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Existence and Decay of Global Strong Solution to 3D Density-Dependent Boussinesq Equations with Vacuum*

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Abstract This paper is concerned with the initial boundary problem for the three-dimensional density-dependent Boussinesq equations with vacuum. We obtain the existence of the global strong solution under the initial density in the norm L^∞ is small enough without any smallness condition of u and θ. Furthermore, the exponential decay rates of the solution and their derivatives in some norm was established. In addition, we show that the solution and their derivatives are monotonically decreasing with respect to time t on $[0, T]$.

Keywords Boussinesq equation, vacuum, global strong solution, exponential decay-in-time

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1. Introduction

The density-dependent Boussinesq equations with vacuum were presented as

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P - \mathrm{div}(\mu(\rho, \theta)\nabla u) = \rho\theta e_3, \\ \rho\theta_t + \rho u \cdot \nabla\theta = \mathrm{div}(\kappa(\rho, \theta)\nabla\theta), \\ \mathrm{div} u = 0 \end{cases}$$

in $\Omega \in \mathbb{R}^3$, where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the fluid velocity vector field, $P(x,t)$, $\rho(x,t)$ and $\theta(x,t)$ are the scalar pressure, density and temperature, respectively. $e_3 = (0, 0, 1)$. The constants μ and κ are the viscosity and the thermal diffusivity, respectively.

The Boussinesq equation [2,4,10] is an important model in mathematics physics. This system describes the influence of the convection phenomena on the dynamics of the ocean or the atmosphere. Fan and Ozawa [3] obtained the local existence

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of the strong solution to the Cauchy problem for the system (1.1)-(1.2) in $\mathbb{R}^3$, and they also established some blow-up criteria $u \in L^{2/(1-r)}(0,T;\dot{X}_r)$, $(0 < r < 1)$ or $u \in L^{2q/(q-3)}(0,T;L^q)$, $3 < q \leq \infty$. Later, Zhang [15] proved the regularity criterion in BMO space $u \in L^2(0,T;BMO)$. In [11], they established the local wellposedness for the incompressible Boussinesq system without dissipation terms under the framework of the Besov spaces in dimension $N \geq 2$. They also obtained a Beale-Kato-Majda type regularity criterion. Zhong [17] considered the Cauchy problem of the 2D density-dependent Boussinesq equations without a dissipation term in the temperature equation with vacuum as far field density. He proved that there exists a unique local strong solution provided the initial density and the initial temperature decay not too slow at infinity. Global well-posedness of two-dimensional density-dependent boussinesq equations with large initial data and vacuum was investigated by Zhong in [18]. In [12], Ye and Zhu got the zero limit of thermal diffusivity for the 2D density-dependent Boussinesq equations with vacuum.

When $\rho = C$, system (1.1) reduces to the classical homogeneous incompressible Boussinesq system which is widely studied. Chae [1](see also [8]) proved the global in time regularity for the 2D Boussinesq system with either the zero diffusivity or the zero viscosity. He [6] studied the blow-up criterion of classical solution to the Boussinesq equations with temperature-dependent viscosity and zero thermal diffusivity in $\mathbb{R}^2$ and $\mathbb{R}^3$. Larios and Pei [9] studied the local well-posedness of solutions to the 3D Boussinesq-MHD system. Some regularity criteria were also investigated in [9]. Later, Zhao [16] investigated the well-posedness of the Cauchy problem to the Boussinesq-MHD system with partial viscosity and zero magnetic diffusion.

Inspired by [5, 13, 14], we consider the following density-dependent Boussinesq equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P - \operatorname{div}(\mu(\rho,\theta)\nabla u) = \rho\theta e_3, \\ \rho\theta_t + \rho u \cdot \nabla\theta = \operatorname{div}(\kappa(\rho,\theta)\nabla\theta), \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

where $\mu(\rho,\theta)$ and $\kappa(\rho,\theta)$ are all function of ρ and θ, which are assumed to satisfy

$$(\mu(\rho,\theta),\kappa(\rho,\theta)) \in C^1[0,\infty), 0 < \underline{\kappa} \leq \kappa(\rho,\theta) \leq C < \infty, 0 < \underline{\mu} \leq \mu(\rho,\theta) \leq C < \infty, \tag{1.2}$$

and

$$(\mu_\rho(\rho,\theta), \mu_\theta(\rho,\theta), \kappa_\rho(\rho,\theta), \kappa_\theta(\rho,\theta)) \leq C \tag{1.3}$$

for some positive constants $\underline{\mu}$, $\underline{\kappa}$ and C.

The initial and boundary conditions satisfy that

$$(\rho,u,\theta)|_{t=0} = (\rho_0,u_0,\theta_0)(x), \quad x \in \Omega; \quad (u,\theta)|_{x\in\partial\Omega} = 0. \tag{1.4}$$

Our main purpose is to study the existence of the global strong solution to the initial boundary value problem of (1.1)-(1.3). Now, we present our results as follows:

Theorem 1.1. *Assume that the initial data* (ρ_0,u_0,θ_0) *satisfies*

$$0 \leq \rho_0 \leq \bar{\rho}, \ \nabla\rho_0 \in L^p(p>3), \ (u_0,\theta_0) \in H_0^1 \cap H^2$$

and the compatibility condition

$$-\mathrm{div}(\mu(\rho_0,\theta_0)\nabla u_0)+\nabla P_0=\rho_0^{1/2}f,\quad -\mathrm{div}(\kappa(\rho_0,\theta_0)\nabla\theta_0)=\rho_0^{1/2}g$$

for some $(P_0,f,g)\in H^1\times L^2\times L^2$. *Then there exists a small positive constant* ε_0, *depending on* Ω, p, q, $\mu(\rho,\theta)$, $\kappa(\rho,\theta)$, f, g, $\|\nabla\rho_0\|_{L^p}$, $\|\sqrt{\rho_0}u_0\|_{L^2}$, $\|\nabla u_0\|_{L^2}$, $\|\sqrt{\rho_0}\theta_0\|_{L^2}$, $\|\nabla\theta_0\|_{L^2}$, *such that if* $\bar\rho\le\varepsilon_0$, *the initial boundary value problem* (1.1)-(1.4) *has a global strong solution satisfying*

$$\begin{cases}0\le\rho\le\bar\rho,\quad \nabla\rho\in L^\infty(0,T;L^p),\\ (\sqrt{\rho}u,\sqrt{\rho}\theta,\nabla u,\nabla\theta,\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\in L^\infty(0,T;L^2),\\ (\nabla u,\nabla\theta,\sqrt{\rho}u_t,\sqrt{\rho}\theta_t,\nabla u_t,\nabla\theta_t)\in L^2(0,T;L^2)\end{cases}$$

and the following decay rates

$$\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)(t)\le Ce^{-C_1t},\qquad \textit{for all}\quad t>0$$

and

$$\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)(t)\le Ce^{-C_2t},\qquad \textit{for all}\quad t>0.$$

Remark 1.1. If the smoothness condition of the given initial condition is higher, we can obtain the stronger regularity and the decay estimate of the higher-order derivative of the solution for the problem (1.1)-(1.4).

Remark 1.2. With the method of reference [7], we can get similar results of Theorem (1.1) for the Cauchy problem (1.1)-(1.2) in $\mathbb{R}^3$.

2. Preliminaries

To derive the estimates of the derivatives of the solutions, we need the following Lemmas, whose proof can be proved using the similar method as in [5,13].

Lemma 2.1. *For any* $3<p<\infty$, *assume that* $\mu(\rho,\theta)\in W^{1,p}$ *satisfies* (1.4) *with* $0\le\rho\le\bar\rho$, $\theta\le C$. *Let* $(u,P)\in H^1_{0,\sigma}\times L^2$ *be the unique weak solution to the problem:*

$$-\mathrm{div}(\mu(\rho,\theta)\nabla u)+\nabla P=F,\quad \mathrm{div}u=0\quad in\quad \Omega,\quad \int Pdx=0.$$

There exists a generic positive constant C, *depending only on* Ω,p,r *and* $\mu(\rho,\theta)$, *such that*
(i) if $F\in L^2$, *then* $(u,P)\in H^2\times H^1$ *and*

$$\|u\|_{H^2}+\|P/\mu(\rho,\theta)\|_{H^1}\le C\left(1+\|\nabla\mu(\rho,\theta)\|_{L^p}^{\alpha_2}\right)\|F\|_{L^2};\tag{2.1}$$

(ii) if $F\in L^r$ *for some* $r\in(3,p)$, *then* $(u,P)\in W^{2,r}\times W^{1,r}$ *and*

$$\|u\|_{W^{2,r}}+\|P/\mu(\rho,\theta)\|_{W^{1,r}}\le C\left(1+\|\nabla\mu(\rho,\theta)\|_{L^p}^{\alpha_r}\right)\|F\|_{L^r},\tag{2.2}$$

where

$$\alpha_2=\frac{p}{p-3}\quad and\quad \alpha_r=\frac{(5r-6)p}{2r(p-3)}.$$

3. Proof of Theorem 1.1

Proposition 3.1. *Under the conditions of Theorem 1.1, if* (ρ, u, θ) *is a smooth solution of* (1.1)-(1.4) *satisfying*

$$\sup_{0\le t\le T}\|\nabla\theta\|_{L^p}\le 2K_1,\quad \sup_{0\le t\le T}\|\nabla\rho\|_{L^p}\le 3K_2, \tag{3.1}$$

$$\sup_{0\le t\le T}\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\kappa(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)+\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\le 3K_3, \tag{3.2}$$

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\theta_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\right)dt\le 3K_4, \tag{3.3}$$

then the following estimates hold

$$\sup_{0\le t\le T}\|\nabla\theta\|_{L^p}\le K_1,\quad \sup_{0\le t\le T}\|\nabla\rho\|_{L^p}\le 2K_2,$$

$$\sup_{0\le t\le T}\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\kappa(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)+\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\le 2K_3,$$

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\min\{\underline{\mu},\underline{\kappa}\}\int_0^T\left(\|\nabla\theta_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\right)dt\le 2K_4$$

under the initial density is small enough. Here, K_1 *is a positive constant,*

$$K_3\triangleq\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\mu(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)\Big|_{t=0},$$

$$K_2\triangleq\|\nabla\rho_0\|_{L^p},\qquad K_4\triangleq\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)\Big|_{t=0}.$$

Lemma 3.1. *Under the conditions of Proposition 3.1, let* (ρ,u,θ) *be a strong solution of* (1.1)-(1.4) *on* $\Omega\times(0,T)$. *Then*

$$\|\rho\|_{L^\infty}\le\|\rho_0\|_{L^\infty}\le\bar{\rho}, \tag{3.4}$$

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)+\int_0^T e^{C_1t}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)dt\le Ce^{-C_1t} \tag{3.5}$$

and $\|(\sqrt{\rho}u)(t)\|_{L^2}^2+\|(\sqrt{\rho}\theta)(t)\|_{L^2}^2$ *is decreasing on [0,T].*

Proof. Firstly, it is easy to deduce (3.4), thus the process of the proof is omitted here. Then, multiplying $(1.1)_2$,$(1.1)_3$ by u and θ in L^2, respectively, integrating it by parts, by Cauchy-Schwarz, Poincaré inequality, (3.2), (3.4) and choosing $\bar{\rho}$ small enough, we immediately get

$$\frac{d}{dt}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)+C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\le 0, \tag{3.6}$$

which shows that $\|(\sqrt{\rho}u)(t)\|_{L^2}^2+\|(\sqrt{\rho}\theta)(t)\|_{L^2}^2$ is decreasing on [0,T]. And by Pioncaré inequality, it is easy to get

$$\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\le\bar{\rho}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right).$$

This, combined with (3.6), and by choosing $\bar{\rho}$ to be sufficiently small, yields

$$\begin{aligned}&\frac{d}{dt}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)\\&+C_1\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)+C\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\le 0.\end{aligned}$$

Multiplying the above inequality by e^{C_1t}, we have

$$\frac{d}{dt}\left(e^{C_1t}\left(\|\sqrt{\rho}u\|_{L^2}^2+\|\sqrt{\rho}\theta\|_{L^2}^2\right)\right)+Ce^{C_1t}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2\right)\le 0. \tag{3.7}$$

Integrating (3.7) over $[0,T]$, the estimate of (3.5) is obtained. Hence, we finish the proof of Lemma 3.1. □

Lemma 3.2. *Under the conditions of Proposition 3.1, let (ρ,u,θ) be a strong solution of (1.1)-(1.4) on $\Omega\times(0,T)$. Then*

$$\sup_{0\le t\le T}\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\kappa(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)+\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\le 2K_3.$$

Proof. Multiplying $(1.1)_1$, $(1.1)_2$ by u_t and θ_t in L^2, respectively, integrating by parts, we infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int\mu(\rho,\theta)|\nabla u|^2dx+\int\kappa(\rho,\theta)|\nabla\theta|^2dx\right)+\int\left(\rho|u_t|^2+\rho|\theta_t|^2\right)dx\\=&\frac{1}{2}\int\left(\mu_t(\rho,\theta)|\nabla u|^2+\kappa_t(\rho,\theta)|\nabla\theta|^2\right)dx-\int\rho u\cdot\nabla u\cdot u_tdx\\&-\int\rho u\cdot\nabla\theta\theta_tdx+\int\rho\theta e_3\cdot u_tdx\triangleq\sum_{i=1}^4 I_i.\end{aligned} \tag{3.8}$$

Using Sobolev inequality, Poincaré inequality, (3.1), (3.2), (3.4) and $(1.1)_1$, we have the following inequalities:

$$\begin{aligned}I_1=&\frac{1}{2}\int\left[\left(\frac{\partial\mu}{\partial\rho}\rho_t+\frac{\partial\mu}{\partial\theta}\theta_t\right)|\nabla u|^2+\left(\frac{\partial\kappa}{\partial\rho}\rho_t+\frac{\partial\kappa}{\partial\theta}\theta_t\right)|\nabla\theta|^2\right]dx\\\le& C\|\nabla\rho\|_{L^p}\|u\|_{L^3}\left(\|\nabla u\|_{L^{6p/(2p-3)}}^2+\|\nabla\theta\|_{L^{6p/(2p-3)}}^2\right)\\&+\|\theta_t\|_{L^6}\left(\|\nabla u\|_{L^2}\|\nabla u\|_{L^3}+\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{L^3}\right)\\\le& C\left(\|\nabla u\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2\right)+C\|\nabla\theta_t\|_{L^2}\left(\|\nabla u\|_{H^1}+\|\nabla\theta\|_{H^1}\right),\end{aligned} \tag{3.9}$$

$$\begin{aligned}I_2+I_3\le& C\left(\bar{\rho}^{1/2}\|\sqrt{\rho}u_t\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^3}+\bar{\rho}^{1/2}\|\sqrt{\rho}\theta_t\|_{L^2}\|u\|_{L^6}\|\nabla\theta\|_{L^3}\right)\\\le&\frac{1}{4}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+C\bar{\rho}\left(\|\nabla u\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2\right)\end{aligned} \tag{3.10}$$

and

$$I_4\le\frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2+\bar{\rho}\|\theta\|_{L^2}^2\le\frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2+\bar{\rho}\|\nabla\theta\|_{L^2}^2. \tag{3.11}$$

One derives from (2.1), (3.1) and Poincaré inequality that

$$\|\nabla u\|_{H^1}^2$$

$$
\begin{aligned}
&\leq C\left(1+\|\nabla\rho\|_{L^p}^{2p/(p-3)}+\|\nabla\theta\|_{L^p}^{2p/(p-3)}\right)\left(\bar\rho^{1/2}\|\sqrt{\rho}u_t\|_{L^2}^2+\bar\rho\|u\cdot\nabla u\|_{L^2}^2+\bar\rho\|\theta\|_{L^2}^2\right)\\
&\leq C\left(\bar\rho^{1/2}\|\sqrt{\rho}u_t\|_{L^2}^2+\bar\rho\|\nabla u\|_{H^1}^2+\bar\rho\|\nabla\theta\|_{L^2}^2\right).
\end{aligned}\tag{3.12}
$$

In a similar manner, for equation $(1.1)_3$, we conclude that

$$
\begin{aligned}
\|\nabla\theta\|_{H^1}^2 &\leq C(1+\|\nabla\rho\|_{L^p}^{2p/(p-3)})\left(\bar\rho^{1/2}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\bar\rho\|u\cdot\nabla\theta\|_{L^2}^2\right)\\
&\leq C\left(\bar\rho^{1/2}\|\sqrt{\rho}\theta_t\|_{L^2}^2+\bar\rho\|\nabla\theta\|_{H^1}^2\right),
\end{aligned}
$$

which together with (3.4), (3.12) and choosing $\bar\rho$ small enough, yields

$$
\|\nabla u\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2\leq C\bar\rho^{1/2}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right).\tag{3.13}
$$

Substituting (3.9)-(3.11) and (3.13) into (3.8), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\kappa(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)+\frac{1}{2}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)\\
&\leq C\bar\rho^{1/2}\|\nabla\theta_t\|_{L^2}^2,
\end{aligned}\tag{3.14}
$$

which integrated respect to t over $(0,T)$, yields

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\kappa(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)+\int_0^T\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)dt\\
&\qquad\leq K_3+C_1\bar\rho^{1/2}.
\end{aligned}
$$

Now, we can take $\bar\rho$ sufficiently small, such that

$$
\sup_{0\leq t\leq T}\left(\|\sqrt{\mu(\rho,\theta)}\nabla u\|_{L^2}^2+\|\sqrt{\kappa(\rho,\theta)}\nabla\theta\|_{L^2}^2\right)+\int_0^T\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2dt\leq 2K_3.
$$

$\square$

Lemma 3.3. *Under the conditions of Proposition 3.1, let* (ρ,u,θ) *be a strong solution of* (1.1)-(1.4) *on* $\Omega\times(0,T)$. *Then*

$$
\sup_{0\leq t\leq T}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\int_0^T\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)dt\leq 2K_4.\tag{3.15}
$$

Proof. Differentiating $(1.1)_{1,2}$ with respect to t, and multiplying them by u_t and θ_t in L^2, respectively, it has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\int\left(\mu(\rho,\theta)|\nabla\theta_t|^2+\kappa(\rho,\theta)|\nabla u_t|^2\right)dt\\
&=-2\int\left(\rho u\cdot u_t\cdot\nabla u_t+\rho u\cdot\nabla\theta_t\theta_t\right)dx-\int\rho u_t\cdot\left(\nabla u\cdot u_t+\nabla\theta\theta_t\right)dx\\
&\quad-\int\rho u\cdot\left(\nabla(u\cdot\nabla u\cdot u_t)+\nabla(u\cdot\nabla\theta\theta_t)\right)dx\\
&\quad-\int\left(\mu_t(\rho,\theta)\cdot\nabla u\cdot\nabla u_t+\kappa_t(\rho,\theta)\cdot\nabla\theta\cdot\nabla\theta_t\right)dx+\int(\rho\theta)_te_3\cdot u_tdx
\end{aligned}
$$

$$\triangleq \sum_{i=0}^{5} M_i. \tag{3.16}$$

Using (3.1) (3.4), (3.3), $(1.1)_1$, Sobolev inequality and Poincaré inequality, we estimate each $M_i (i=1,...,5)$ in the following way

$$\begin{aligned}M_1+M_2 \leq& C\left(\bar{\rho}\|u\|_{L^6}\left(\|u_t\|_{L^3}\|\nabla u_t\|_{L^2}+\|\theta_t\|_{L^3}\|\nabla\theta_t\|_{L^2}\right)\right.\\&\left.+\bar{\rho}\|u_t\|_{L^3}\left(\|\nabla u\|_{L^2}\|u_t\|_{L^6}+\|\nabla\theta\|_{L^2}\|\theta_t\|_{L^6}\right)\right)\\ \leq& C\bar{\rho}^2\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right),\end{aligned} \tag{3.17}$$

$$\begin{aligned}M_3 \leq& C\bar{\rho}\|u\|_{L^6}\left(\|\nabla u\|_{L^3}^2\|u_t\|_{L^6}+\|\nabla\theta\|_{L^3}^2\|\theta_t\|_{L^6}\right)+C\bar{\rho}\|u\|_{L^6}^2\left(\|\nabla^2 u\|_{L^2}\|u_t\|_{L^6}\right.\\&\left.+\|\nabla^2\theta\|_{L^2}\|\theta_t\|_{L^6}+\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2}+\|\nabla\theta\|_{L^6}\|\nabla\theta_t\|_{L^2}\right)\\ \leq& C\bar{\rho}\left(\|\nabla u\|_{H^1}^4+\|\nabla u\|_{H^1}^2\|\nabla\theta\|_{H^1}^2+\|\nabla\theta\|_{H^1}^4\right)\\&+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\\ \leq& C\bar{\rho}\left(\|\sqrt{\rho}u_t\|_{L^2}^4+\|\sqrt{\rho}\theta_t\|_{L^2}^4\right)+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\\ \leq& C\bar{\rho}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right),\end{aligned} \tag{3.18}$$

$$\begin{aligned}M_4 =& \int\left(\left(\mu_\rho\rho_t+\mu_\theta\theta_t\right)\cdot\nabla u\cdot\nabla u_t+\left(\kappa_\rho\rho_t+\kappa_\theta\theta_t\right)\cdot\nabla\theta\cdot\nabla\theta_t\right)dt\\ \leq& C\|u\|_{L^\infty}\|\nabla\rho\|_{L^p}\left(\|\nabla u\|_{L^{2p/(p-2)}}\|\nabla u_t\|_{L^2}+\|\nabla\theta\|_{L^{2p/(p-2)}}\|\nabla\theta_t\|_{L^2}\right)\\&+C\|\theta_t\|_{L^6}(\|\nabla u\|_{L^3}+\|\nabla\theta\|_{L^3})(\|\nabla u_t\|_{L^2}+\|\nabla\theta_t\|_{L^2})\\ \leq& C\left(\|\nabla u\|_{H^1}^2+\|\nabla\theta\|_{H^1}^2\right)\left(\|\nabla u\|_{H^1}^2+\nabla\theta_t\|_{L^2}^2\right)\\&+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\\ \leq& C\bar{\rho}^{1/2}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)\left(\|\nabla u\|_{H^1}^2+\nabla\theta_t\|_{L^2}^2\right)\\&+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\\ \leq& C\bar{\rho}^{1/2}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\left(\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right),\end{aligned} \tag{3.19}$$

$$\begin{aligned}M_5 =& \int\rho u\cdot\nabla(\theta e_3\cdot u_t)dx+\int\rho\theta_t e_3\cdot u_t dx\\ \leq& C\bar{\rho}\left(\|u\|_{L^3}\|\nabla\theta\|_{L^2}\|u_t\|_{L^6}+\|u\|_{L^3}\|\theta\|_{L^6}\|\nabla u_t\|_{L^2}+\|\nabla\theta_t\|_{L^2}\|\nabla u_t\|_{L^2}\right)\\ \leq& C\bar{\rho}^2\left(\|\nabla u\|_{L^2}^2\|\nabla\theta\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\|\nabla u_t\|_{L^2}^2\\ \leq& C\bar{\rho}^2\|\nabla\theta_t\|_{L^2}^2+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{8}\|\nabla u_t\|_{L^2}^2.\end{aligned} \tag{3.20}$$

Substituting (3.17)-(3.20) into (3.16) and choosing $\bar{\rho}$ sufficiently small, yield

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)+\frac{\min\{\underline{\mu},\underline{\kappa}\}}{2}\left(\|\nabla\theta_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\right)\le 0. \tag{3.21}$$

Integrating the above equality about t over [0,T], we immediately deduce the result as in Lemma 3.3. □

Lemma 3.4. *Under the conditions of Proposition 3.1, let (ρ,u,θ) be a strong solution of (1.1)-(1.4) on $\Omega\times(0,T)$. Then*

$$\sup_{0\le t\le T}\|(\nabla u,\nabla\theta,\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\|_{L^2}^2+\int_0^T e^{C_2t}\|(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t,\nabla u_t,\nabla\theta_t)\|_{L^2}^2dt\le Ce^{-C_2t} \tag{3.22}$$

and $\|(\nabla u,\nabla\theta,\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)(t)\|_{L^2}^2$ is monotonically decreasing on [0,T].

Proof. Summing up the inequalities of (3.14) and (3.21), we obtain from (1.4) that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)\\&\quad+C\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\le 0\end{aligned} \tag{3.23}$$

It follows from the above inequality that

$$\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)(t)$$

is decreasing on $[0,T]$. Due to (3.13), by Poincaré inequality, we get

$$\begin{aligned}&\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\\&\le C\bar{\rho}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right).\end{aligned} \tag{3.24}$$

From (3.23), (3.24), choosing $\bar{\rho}$ appropriately small, we deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)\\&\quad+C_2\left(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2+\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)\\&\quad+C\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^2\right)\le 0\end{aligned}$$

which is multiplied by e^{C_2t}. Integrating it about t, it is easy to deduce the result of Lemma 3.4. □

Lemma 3.5. *Under the conditions of Proposition 3.1, let (ρ,u,θ) be a strong solution of (1.1)-(1.4) on $\Omega\times(0,T)$. Then*

$$\int_0^T\left(\|\nabla u\|_{L^\infty}+\|\nabla u\|_{W^{1,p}}\right)dt\le C\max\{\bar{\rho}^{(5r-6)/4r},\bar{\rho}\}. \tag{3.25}$$

Proof. It infers from Sobolev inequality, Pioncaré inequality, (2.2), (3.15) and (3.22) that

$$\int_0^T\|\nabla u\|_{L^\infty}dt\le C\int_0^T\|\nabla u\|_{W^{1,p}}dt\le C\int_0^T\|\rho u_t\|_{L^r}+\|\rho\theta_t\|_{L^r}+\|\rho u\cdot\nabla u\|_{L^r}dt$$

$$
\begin{aligned}
&\leq C\int_0^T \|\rho u_t\|_{L^2}^{(6-r)/2r}\|\rho u_t\|_{L^6}^{(3r-6)/2r}+\bar\rho\|\rho\theta_t\|_{L^2}^{(6-r)/2r}\|\rho\theta_t\|_{L^6}^{(3r-6)/2r}\\
&\quad+\bar\rho\|u\cdot\nabla u\|_{L^r}dt\\
&\leq C\bar\rho^{(5r-6)/4r}\left(\int_0^T\left(e^{C_2t/2}\|(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\|_{L^2}\right)^{(6-r)/2r}\times\right.\\
&\quad\left.\left(e^{C_2t/2}\|(\nabla u_t,\nabla\theta_t)\|_{L^2}\right)^{(3r-6)/2r}e^{-C_2t/2}dt\right)+\int_0^T\bar\rho\|\nabla u\|_{H^1}^2dt\\
&\leq C\bar\rho^{(5r-6)/4r}\left(\sup_{0\leq t\leq T}\left(e^{C_2t/2}\|(\sqrt{\rho}u_t,\sqrt{\rho}\theta_t)\|_{L^2}\right)^{(6-r)/2r}\times\right.\\
&\quad\left.\left(\int_0^T e^{C_2t}\|(\nabla u_t,\nabla\theta_t)\|_{L^2}^2dt\right)^{(3r-6)/4r}\left(\int_0^T e^{-2C_2tr/(r+6)}dt\right)^{(r+6)/4r}\right)\\
&+\int_0^T\bar\rho\|\nabla u\|_{H^1}^2dt\leq C\max\{\bar\rho^{(5r-6)/4r},\bar\rho\}.
\end{aligned}
$$

□

Proof of Proposition 3.1. Taking operator ∇ to the equation of $(1.1)_1$, multiplying it by $|\nabla\rho|^{p-2}\nabla\rho$, and integrating by parts, by Gronwall inequality and (3.25), we have

$$
\begin{aligned}
\sup_{0\leq t\leq T}\|\nabla\rho\|_{L^p}&\leq C\|\nabla\rho_0\|_{L^p}\exp\{\int_0^T\|\nabla u\|_{L^\infty}dt\}\\
&\leq C\|\nabla\rho_0\|_{L^p}\exp\left\{\min\{\bar\rho^{(5r-6)/4r},\bar\rho\}\right\}.
\end{aligned}
$$

Hence, we can choose $\bar\rho$ appropriately small to obtain

$$
\sup_{0\leq t\leq T}\|\nabla\rho\|_{L^p}\leq 2K_1. \tag{3.26}
$$

To close $\|\nabla\theta\|_{L^p}$, it follows from $(1.1)_3$, (3.4) and (3.3) that

$$
\begin{aligned}
\|\nabla\theta\|_{L^p}&\leq C\bar\rho^{1/4}\left(\|\sqrt{\rho}u_t\|_{L^2}^2+\|\sqrt{\rho}\theta_t\|_{L^2}^2\right)^{1/2}\\
&\leq C_2\bar\rho^{1/4}.
\end{aligned}\tag{3.27}
$$

Therefore, collecting with Lemma 3.2, Lemma 3.3, (3.26) and (3.27), we complete the proof of the Proposition 3.1.

Proof of Theorem 1.1. Combining the local strong solution and the global a priori estimates in Lemmas 3.1-3.4, by continuity arguments, we can obtain the existence of global strong solution for (1.1) when the initial density is suitable small. From (3.5) and (3.25), the decay rates of the norm $\|u(t)\|_{H^1}^2+\|\theta(t)\|_{H^1}^2$ is proved.

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References

[1] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Advances in Mathematics, 2006, 203(2), 497–515.

[2] P. Constantin and C. Doering, *Infinite Prandtl number convection*, Journal of Statistical Physics, 1999, 94(1-2), 159–172.

[3] J. Fan and T. Ozawa, *Regularity criteria for the 3D density-dependent Boussinesq equations*, Nonlinearity, 2009, 22(3), 553–568.

[4] A. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, London, 1982.

[5] X. Huang and Y. Wang, *Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity*, Journal of Differential Equations, 2015, 259(2), 1606–1627.

[6] M. He, *On the blowup criteria and global regularity for the non-diffusive Boussinesq equations with temperature-dependent viscosity coefficient*, Nonlinear Analysis. Theory, Methods & Applications, 2016, 144, 93–109.

[7] C. He, J. Li and B. Q. Lü, *Global well-posedness and exponential stability of 3D Navier-Stokes equations with density-dependent viscosity and vacuum in unbounded domains*, Archive for Rational Mechanics and Analysis, 2021, 239(3), 1809–1835.

[8] T. Y. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete and Continuous Dynamical Systems, 2005, 12(1), 1–12.

[9] A. Larios and Y. Pei, *On the local well-posedness and a Prodi-Serrin-type regularity criterion of the three-dimensional MHD-Boussinesq system without thermal diffusion*, Journal of Differential Equations, 2017, 263(2), 1419–1450.

[10] J. Pedlosky, *Geophysical Fluid Dyanmics*, Springer-Verlag, New York, 1987.

[11] H. Qiu and Z. Yao, *Well-posedness for density-dependent Boussinesq equations without dissipation terms in Besov spaces*, Computers & Mathematics with Applications, 2017, 73(9), 1920–1931.

[12] X. Ye and M. Zhu, *The zero limit of thermal diffusivity for the 2D density-dependent Boussinesq equations with vacuum*, ZAMM. Zeitschrift für Angewandte Mathematik und Mechanik, 2020, 100(1), Article ID e201900064, 8 pages.

[13] J. Zhang, *Global well-posedness for the incompressible Navier-Stokes equations with density-dependent viscosity coefficient*, Journal of Differential Equations, 2015, 259(5), 1722–1742.

[14] M. Zhang, *On the Cauchy problem of 3D nonhomogeneous magnetohydrodynamic equations with density-dependent viscosity and vacuum*,Zeitschrift für Angewandte Mathematik und Physik,2020,71(4),Article ID 106, 27 pages.

[15] Z. Zhang, *3D Density-Dependent Boussinesq Equations with Velocity Field in BMO Spaces*, Acta Applicandae Mathematicae, 2016, 142, 1–8.

[16] X. Zhao, *Global wellposedness of magnetohydrodynamics for Boussinesq system with partial viscosity and zero magnetic diffusion*, Journal of Mathematical Analysis and Applications, 2018, 461(1), 97–127.

[17] X. Zhong, *Strong solutions to the 2D Cauchy problem of density-dependent viscous Boussinesq equations with vacuum*, Journal of Mathematical Physics, 2019, 60(5), Article ID 051505, 15 pages.

[18] X. Zhong, *Global well-posedness to the Cauchy problem of two-dimensional density-dependent Boussinesq equations with large initial data and vacuum*, Discrete and Continuous Dynamical Systems, 2019, 39(11), 6713–6745.