{L^2} \\
& \lesssim (\|H\|_{L^\infty} \| \nabla^N H \|_{L^2} + \| \nabla^{N-1} H \|_{L^6} \| \nabla H \|_{L^3}) \| \nabla^N \sigma \|_{L^2} \\
& \lesssim \delta \| \nabla^N (\sigma, H) \|_{L^2}^2.
\end{aligned} \tag{2.42}
$$

Substituting (2.35)-(2.36), (2.38) and (2.40)-(2.42) into (2.34) yields immediately

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \nabla^{N-1} z^h \nabla^N \sigma^h \mathrm{d}x + \frac{\gamma}{2} \| \nabla^N \sigma^h \|_{L^2}^2 + \frac{\bar\rho}{\gamma} \| \nabla^N (\nabla \Phi)^h \|_{L^2}^2 \\
& \lesssim \| \nabla^{N+1} z^h \|_{L^2}^2 + \delta (\| \nabla^N (\sigma, z, H) \|_{L^2}^2 + \| \nabla^{N+1} z \|_{L^2}^2).
\end{aligned} \tag{2.43}
$$

Now, to close the above energy estimate, we define the following temporary energy functional:

$$
\mathfrak{E}(\mathrm{t}) = D_0 \| \nabla^N (\sigma^h, z^h, H^h, \nabla \Phi^h) \|_{L^2}^2 + \int_{\mathbb{R}^3} \nabla^{N-1} z^h \nabla^N \sigma^h \mathrm{d}x, \tag{2.44}
$$

where D_0 is a large enough positive number. Then we have the equivalence relationship as follows:

$$c_1\|\nabla^N(\sigma^h, z^h, H^h, \nabla\Phi^h)\|_{L^2}^2 \le \mathfrak{E}(\mathfrak{t}) \le C_1\|\nabla^N(\sigma^h, z^h, H^h, \nabla\Phi^h)\|_{L^2}^2, \tag{2.45}$$

where c_1 and C_1 are two positive constants which are independent of time. Together with the estimates (2.33) and (2.43), applying Lemma 3.1 and the smallness of δ, we obtain

$$\begin{aligned} &\frac{d}{dt}\mathfrak{E}(\mathfrak{t}) + \|\nabla^N\sigma^h\|_{L^2}^2 + \|\nabla^{N+1}z^h\|_{L^2}^2 + \|\nabla^{N+1}H^h\|_{L^2}^2 + \|\nabla^N(\nabla\Phi)^h\|_{L^2}^2 \\ &\lesssim \|(\nabla^N(\sigma^l, z^l, H^l)\|_{L^2}^2. \end{aligned} \tag{2.46}$$

Employing the property of the high-frequency part, we have

$$\|\nabla^N\sigma^h\|_{L^2}^2 + \|\nabla^{N+1}z^h\|_{L^2}^2 + \|\nabla^{N+1}H^h\|_{L^2}^2 + \|\nabla^N(\nabla\Phi)^h\|_{L^2}^2 \ge C_1\mathfrak{E}(\mathfrak{t}). \tag{2.47}$$

Then taking full advantage of the Gronwall inequality and Lemma 2.3, we can obtain

$$\begin{aligned} \mathfrak{E}(\mathfrak{t}) &\le \mathfrak{E}(0)e^{-C_1t} + \int_0^t e^{-C_1(t-s)}\|\nabla^N(\sigma^l, z^l, H^l)(s)\|_{L^2}^2 \mathrm{d}s \\ &\le \mathfrak{E}(0)e^{-C_1t} + C\int_0^t e^{-C_1(t-s)}(1+s)^{-\frac{3}{2}-N}\Big(\|U_0\|_{L^1}^2 + \delta^2\mathfrak{M}(s)\Big)\mathrm{d}s \\ &\le \mathfrak{E}(0)e^{-C_1t} + C\Big(\|U_0\|_{L^1}^2 + \delta^2\mathfrak{M}(t)\Big)\int_0^t e^{-C_1(t-s)}(1+s)^{-\frac{3}{2}-N}\mathrm{d}s \\ &\le C(1+t)^{-\frac{3}{2}-N}(\mathfrak{E}(0) + \|U_0\|_{L^1}^2 + \delta^2\mathfrak{M}(t)). \end{aligned} \tag{2.48}$$

Therefore, due to the equivalence relationships of $\mathfrak{E}(\mathfrak{t})$, we directly complete the proof of Lemma 2.4. $\square$

Finally, by Lemma 2.3 and Lemma 2.4, it holds that

$$\begin{aligned} &\|\nabla^N(\sigma, z, H, \nabla\Phi)(t)\|_{L^2}^2 \\ \le &\|\nabla^N(\sigma^h, z^h, H^h, \nabla\Phi^h)(t)\|_{L^2}^2 + \|\nabla^N(\sigma^l, z^l, H^l, \nabla\Phi^l)(t)\|_{L^2}^2 \\ \le &C(1+t)^{-\frac{3}{2}-N}(\mathfrak{E}(0) + \|U_0\|_{L^1}^2 + \delta^2\mathfrak{M}(t)). \end{aligned} \tag{2.49}$$

Following the definition (2.4) of $\mathfrak{M}(t)$ and estimate (2.49), one can deduce that

$$\mathfrak{M}(t) \le C(\mathfrak{E}(0) + \|U_0\|_{L^1}^2 + \delta^2\mathfrak{M}(t)).$$

Together with the smallness of δ, it implies that $\mathfrak{M}(t) \le C(\mathfrak{E}(0) + \|U_0\|_{L^1}^2)$. This completes the proof of Proposition 2.1. What's more, note that $\sigma = \mathrm{div}\nabla\Phi$, and then we have (1.10).

3. Appendix

In our appendix, we will give some important analytical tools which are available to our proof mentioned above. To begin with, let's introduce the properties about high-low frequency decomposition.

Lemma 3.1. *If $F(x) \in H^\ell(\mathbb{R}^3)$ (ℓ is an integer), it holds that*

$$\|\nabla^i F\|_{L^2} \le \|\nabla^i F^h\|_{L^2} + \|\nabla^i F^l\|_{L^2}, \quad i \ge 0, \tag{3.1}$$

$$\|\nabla^i F^l\|_{L^2} \le \|\nabla^{i-1} F^l\|_{L^2}, \quad i \ge 1, \tag{3.2}$$

$$\|\nabla^i F^h\|_{L^2} \le \|\nabla^{i+1} F^h\|_{L^2}, \quad i \ge 1. \tag{3.3}$$

In the next few lines, we will give the well-known theorem, known as the Gagliardo-Nirenberg-Sobolev inequality.

Lemma 3.2. *Let $0 \le i, j \le \ell$. Then we have*

$$\|\nabla^i F\|_{L^p} \lesssim \|\nabla^j F\|_{L^q}^{1-\theta} \|\nabla^\ell F\|_{L^r}^{\theta},$$

where $\theta \in [\frac{i}{\ell}, 1]$ and satisfies

$$\frac{1}{p} - \frac{i}{3} = \Big(\frac{1}{q} - \frac{j}{3}\Big)(1-\theta) + \Big(\frac{1}{r} - \frac{\ell}{3}\Big)\theta.$$

In particular, taking $p = q = r = 2$, one has

$$\|\nabla^i F\|_{L^2} \lesssim \|\nabla^j F\|_{L^2}^{\frac{\ell-i}{\ell-j}} \|\nabla^\ell F\|_{L^2}^{\frac{i-j}{\ell-j}}.$$

Proof. See the article [7] to get more details about this proof. □

To facilitate the estimation of cross terms, we list the following Lemma, cf [5].

Lemma 3.3. *It holds that for $\ell \ge 0$,*

$$\|\nabla^\ell (F_1 F_2)\|_{L^p} \lesssim \|F_1\|_{L^{p_1}} \|\nabla^\ell F_2\|_{L^{p_2}} + \|F_2\|_{L^{p_3}} \|\nabla^\ell F_1\|_{L^{p_4}},$$

where $p, p_2, p_3 \in (1, +\infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Furthermore, with the help of Lemma 3.2-Lemma 3.3, we can derive the following estimate.

Lemma 3.4. *Let integer $\ell \ge 1$. For any pair of functions F_1 and F_2 belonging to $H^\ell \cap L^\infty$, we can define the following commutator*

$$[\nabla^\ell, F_1] F_2 = \nabla^\ell (F_1 F_2) - F_1 \nabla^\ell F_2. \tag{3.4}$$

Then we have

$$\|[\nabla^\ell, F_1] F_2\|_{L^p} \lesssim \|\nabla F_1\|_{L^{p_1}} \|\nabla^{\ell-1} F_2\|_{L^{p_2}} + \|\nabla^\ell F_1\|_{L^{p_3}} \|F_2\|_{L^{p_4}}. \tag{3.5}$$

Here p, p_1, p_2, p_3 are defined in Lemma 3.2.

For convenience, we will present some Sobolev embedding inequalities.

Lemma 3.5. *(i) Assume that $F(x) \in H^1(\mathbb{R}^3)$. Then we have:*

$$\|F\|_{L^6} \le C \|\nabla F\|_{L^2},$$

$$\|F\|_{L^3} \le C(\|F\|_{L^2} + \|F\|_{L^6}) \le C \|F\|_{H^1}.$$

(ii) If $F(x) \in H^2(\mathbb{R}^3)$, then

$$\|F\|_{L^\infty} \le C \|\nabla F\|_{H^1}.$$

In the end, we present the following Moser-type inequality.

Lemma 3.6. *Assume that $F(f)$ is a smooth function of f with bounded derivatives of any order and f belongs to H^ℓ for any integer $\ell \geq 3$. Then it holds that*

$$\|\nabla^\ell F(f)\|_{L^2} \lesssim \sup_{1\leq j\leq \ell} \|F^{(j)}(f)\|_{L^\infty}\Big(\sum_{2\leq k\leq \ell} \|f\|_{L^2}^{k-1-\frac{3(k-1)}{2\ell}} \|\nabla^\ell f\|_{L^2}^{1+\frac{3(k-1)}{2\ell}} + \|\nabla^\ell f\|_{L^2}\Big). \tag{3.6}$$

Specifically, if f has the lower and upper bounds with $\|f\|_{H^\ell} \leq 1$, then we have

$$\|\nabla^\ell F(f)\|_{L^2} \lesssim \|\nabla^\ell f\|_{L^2}. \tag{3.7}$$

Proof. *See the article [2] to get more details about this proof.* □

Remark 3.1. Observe that $\frac{\mu}{\sigma+\bar{\rho}} - \mu_1$, $\frac{\lambda+\mu}{\sigma+\bar{\rho}} - \mu_2$, $\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\sigma+\bar{\rho})}{(\sigma+\bar{\rho})}$ and $\frac{\bar{\rho}}{\sigma+\bar{\rho}}$are all smooth functions of σ. By virtue of Lemma 3.6, we can get

$$\|\nabla^i\Big(\frac{\mu}{\sigma+\bar{\rho}} - \mu_1\Big)\|_{L^2} \lesssim \|\nabla^i \sigma\|_{L^2}, \quad \|\nabla^i\Big(\frac{\lambda+\mu}{\sigma+\bar{\rho}} - \mu_2\Big)\|_{L^2} \lesssim \|\nabla^i \sigma\|_{L^2},$$

and

$$\|\nabla^i\Big(\frac{P'(\bar{\rho})}{\bar{\rho}} - \frac{P'(\sigma+\bar{\rho})}{(\sigma+\bar{\rho})}\Big)\|_{L^2} \lesssim \|\nabla^i \sigma\|_{L^2}, \quad \|\nabla^i\Big(\frac{\bar{\rho}}{\sigma+\bar{\rho}}\Big)\|_{L^2} \lesssim \|\nabla^i \sigma\|_{L^2}.$$

Conclusion: In this paper, we have studied the optimal decay rates for the highest-order derivatives of solutions for the compressible MHD equations under the influence of Coulomb force. In the future, we will derive the optimal decay rates of the solutions for this model in the non-isentropic case. It's worth pointing out that there is no information about the solutions to the linear systems for this case, and we will address this difficulty in our future work.

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References

[1] Y. Chen, R. Pan and L. Tong, *The sharp time decay rate of the isentropic Navier-Stokes system in $\mathbb{R}^3$,* Electronic Research Archive, 2021, 29(2), 1945–1967.

[2] Q. Chen and Z. Tan, *Time decay of solutions to the compressible Euler equations with damping*, Kinetic & Related Models, 2014, 7(4) 605–619.

[3] Q. Chen, G. Wu and Y. Zhang, *Optimal large time behavior of the compressible bipolar Navier-Stokes-Poisson system*, 2022 21(2), 323–349.

[4] L. He and Y, Zhou, *Weak-strong uniqueness for compressible magnetohydrodynamic equations with Coulomb force*, Advances in Mathematical Physics, 2021, 2021, 1–10.

[5] N. Ju, *Existence and uniqueness of the solution to the dissipative 2D quasi–geostrophic equations in the Sobolev space*, Communications in Mathematical Physics, 2004, 251, 365–376.

[6] F. Jiang, Z. Tan and H. Wang, *A note on global existence of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force*, Journal of mathematical analysis and applications, 2011, 379(1), 316–324.

[7] L. Nirenberg, *On elliptic partial differential equations*, Ann.scuola Norm.sup.pisa, 1959.

[8] Z. Tan and Y. Wang, *Global existence and large-time behavior of weak solutions to the compressible magnetohydrodynamic equations with Coulomb force*, Nonlinear Analysis: Theory, Methods & Applications, 2009, 71(11), 5866–5884.

[9] W. Wang, *Decay estimates for compressible magnetohydrodynamic equations with Coulomb force*, (Chinese) Xiamen Daxue Xuebao Ziran Kexue Ban, 2012, 51, 807–812.

[10] W. Zheng and Z. Tan, *The Decay Estimates for Magnetohydrodynamic Equations with Coulomb Force*, Acta Mathematica Scientia, 2020, 40(6), 1928–1940.

[11] Z, Tan, L, Tong and Y. Wang, *Large time behavior of the compressible magnetohydrodynamic equations with Coulomb force*, Journal of Mathematical Analysis and Applications, 2015, 427(2), 600–617.

[12] Z, Tan, Y. Wang and L, Tong, *The asymptotic behavior of globally smooth solutions to the compressible magnetohydrodynamic equations with Coulomb force*, Analysis and Applications, 2017, 15(4), 571–594.

[13] G. Wu, Y. Zhang and L. Zou, *Optimal large-time behavior of the two-phase fluid model in the whole space*, SIAM Journal on Mathematical Analysis, 2020, 52(6), 5748–5774.

[14] G. Wu, Y. Zhang and W. Zou, *Optimal time-decay rates for the 3D compressible magnetohydrodynamic flows with discontinuous initial data and large oscillations*, Journal of the London Mathematical Society, 2021, 103(3), 817–845.