Journal of Nonlinear Modeling and Analysis
Volume 6, Number 2, June 2024, 320–332
http://jnma-online.com
DOI:10.12150/jnma.2024.320

Well-Posedness of MHD Equations in Sobolev-Gevery Space

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Abstract This paper is devoted to the study of the 3D incompressible magnetohydrodynamic system. We prove the local in time well-posedness for any large initial data in $\dot{H}^1_{a,1}(\mathbb{R}^3)$ or $H^1_{a,1}(\mathbb{R}^3)$. Furthermore, the global well-posedness of a strong solution in $\tilde{L}^\infty(0,T;H^1_{a,1}(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1_{a,1}(\mathbb{R}^3) \cap \dot{H}^2_{a,1}(\mathbb{R}^3))$ with initial data satisfying a smallness condition is established.

Keywords MHD equation, Sobolev-Gevery space, well-posedness

MSC(2010) 35Q35, 35D35, 76W05.

1. Introduction

The magnetohydrodynamic equations reflect the basic physics laws governing the dynamics of electrically conducting fluids. The velocity field obeys the Navier-Stokes equations, and the magnetic field satisfies the Maxwell's equations of electromagnetism. The magnetohydrodynamic equations play important roles in the study of many phenomena in geophysics, astrophysics, and cosmology(see, [1–3]). In this paper, we consider the 3D incompressible magnetohydrodynamic (short written MHD) equations, which can be written as:

$$\begin{cases} \partial_t u - \mu\triangle u + u\cdot\nabla u + \nabla p = b\cdot\nabla b, \\ \partial_t b - \nu\triangle b + u\cdot\nabla b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \nabla\cdot b = 0, \\ u(x,0) = u_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}^3$. We denote $u = u(x,t)$, $b = b(x,t)$ and $p = p(x,t)$ the velocity field, magnetic field and scalar pressure respectively. The constants μ and ν are the viscosity and resistivity coefficient, u_0 and b_0 are the initial velocity field and initial magnetic field satisfying $\nabla\cdot u_0 = 0$, $\nabla\cdot b_0 = 0$. When $b = 0$, equation (1.1) reduces to the classical Navier-Stokes equation, which has been investigated in many exciting results. Leray [4] and Hopf [5] established the global existence of weak solutions. Fujita and Kato [6] obtained the local well-posedness for large initial data and the global well-posedness for small initial data in Sobolev space. And Kato [7] established similar results in $L^n(\mathbb{R}^n)$. Lei and Lin [8] proved the existence of global mild solution with small initial data in the critical space $\chi^{-1}(\mathbb{R}^3)$. Benameur

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and Jlali [16] studied the long time decay of global solution in Sobolev-Gevery spaces $\dot{H}^1_{a,\sigma}(\mathbb{R}^3)$. Sun and Liu [20] proved that if $u \in C([0,+\infty), \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3))$ is a global solution of the 3D fractional Navier-Stokes equation, then $\|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)}$ decays to zero as time approaches infinity. More results about the solutions to the Navier-Stokes equations in Sobolev-Gevery spaces can be found in ([17–19]). For the MHD equation (1.1), there are several important results. Duvaut and Lions [9] constructed a global weak solution and local strong solution. Sermange and Temam [10] established the local well-posedness of equation (1.1) in Sobolev space for any initial data. Chaabani [21] proved the local in time well-posedness for any large initial data in $H^{\frac{1}{2}}_{a,\sigma}(\mathbb{T}^3)$ as well as global in time well-posedness when initial data satisfies a smallness condition. More researches on the well-posedness for MHD euqations can be referred to ([11–14]).

In this paper, we study not only the well-posedness of the local solution in $C([0,T];\dot{H}^1_{a,1}(\mathbb{R}^3))$ and $C([0,T];H^1_{a,1}(\mathbb{R}^3))$ for large initial data but also the well-posedness of the global solution in $\tilde{L}^\infty(0,T;H^1_{a,1}(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1_{a,1}(\mathbb{R}^3) \cap \dot{H}^2_{a,1}(\mathbb{R}^3))$ for small initial data. We use the Banach contraction mapping principle to prove it. Although it is considered to be valid, the construction of the work space in the process of proof is delicate and not easy. We present it in this paper. Our results are stated in the following theorems.

Theorem 1.1. *Consider the MHD equation (1.1) with $\mu > 0$, and $\nu > 0$. Assume $(u_0, b_0) \in \dot{H}^1_{a,1}(\mathbb{R}^3)$ with $divu_0 = divb_0 = 0$. There exists a time $T = T(\|u_0\|_{\dot{H}^1_{a,1}}, \|b_0\|_{\dot{H}^1_{a,1}}) > 0$, such that (1.1) has a unique solution $(u,b) \in C([0,T]; \dot{H}^1_{a,1}(\mathbb{R}^3))$.*

Theorem 1.2. *Consider the MHD equation (1.1) with $\mu > 0$, and $\nu > 0$. Assume $(u_0, b_0) \in H^1_{a,1}(\mathbb{R}^3)$ with $divu_0 = divb_0 = 0$. There exists a time $T = T(\|u_0\|_{H^1_{a,1}}, \|b_0\|_{H^1_{a,1}}) > 0$, such that (1.1) has a unique solution $(u,b) \in C([0,T]; H^1_{a,1}(\mathbb{R}^3))$.*

Theorem 1.3. *Consider the MHD equation (1.1) with $\mu > 0$, and $\nu > 0$. Assume $(u_0, b_0) \in H^1_{a,1}(\mathbb{R}^3)$ with $divu_0 = divb_0 = 0$. There exists a small enough constant $\varepsilon > 0$ such that if $\|(u_0,b_0)\|_{H^1_{a,1}} \le \varepsilon$, then system (1.1) has a unique global solution $(u,b) \in \tilde{L}^\infty(0,T;H^1_{a,1}(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1_{a,1}(\mathbb{R}^3) \cap \dot{H}^2_{a,1}(\mathbb{R}^3))$, for any $T > 0$.*

2. Notations and lemmas

In this section, we first introduce some notations and definitions that will be used later, then we present several tool lemmas which serve as preparation for the proof of our main results.

- The Fourier transformation is defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{R^3} e^{-ix\cdot\xi} f(x)dx.$$

- The homogeneous Sobolev space is defined as

$$\dot{H}^s = \{f \in \mathcal{S}'(\mathbb{R}^3);\ \hat{f} \in L^1_{loc} \text{ and } |\xi|^s \hat{f} \in L^2(\mathbb{R}^3)\}.$$

• The Sobolev-Gevery space is defined as: for $a > 0$, $s \geq 0$, $\sigma \geq 1$,

$$\dot{H}^s_{a,\sigma}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : e^{a\Lambda^{\frac{1}{\sigma}}} f \in \dot{H}^s\},$$

with the norm

$$\|f(t)\|_{\dot{H}^s_{a,\sigma}} = (\int_{R^3} |\xi|^{2s} e^{2a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(t,\xi)|^2 d\xi)^{\frac{1}{2}} = \|e^{a\Lambda^{\frac{1}{\sigma}}} f\|_{\dot{H}^s}.$$

Here the Zygmund operator $\Lambda \triangleq (-\triangle)^{\frac{1}{2}}$ is defined via Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$.

• The Lebesgue-Gevery space $L^p_{a,\sigma}$ is defined as follows. For $1 \leq p \leq \infty$, $a > 0$, $\sigma \geq 1$,

$$L^p_{a,\sigma}(\mathbb{R}^3) = \{f \in L^p(\mathbb{R}^3) : e^{a\Lambda^{\frac{1}{\sigma}}} f \in L^p\},$$

with the norm

$$\|f\|_{L^p_{a,\sigma}} = (\int_{R^3} (e^{a|\xi|^{\frac{1}{\sigma}}} |\hat{f}(t,\xi)|)^p d\xi)^{\frac{1}{p}} = \|e^{a\Lambda^{\frac{1}{\sigma}}} f\|_{L^p}.$$

We shall use the Banach contraction mapping principle which can be found in [15], Lemma 5.5.

Lemma 2.1. *Let E be a Banach space, $\mathcal{B}$ a continuous bilinear map from $E \times E$ to E, and α a positive real number such that*

$$\alpha < \frac{1}{4\|\mathcal{B}\|} \text{ with } \|\mathcal{B}\| = \sup_{\|f\|,\|g\| \leq 1} \|\mathcal{B}(f,g)\|.$$

For any a in the ball $B(0,\alpha)$ (with center 0 and radius α) in E, a unique x then exists in $B(0,2\alpha)$ such that

$$x = a + B(x,x).$$

The following Lemma 2.2 is a Hölder inequality in the Lebesgue-Gevery spaces.

Lemma 2.2. *Let $a > 0$, $\sigma \geq 1$, $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then*

$$\|u \otimes v\|_{L^2_{a,\sigma}(\mathbb{R}^3)} \leq \|u\|_{L^p_{a,\sigma}(\mathbb{R}^3)} \|v\|_{L^q_{a,\sigma}(\mathbb{R}^3)}. \tag{2.1}$$

Proof.

$$\|u \otimes v\|_{L^2_{a,\sigma}} = (\int_\xi e^{2a|\xi|^{\frac{1}{\sigma}}} |\widehat{u \otimes v}(\xi)|^2 \mathrm{d}\xi)^{\frac{1}{2}} \leq (\int_\xi (\int_\eta e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi-\eta)||\widehat{v}(\eta)| \mathrm{d}\eta)^2 \mathrm{d}\xi)^{\frac{1}{2}}. \tag{2.2}$$

Using the inequality $e^{a|\xi|^{\frac{1}{\sigma}}} \leq e^{a|\xi-\eta|^{\frac{1}{\sigma}}} e^{a|\eta|^{\frac{1}{\sigma}}}$, we get

$$\begin{aligned}\|u \otimes v\|_{L^2_{a,\sigma}} &\leq (\int_\xi (\int_\eta e^{a|\xi-\eta|^{\frac{1}{\sigma}}} |\widehat{u}(\xi-\eta)| e^{a|\eta|^{\frac{1}{\sigma}}} |\widehat{v}(\eta)| \mathrm{d}\eta)^2 \mathrm{d}\xi)^{\frac{1}{2}} \\ &\leq \|UV\|_{L^2} \leq \|U\|_{L^p} \|V\|_{L^q} \leq \|u\|_{L^p_{a,\sigma}} \|v\|_{L^q_{a,\sigma}},\end{aligned}$$

where $\widehat{U} = e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)|$, and $\widehat{V} = e^{a|\xi|^{\frac{1}{\sigma}}} |\widehat{u}(\xi)|$. □

The similar argument can also deduce the following Lemma 2.3.

Lemma 2.3. *([17]) For any* $(\alpha,\beta)\in(-\frac{3}{2},\frac{3}{2})$*, a constant* C *exists such that if* $\alpha+\beta$ *is positive, then we have*

$$\|fg\|_{\dot{H}^{\alpha+\beta-\frac{3}{2}}_{a,1}(\mathbb{R}^3)}\le C\|f\|_{\dot{H}^{\alpha}_{a,1}(\mathbb{R}^3)}\|g\|_{\dot{H}^{\beta}_{a,1}(\mathbb{R}^3)}. \tag{2.3}$$

The following Lemma 2.4 is the classical L^p-L^q estimates of heat operator.

Lemma 2.4. *([22]) Let* $s\ge 0$*,* $1\le p\le q\le\infty$*. Then*

$$\left\|\Lambda^s e^{t\Delta}f\right\|_{L^q(\mathbb{R}^d)}\le Ct^{-\frac{s}{2}}t^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{2.4}$$

The following Lemma 2.5 is an interpolation inequality in the Sobolev-Gevery spaces.

Lemma 2.5. *Let* $s_1\le s\le s_2$*,* $a>0$*,* $\sigma\ge 1$*,* $0<\theta<1$*. Then,* $\dot{H}^{s_1}_{a,\sigma}\cap\dot{H}^{s_2}_{a,\sigma}$ *is included in* $\dot{H}^{s}_{a,\sigma}$*, and we have*

$$\|u\|_{\dot{H}^{s}_{a,\sigma}}\le\|u\|^{1-\theta}_{\dot{H}^{s_1}_{a,\sigma}}\|u\|^{\theta}_{\dot{H}^{s_2}_{a,\sigma}}\ \textit{with}\ s=(1-\theta)s_1+\theta s_2.$$

Proof. Using the Sobolev interpolation inequality which can be found in [15], Proposition 1.32, we have

$$\|f\|_{\dot{H}^s}\le\|f\|^{1-\theta}_{\dot{H}^{s_1}}\|f\|^{\theta}_{\dot{H}^{s_2}},\ s=(1-\theta)s_1+\theta s_2.$$

Let $f=e^{a\Lambda^{\frac{1}{\sigma}}}u$, and the Lemma 2.5 is proved. □

The similar argument can also infer the following Lemma 2.6.

Lemma 2.6. *Let* $a>0$*. Then*

$$\|u\|_{L^4_{a,1}}\le\|u\|^{\frac{1}{4}}_{L^2_{a,1}}\|u\|^{\frac{3}{4}}_{\dot{H}^1_{a,1}}. \tag{2.5}$$

Proof. Using the Lebesgue interpolation inequality and $\|f\|_{L^6}\le\|f\|_{\dot{H}^1}$, we have

$$\|f\|_{L^4}\le\|f\|^{\frac{1}{4}}_{L^2}\|f\|^{\frac{3}{4}}_{L^6}\le\|f\|^{\frac{1}{4}}_{L^2}\|f\|^{\frac{3}{4}}_{\dot{H}^1}.$$

Let $f=e^{a\Lambda}u$, and the Lemma 2.6 is proved. □

3. Proof of Theorem 1.1

Taking the integral form of system (1.1), we have

$$u(t,x)=\mathrm{e}^{t\Delta}u_0(x)-\int_0^t\mathrm{e}^{(t-s)\Delta}\mathbb{P}(u\cdot\nabla u)(s)\mathrm{d}s+\int_0^t\mathrm{e}^{(t-s)\Delta}\mathbb{P}(b\cdot\nabla b)(s)\mathrm{d}s, \tag{3.1}$$

$$b(t,x)=\mathrm{e}^{t\Delta}b_0(x)-\int_0^t\mathrm{e}^{(t-s)\Delta}(u\cdot\nabla b)(s)\mathrm{d}s+\int_0^t\mathrm{e}^{(t-s)\Delta}(b\cdot\nabla u)(s)\mathrm{d}s, \tag{3.2}$$

where $\mathbb{P}:=Id-\nabla(-\triangle)^{-1}div$ stands for the Leray projector onto a divergence-free vector field.

We will construct a local solution in $\dot{H}^1_{a,1}(\mathbb{R}^3)$. We first prove the estimate of $\|(u,b)\|_{L^\infty(\dot{H}^1_{a,1})}$. Applying (3.1) and (3.2) by $\Lambda e^{a\Lambda}$, we get

$$\begin{aligned}\Lambda e^{a\Lambda}u(t,x) =&\mathrm{e}^{t\Delta}\Lambda e^{a\Lambda}u_0(x) - \int_0^t \Lambda \mathrm{e}^{(t-s)\Delta}e^{a\Lambda}\nabla\mathbb{P}(u\otimes u)(s)\mathrm{d}s\\ &+\int_0^t \Lambda \mathrm{e}^{(t-s)\Delta}e^{a\Lambda}\nabla\mathbb{P}(b\otimes b)(s)\mathrm{d}s\\ \triangleq& I_1+I_2+I_3,\end{aligned}\tag{3.3}$$

$$\begin{aligned}\Lambda e^{a\Lambda}b(t,x) =&\mathrm{e}^{t\Delta}\Lambda e^{a\Lambda}b_0(x) - \int_0^t \Lambda \mathrm{e}^{(t-s)\Delta}\nabla e^{a\Lambda}(u\otimes b)(s)\mathrm{d}s\\ &+\int_0^t \Lambda \mathrm{e}^{(t-s)\Delta}\nabla e^{a\Lambda}(b\otimes u)(s)\mathrm{d}s\\ \triangleq& J_1+J_2+J_3.\end{aligned}\tag{3.4}$$

Taking the L^2-norm, by Lemma 2.4 with $p=q=2$,

$$\|I_1\|_{L^2}\le\|u_0\|_{\dot{H}^1_{a,1}},$$

$$\|J_1\|_{L^2}\le\|b_0\|_{\dot{H}^1_{a,1}}.$$

By Lemmas 2.4 and 2.3 with $\alpha=\beta=1$, we have

$$\begin{aligned}\|I_2\|_{L^2}&\le\int_0^t\|\Lambda^{\frac32}\mathrm{e}^{(t-s)\Delta}\Lambda^{\frac12}e^{a\Lambda}\mathbb{P}(u\otimes u)(s)\|_{L^2}\,\mathrm{d}s\\ &\le C\int_0^t(t-s)^{-\frac34}\|\Lambda^{\frac12}e^{a\Lambda}\mathbb{P}(u\otimes u)(s)\|_{L^2}\,\mathrm{d}s\\ &\le C\int_0^t(t-s)^{-\frac34}\|u\otimes u\|_{\dot{H}^{\frac12}_{a,1}}\,\mathrm{d}s\\ &\le CT^{\frac14}\|u\|^2_{L^\infty(\dot{H}^1_{a,1})}.\end{aligned}\tag{3.5}$$

By a similar argument it can be deduced

$$\|I_3\|_{L^2}\le CT^{\frac14}\|b\|^2_{L^\infty(\dot{H}^1_{a,1})},\tag{3.6}$$

$$\|J_2\|_{L^2}\le CT^{\frac14}\|u\|_{L^\infty(\dot{H}^1_{a,1})}\|b\|_{L^\infty(\dot{H}^1_{a,1})},\tag{3.7}$$

and

$$\|J_3\|_{L^2}\le CT^{\frac14}\|b\|_{L^\infty(\dot{H}^1_{a,1})}\|u\|_{L^\infty(\dot{H}^1_{a,1})}.\tag{3.8}$$

Thus we obtain

$$\|(u,b)\|_{L^\infty(\dot{H}^1_{a,1})}\le\|(u_0,b_0)\|_{\dot{H}^1_{a,1}}+CT^{\frac14}\|(u,b)\|^2_{L^\infty(\dot{H}^1_{a,1})}.\tag{3.9}$$

By Lemma 2.1, take $\|u_0\|_{\dot{H}^1_{a,1}} + \|b_0\|_{\dot{H}^1_{a,1}} \le \frac{1}{4C_0T^{\frac{1}{4}}}$ for $C_0 > C$. Thus the local existence time $T \le \left[\frac{1}{4C_0(\|u_0\|_{\dot{H}^1_{a,1}}+\|b_0\|_{\dot{H}^1_{a,1}})}\right]^4$, and we construct a local solution on $[0,T]$.

By (3.1) and (3.2), $u(t,x)$ and $b(t,x)$ are continuous with respect to t, and their $\dot{H}^1_{a,1}\left(\mathbb{R}^3\right)$ is bounded. By the Lebesgue dominated convergence theorem, we obtain that $(u,b) \in C\left([0,T];\dot{H}^1_{a,1}\left(\mathbb{R}^3\right)\right)$. The proof of Theorem 1.1 is finished.

4. Proof of Theorem 1.2

We can also construct a local solution in $H^1_{a,1}(\mathbb{R}^3)$. We use the estimate of $\|(u,b)\|_{L^\infty(\dot{H}^1_{a,1})}$ in section 3. Next, we prove the estimate of $\|(u,b)\|_{L^\infty(L^2_{a,1})}$. Applying (3.1) and (3.2) by $e^{a\Lambda}$, we get

$$\begin{aligned} e^{a\Lambda}u(t,x) =& \mathrm{e}^{t\Delta}e^{a\Lambda}u_0(x) - \int_0^t \nabla \mathrm{e}^{(t-s)\Delta}e^{a\Lambda}\mathbb{P}(u\otimes u)(s)\mathrm{d}s \\ &+ \int_0^t \nabla \mathrm{e}^{(t-s)\Delta}e^{a\Lambda}\mathbb{P}(b\otimes b)(s)\mathrm{d}s \\ \triangleq& K_1 + K_2 + +K_3, \end{aligned} \tag{4.1}$$

$$\begin{aligned} e^{a\Lambda}b(t,x) =& \mathrm{e}^{t\Delta}e^{a\Lambda}b_0(x) - \int_0^t \nabla \mathrm{e}^{(t-s)\Delta}e^{a\Lambda}(u\otimes b)(s)\mathrm{d}s \\ &+ \int_0^t \nabla \mathrm{e}^{(t-s)\Delta}e^{a\Lambda}(b\otimes u)(s)\mathrm{d}s \\ \triangleq& M_1 + M_2 + +M_3. \end{aligned} \tag{4.2}$$

Taking the L^2-norm, we have

$$\|K_1\|_{L^2} \le \|u_0\|_{L^2_{a,1}},$$

$$\|M_1\|_{L^2} \le \|b_0\|_{L^2_{a,1}}.$$

By Lemma 2.4 with $p=q=2$ and Lemma 2.3 with $\alpha=\beta=1$, we get

$$\begin{aligned} \|K_2\|_{L^2} &\le \int_0^t \|\Lambda^{\frac{1}{2}}\mathrm{e}^{(t-s)\Delta}\Lambda^{\frac{1}{2}}e^{a\Lambda}\mathbb{P}(u\otimes u)(s)\|_{L^2}\ \mathrm{d}s \\ &\le C\int_0^t (t-s)^{-\frac{1}{4}}\|\Lambda^{\frac{1}{2}}e^{a\Lambda}\mathbb{P}(u\otimes u)(s)\|_{L^2}\ \mathrm{d}s \\ &\le C\int_0^t (t-s)^{-\frac{1}{4}}\|u\otimes u\|_{\dot{H}^{\frac{1}{2}}_{a,1}}\ \mathrm{d}s \\ &\le CT^{\frac{3}{4}}\|u\|^2_{L^\infty(\dot{H}^1_{a,1})}. \end{aligned} \tag{4.3}$$

Arguing similarly to the above inequality (4.3), we obtain

$$\|K_3\|_{L^2} \le CT^{\frac{3}{4}}\|b\|^2_{L^\infty(\dot{H}^1_{a,1})}, \tag{4.4}$$

$$\|M_2\|_{L^2} \le CT^{\frac{3}{4}}\|u\|_{L^\infty(\dot{H}^1_{a,1})}\|b\|_{L^\infty(\dot{H}^1_{a,1})}, \tag{4.5}$$

and

$$\|M_3\|_{L^2} \le CT^{\frac{3}{4}}\|b\|_{L^\infty(\dot{H}^1_{a,1})}\|u\|_{L^\infty(\dot{H}^1_{a,1})}. \tag{4.6}$$

Thus we obtain

$$\|(u,b)\|_{L^\infty(L^2_{a,1})} \le \|(u_0,b_0)\|_{L^2_{a,1}} + CT^{\frac{3}{4}}\|(u,b)\|^2_{L^\infty(\dot{H}^1_{a,1})}. \tag{4.7}$$

Combining the estimates (3.9) and (4.7), we have

$$\|(u,b)\|_{L^\infty(H^1_{a,1})} \le \|(u_0,b_0)\|_{H^1_{a,1}} + C(T^{\frac{3}{4}}+T^{\frac{1}{4}})\|(u,b)\|^2_{L^\infty(H^1_{a,1})}. \tag{4.8}$$

By Lemma 2.1, take $\|u_0\|_{H^1_{a,1}} + \|b_0\|_{H^1_{a,1}} \le \frac{1}{4C_0(T^{\frac{3}{4}}+T^{\frac{1}{4}})}$ for $C_0 > C$. Thus we can obtain the local existence time $T(\|u_0\|_{H^1_{a,1}}, \|b_0\|_{H^1_{a,1}})$, and we construct a local solution on $[0,T]$.

By (3.1) and (3.2), $u(t,x)$ and $b(t,x)$ are continuous with respect to t, and their $H^1_{a,1}(\mathbb{R}^3)$ is bounded. By the Lebesgue dominated convergence theorem, we obtain that $(u,b) \in C([0,T]; H^1_{a,1}(\mathbb{R}^3))$. The proof of Theorem 1.2 is finished.

5. Proof of Theorem 1.3

Taking the Fourier transform to the integral form of system (1.1), we have

$$\begin{aligned}\hat{u}(t,\xi) =& e^{-\mu t|\xi|^2}\hat{u}_0 - \int_0^t \mathrm{e}^{-\mu(t-s)|\xi|^2}\mathbb{P}i\xi\cdot\widehat{u\otimes u}(s)\mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{-\mu(t-s)|\xi|^2}\mathbb{P}i\xi\cdot\widehat{b\otimes b}(s)\mathrm{d}s,\end{aligned} \tag{5.1}$$

$$\begin{aligned}\hat{b}(t,\xi) =& e^{-\nu t|\xi|^2}\hat{b}_0 - \int_0^t \mathrm{e}^{-\nu(t-s)|\xi|^2}i\xi\cdot\widehat{u\otimes b}(s)\mathrm{d}s \\ &+ \int_0^t \mathrm{e}^{-\nu(t-s)|\xi|^2}i\xi\cdot\widehat{b\otimes u}(s)\mathrm{d}s.\end{aligned} \tag{5.2}$$

We first prove the estimate of $\|(u,b)\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}$. Multiplying (5.1) and (5.2) by $|\xi|e^{a|\xi|}$, and summing up the resulting equations, we get

$$\begin{aligned}&|\xi|e^{a|\xi|}|\hat{u}(t,\xi)| + |\xi|e^{a|\xi|}|\hat{b}(t,\xi)| \\ \le& e^{-\mu t|\xi|^2}|\xi|e^{a|\xi|}|\hat{u}_0| + e^{-\nu t|\xi|^2}|\xi|e^{a|\xi|}|\hat{b}_0| + \int_0^t |\xi|\mathrm{e}^{-\mu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{u\otimes u}(s)|\mathrm{d}s \\ &+ \int_0^t |\xi|\mathrm{e}^{-\mu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{b\otimes b}(s)|\mathrm{d}s + \int_0^t |\xi|\mathrm{e}^{-\nu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{u\otimes b}(s)|\mathrm{d}s \\ &+ \int_0^t |\xi|\mathrm{e}^{-\nu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{b\otimes u}(s)|\mathrm{d}s.\end{aligned} \tag{5.3}$$

Taking the L^∞-norm in time to (5.3), by Young inequality for convolution, we conclude that

$$\begin{aligned}
&\sup_{0\le t<\infty}|\xi|e^{a|\xi|}|\hat{u}(t,\xi)|+\sup_{0\le t<\infty}|\xi|e^{a|\xi|}|\hat{b}(t,\xi)| \\
\le&|\xi|e^{a|\xi|}|\hat{u}_0|+|\xi|e^{a|\xi|}|\hat{b}_0|+\frac{1}{\sqrt{2\mu}}\left(\int_0^t|\xi|^2e^{2a|\xi|}|\widehat{u\otimes u}(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\sqrt{2\mu}}\left(\int_0^t|\xi|^2e^{2a|\xi|}|\widehat{b\otimes b}(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2\nu}}\left(\int_0^t|\xi|^2e^{2a|\xi|}|\widehat{u\otimes b}(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\sqrt{2\nu}}\left(\int_0^t|\xi|^2e^{2a|\xi|}|\widehat{b\otimes u}(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}. \qquad (5.4)
\end{aligned}$$

Taking L^2-norm with respect to ξ for (5.4), we obtain

$$\begin{aligned}
&\|u\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}+\|b\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})} \\
\le&\|u_0\|_{\dot{H}^1_{a,1}}+\|b_0\|_{\dot{H}^1_{a,1}}+\frac{1}{\sqrt{2\mu}}\left(\int_0^t\|u\otimes u\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\sqrt{2\mu}}\left(\int_0^t\|b\otimes b\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2\nu}}\left(\int_0^t\|u\otimes b\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\sqrt{2\nu}}\left(\int_0^t\|b\otimes u\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}. \qquad (5.5)
\end{aligned}$$

By Lemma 2.3 with $\alpha=\beta=\frac{5}{4}$ and Lemma 2.5, we have

$$\begin{aligned}
\left(\int_0^t\|u\otimes u\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}&\le C\left(\int_0^t\|u\|^4_{\dot{H}^{\frac{5}{4}}_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C\left(\int_0^t\|u\|^2_{\dot{H}^1_{a,1}}\|u\|_{\dot{H}^1_{a,1}}\|u\|_{\dot{H}^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C\|u\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\left(\int_0^t\|u\|_{\dot{H}^1_{a,1}}\|u\|_{\dot{H}^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C\|u\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|\nabla u\|_{L^2(H^1_{a,1})}. \qquad (5.6)
\end{aligned}$$

Using a similar argument, we deduce that

$$\left(\int_0^t\|b\otimes b\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\le C\|b\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|\nabla b\|_{L^2(H^1_{a,1})}, \qquad (5.7)$$

$$\begin{aligned}
\left(\int_0^t\|u\otimes b\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\le\frac{C}{2}\big(&\|u\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|\nabla u\|_{L^2(H^1_{a,1})} \\
&+\|b\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|\nabla b\|_{L^2(H^1_{a,1})}\big), \qquad (5.8)
\end{aligned}$$

and

$$\left(\int_0^t\|b\otimes u\|^2_{\dot{H}^1_{a,1}}(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\le\frac{C}{2}\big(\|u\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|\nabla u\|_{L^2(H^1_{a,1})}$$

$$+\|b\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|\nabla b\|_{L^2(H^1_{a,1})}\Big). \tag{5.9}$$

Inserting the estimates (5.6)-(5.9) into (5.5), we get

$$\begin{aligned}\|(u,b)\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})} &\le \|(u_0,b_0)\|_{\dot{H}^1_{a,1}}\\ &\quad + max\Big(\frac{C}{\sqrt{2\mu}},\frac{C}{\sqrt{2\nu}}\Big)\|(u,b)\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|(\nabla u,\nabla b)\|_{L^2(H^1_{a,1})}. \end{aligned}\tag{5.10}$$

Next, we prove the estimate of $\|(u,b)\|_{\tilde{L}^\infty(L^2_{a,1})}$. Multiplying (5.1) and (5.2) by $e^{a|\xi|}$, and summing up the resulting equations, we get

$$\begin{aligned}&e^{a|\xi|}|\hat{u}(t,\xi)|+e^{a|\xi|}|\hat{b}(t,\xi)|\\ \le& e^{-\mu t|\xi|^2}e^{a|\xi|}|\hat{u}_0|+e^{-\nu t|\xi|^2}e^{a|\xi|}|\hat{b}_0|+\int_0^t|\xi|\mathrm{e}^{-\mu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{u\otimes u}(s)|\mathrm{d}s\\ &+\int_0^t|\xi|\mathrm{e}^{-\mu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{b\otimes b}(s)|\mathrm{d}s+\int_0^t|\xi|\mathrm{e}^{-\nu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{u\otimes b}(s)|\mathrm{d}s\\ &+\int_0^t|\xi|\mathrm{e}^{-\nu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{b\otimes u}(s)|\mathrm{d}s.\end{aligned}\tag{5.11}$$

Taking the L^∞-norm in time to (5.11), we conclude that

$$\begin{aligned}&\sup_{0\le t<\infty}e^{a|\xi|}|\hat{u}(t,\xi)|+\sup_{0\le t<\infty}e^{a|\xi|}|\hat{b}(t,\xi)|\\ \le& e^{a|\xi|}|\hat{u}_0|+e^{a|\xi|}|\hat{b}_0|+\frac{1}{\sqrt{2\mu}}\left(\int_0^t e^{2a|\xi|}|\widehat{u\otimes u}(s)|^2\,\mathrm{d}s\right)^{\frac12}\\ &+\frac{1}{\sqrt{2\mu}}\left(\int_0^t e^{2a|\xi|}|\widehat{b\otimes b}(s)|^2\,\mathrm{d}s\right)^{\frac12}+\frac{1}{\sqrt{2\nu}}\left(\int_0^t e^{2a|\xi|}|\widehat{u\otimes b}(s)|^2\,\mathrm{d}s\right)^{\frac12}\\ &+\frac{1}{\sqrt{2\nu}}\left(\int_0^t e^{2a|\xi|}|\widehat{b\otimes u}(s)|^2\,\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{5.12}$$

Taking the L^2-norm on both sides of (5.12), we have

$$\begin{aligned}&\|u\|_{\tilde{L}^\infty(L^2_{a,1})}+\|b\|_{\tilde{L}^\infty(L^2_{a,1})}\\ \le& \|u_0\|_{L^2_{a,1}}+\|b_0\|_{L^2_{a,1}}+\frac{1}{\sqrt{2\mu}}\left(\int_0^t\|u\otimes u\|^2_{L^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac12}\\ &+\frac{1}{\sqrt{2\mu}}\left(\int_0^t\|b\otimes b\|^2_{L^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac12}+\frac{1}{\sqrt{2\nu}}\left(\int_0^t\|u\otimes b\|^2_{L^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac12}\\ &+\frac{1}{\sqrt{2\nu}}\left(\int_0^t\|b\otimes u\|^2_{L^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac12}.\end{aligned}\tag{5.13}$$

Using Lemma 2.2 with $p=q=4$ and Lemma 2.6, we have

$$\left(\int_0^t\|u\otimes u\|^2_{L^2_{a,1}}(s)\,\mathrm{d}s\right)^{\frac12}\le C\left(\int_0^t\|u\|^4_{L^4_{a,1}}(s)\,\mathrm{d}s\right)^{\frac12}$$

$$\le C\left(\int_0^t \left(\|u\|_{\dot H^1_{a,1}}^{\frac34}\|u\|_{L^2_{a,1}}^{\frac14}\right)^4(s)\ \mathrm{d}s\right)^{\frac12}$$
$$\le C\|u\|_{\tilde L^\infty(H^1_{a,1})}\left(\int_0^t \|u\|^2_{\dot H^1_{a,1}}(s)\ \mathrm{d}s\right)^{\frac12}$$
$$\le C\|u\|_{\tilde L^\infty(H^1_{a,1})}\|\nabla u\|_{L^2(L^2_{a,1})}. \tag{5.14}$$

By an argument similar to (5.14), we obtain

$$\left(\int_0^t \|b\otimes b\|^2_{L^2_{a,1}}(s)\ \mathrm{d}s\right)^{\frac12} \le C\|b\|_{\tilde L^\infty(H^1_{a,1})}\|\nabla b\|_{L^2(L^2_{a,1})}, \tag{5.15}$$

$$\left(\int_0^t \|u\otimes b\|^2_{L^2_{a,1}}(s)\ \mathrm{d}s\right)^{\frac12} \le \frac{C}{2}\big(\|u\|_{\tilde L^\infty(H^1_{a,1})}\|\nabla u\|_{L^2(L^2_{a,1})} + \|b\|_{\tilde L^\infty(H^1_{a,1})}\|\nabla b\|_{L^2(L^2_{a,1})}\big), \tag{5.16}$$

and

$$\left(\int_0^t \|b\otimes u\|^2_{L^2_{a,1}}(s)\ \mathrm{d}s\right)^{\frac12} \le \frac{C}{2}\big(\|u\|_{\tilde L^\infty(H^1_{a,1})}\|\nabla u\|_{L^2(L^2_{a,1})} + \|b\|_{\tilde L^\infty(H^1_{a,1})}\|\nabla b\|_{L^2(L^2_{a,1})}\big). \tag{5.17}$$

Inserting the estimates (5.14)-(5.17) into (5.13), we get

$$\|(u,b)\|_{\tilde L^\infty(L^2_{a,1})} \le \|(u_0,b_0)\|_{L^2_{a,1}} + max(\frac{C}{\sqrt{2\mu}},\frac{C}{\sqrt{2\nu}})\|(u,b)\|_{\tilde L^\infty(H^1_{a,1})}\|(\nabla u,\nabla b)\|_{L^2(L^2_{a,1})}. \tag{5.18}$$

Combining the estimates (5.10) and (5.18), we have

$$\|(u,b)\|_{\tilde L^\infty(H^1_{a,1})} \le \|(u_0,b_0)\|_{H^1_{a,1}} + max(\frac{C}{\sqrt{2\mu}},\frac{C}{\sqrt{2\nu}})\|(u,b)\|_{\tilde L^\infty(H^1_{a,1})}\|(\nabla u,\nabla b)\|_{L^2(H^1_{a,1})}. \tag{5.19}$$

To close the estimate, we prove the estimate of $\|(\nabla u,\nabla b)\|_{L^2(0,T;H^1_{a,1})}$. Firstly, we estimate $\|(\nabla u,\nabla b)\|_{L^2(0,T;\dot H^1_{a,1})}$ which is equal to $\|(u,b)\|_{L^2(0,T;\dot H^2_{a,1})}$. Multiplying (5.1) and (5.2) by $|\xi|^2 e^{a|\xi|}$, we have

$$\begin{aligned}
&|\xi|^2e^{a|\xi|}|\hat u(t,\xi)| + |\xi|^2e^{a|\xi|}|\hat b(t,\xi)|\\
\le& e^{-\mu t|\xi|^2}|\xi|^2e^{a|\xi|}|\hat u_0| + e^{-\nu t|\xi|^2}|\xi|^2e^{a|\xi|}|\hat b_0| + \int_0^t |\xi|^2\mathrm{e}^{-\mu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{u\otimes u}(s)|\mathrm{d}s\\
&+\int_0^t |\xi|^2\mathrm{e}^{-\mu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{b\otimes b}(s)|\mathrm{d}s + \int_0^t |\xi|^2\mathrm{e}^{-\nu(t-s)|\xi|^2}|\xi|e^{a|\xi|}|\widehat{u\otimes b}(s)|\mathrm{d}s\\
&+\int_0^t |\xi|^2\mathrm{e}^{-\nu(t-s)|\xi|}|\xi|e^{a|\xi|}|\widehat{b\otimes u}(s)|\mathrm{d}s.
\end{aligned} \tag{5.20}$$

Taking the L^2-norm with respect to time t in (5.20), we deduce that

$$\begin{aligned}
&\left(\int_0^t |\xi|^4 e^{2a|\xi|}|\hat{u}(t,\xi)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \left(\int_0^t |\xi|^4 e^{2a|\xi|}|\hat{b}(t,\xi)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
\le&\frac{1}{\sqrt{2\mu}}|\xi|e^{a|\xi|}|\hat{u}_0| + \frac{1}{\sqrt{2\nu}}|\xi|e^{a|\xi|}|\hat{b}_0| + \frac{1}{\mu}\left(\int_0^t |\xi|^2 e^{2a|\xi|}|\widehat{u\otimes u}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\mu}\left(\int_0^t |\xi|^2 e^{2a|\xi|}|\widehat{b\otimes b}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \frac{1}{\nu}\left(\int_0^t |\xi|^2 e^{2a|\xi|}|\widehat{u\otimes b}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\nu}\left(\int_0^t |\xi|^2 e^{2a|\xi|}|\widehat{b\otimes u}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}. \qquad (5.21)
\end{aligned}$$

Taking the L^2-norm with respect to ξ on both sides of (5.21), by an argument similar to (5.10), we have

$$\begin{aligned}
\|(\nabla u,\nabla b)\|_{L^2(\dot{H}^1_{a,1})} \le& max(\frac{1}{\sqrt{2\mu}},\frac{1}{\sqrt{2\nu}})\,\|(u_0,b_0)\|_{\dot{H}^1_{a,1}} \\
&+ max(\frac{C}{\mu},\frac{C}{\nu})\|(u,b)\|_{\tilde{L}^\infty(\dot{H}^1_{a,1})}\|(\nabla u,\nabla b)\|_{L^2(H^1_{a,1})}. \qquad (5.22)
\end{aligned}$$

In the following, we estimate the norm $\|(\nabla u,\nabla b)\|_{L^2(0,T;L^2_{a,1})}$ which is equal to the norm $\|(u,b)\|_{L^2(0,T;\dot{H}^1_{a,1})}$. Multiplying (5.1) and (5.2) by $|\xi|e^{a|\xi|}$, and summing up the resulting equations, we get

$$\begin{aligned}
&|\xi|e^{a|\xi|}|\hat{u}(t,\xi)| + |\xi|e^{a|\xi|}|\hat{b}(t,\xi)| \\
\le&e^{-\mu t|\xi|^2}|\xi|e^{a|\xi|}|\hat{u}_0| + e^{-\nu t|\xi|^2}|\xi|e^{a|\xi|}|\hat{b}_0| + \int_0^t |\xi|^2 \mathrm{e}^{-\mu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{u\otimes u}(s)|\mathrm{d}s \\
&+\int_0^t |\xi|^2 \mathrm{e}^{-\mu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{b\otimes b}(s)|\mathrm{d}s + \int_0^t |\xi|^2 \mathrm{e}^{-\nu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{u\otimes b}(s)|\mathrm{d}s \\
&+\int_0^t |\xi|^2 \mathrm{e}^{-\nu(t-s)|\xi|^2}e^{a|\xi|}|\widehat{b\otimes u}(s)|\mathrm{d}s. \qquad (5.23)
\end{aligned}$$

Taking the L^2-norm with respect to time t in (5.23), we deduce that

$$\begin{aligned}
&\left(\int_0^t |\xi|^2 e^{2a|\xi|}|\hat{u}(t,\xi)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \left(\int_0^t |\xi|^2 e^{2a|\xi|}|\hat{b}(t,\xi)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
\le&\frac{1}{\sqrt{2\mu}}e^{a|\xi|}|\hat{u}_0| + \frac{1}{\sqrt{2\nu}}e^{a|\xi|}|\hat{b}_0| + \frac{1}{\mu}\left(\int_0^t e^{2a|\xi|}|\widehat{u\otimes u}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\mu}\left(\int_0^t e^{2a|\xi|}|\widehat{b\otimes b}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} + \frac{1}{\nu}\left(\int_0^t e^{2a|\xi|}|\widehat{u\otimes b}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&+\frac{1}{\nu}\left(\int_0^t e^{2a|\xi|}|\widehat{b\otimes u}(s)|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}. \qquad (5.24)
\end{aligned}$$

Taking the L^2-norm on both sides of (5.24), arguing similarly to (5.18), we have

$$\|(\nabla u,\nabla b)\|_{L^2(L^2_{a,1})} \le max(\frac{1}{\sqrt{2\mu}},\frac{1}{\sqrt{2\nu}})\,\|(u_0,b_0)\|_{L^2_{a,1}}$$

DOI https://doi.org/10.12150/jnma.2024.320 | Generated on 2024-12-19 04:05:34

$$+ max(\frac{C}{\mu}, \frac{C}{\nu})\|(u,b)\|_{\tilde{L}^\infty(H^1_{a,1})}\|(\nabla u, \nabla b)\|_{L^2(L^2_{a,1})}. \quad (5.25)$$

Combining the estimates (5.22) and (5.25), it arrives at

$$\begin{aligned}\|(\nabla u, \nabla b)\|_{L^2(H^1_{a,1})} \le & max(\frac{1}{\sqrt{2\mu}}, \frac{1}{\sqrt{2\nu}})\,\|(u_0,b_0)\|_{H^1_{a,1}} \\ & + max(\frac{C}{\mu}, \frac{C}{\nu})\|(u,b)\|_{\tilde{L}^\infty(H^1_{a,1})}\|(\nabla u, \nabla b)\|_{L^2(H^1_{a,1})}. \quad (5.26)\end{aligned}$$

Combining the estimates (5.19) and (5.26), we have

$$\begin{aligned}& \|(u,b)\|_{\tilde{L}^\infty(H^1_{a,1})} + \|(\nabla u, \nabla b)\|_{L^2(H^1_{a,1})} \\ \le & max(1, \frac{1}{\sqrt{2\mu}}, \frac{1}{\sqrt{2\nu}})\,\|(u_0,b_0)\|_{H^1_{a,1}} \\ & + Cmax(\frac{1}{\sqrt{2\mu}}, \frac{1}{\mu}, \frac{1}{\sqrt{2\nu}}, \frac{1}{\nu})\|(u,b)\|_{\tilde{L}^\infty(H^1_{a,1})}\|(\nabla u, \nabla b)\|_{L^2(H^1_{a,1})}. \quad (5.27)\end{aligned}$$

Applying Lemma 2.1, if we take the initial data small enough,

$$\|(u_0,b_0)\|_{H^1_{a,1}} \le \frac{1}{4C_0 max(1, \frac{1}{\sqrt{2\mu}}, \frac{1}{\sqrt{2\nu}}) max(\frac{1}{\sqrt{2\mu}}, \frac{1}{\mu}, \frac{1}{\sqrt{2\nu}}, \frac{1}{\nu})} = \varepsilon,$$

for $C_0 > C$, then we have a unique global solution $(u,b) \in \tilde{L}^\infty\left(0,T; H^1_{a,1}\left(\mathbb{R}^3\right)\right)$ $\cap L^2\left(0,T; \dot{H}^1_{a,1}\left(\mathbb{R}^3\right) \cap \dot{H}^2_{a,1}\left(\mathbb{R}^3\right)\right)$, thus we complete the proof of Theorem 1.3.

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