

# Long-Time Asymptotics of Complex mKdV Equation with Weighted Sobolev Initial Data\*

Hongyi Zhang<sup>1</sup> and Yufeng Zhang<sup>1,†</sup>

**Abstract** In this paper, we apply  $\bar{\partial}$ -steepest descent method to analyze the long-time asymptotics of complex mKdV equation with the initial value belonging to weighted Sobolev spaces. Firstly, the Cauchy problem of the complex mKdV equation is transformed into the corresponding Riemann-Hilbert problem on the basis of the Lax pair and the scattering data. Then the long-time asymptotics of complex mKdV equation is obtained by studying the solution of the Riemann-Hilbert problem.

**Keywords** Riemann-Hilbert problem, complex mKdV equation,  $\bar{\partial}$ -steepest descent method, long-time asymptotics

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## 1. Introduction

The study of nonlinear partial differential equations (NLPDEs) has played an important role in the development of science and technology. Until now, NLPDEs can be used to explain some complex physical phenomena, including mathematics, fluid mechanics, plasma physics, atmospheric oceans, aerodynamics, etc [2–9]. Nowadays, the inverse scattering transformation [10–13], Hirota bilinear method [14–16], Darboux transformation [17, 18] and so on are effective methods to solve NLPDEs. Especially, the inverse scattering transformation is the first method which was found and used to obtain the exact solution of the soliton equation. In the early 20th century, the solution of Riemann-Hilbert (RH) problem was developed and promoted [19, 20]. In 1993, Deift and Zhou proposed the famous nonlinear steepest descent method to analyze the long-time asymptotic behavior of integrable evolution equations. Deift and Zhou analyzed the long-time asymptotic behavior of the solution to the initial value problem of the famous mKdV equation and Schrödinger equation [21, 22]. Cuccagna studied the asymptotic stability of N-soliton solutions of the defocusing nonlinear Schrödinger equation by  $\bar{\partial}$ -steepest descent method [23]. Robert analyzed the derivative nonlinear Schrödinger equation via  $\bar{\partial}$ -steepest descent method [24]. In addition, Fan, Geng and Ma studied the soliton solutions and long-time asymptotic behavior of some integrable evolution equations based on

<sup>†</sup>the corresponding author.

Email address: a15366763662@163.com, zhangyfcumt@163.com

<sup>1</sup>School of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu, 221116, PR China

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RH problem [25–36]; among them, Ma has already done some work on nonlocal equations [35, 36].

In this paper, we study the equation derived from the Lax pair given by Yishen Li [37]. The Lax pair is

$$\begin{aligned}\psi_x &= -i\lambda\sigma_3\psi + P\psi, \\ \psi_t &= (\zeta\lambda^3 + \eta\lambda^2 + \vartheta\lambda + \iota)\sigma_3\psi + Q\psi,\end{aligned}\quad (1.1)$$

where  $\psi(x, t, \lambda)$  is a  $2 \times 2$  matrix,  $\sigma_3 = \text{diag}(1, -1)$ , and

$$\begin{aligned}P &= \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \\ Q &= i\zeta\lambda^2 P - i\lambda \begin{pmatrix} \frac{i\zeta}{2}uv & -\frac{i\zeta}{2}u_x - \eta u \\ \frac{i\zeta}{2}v_x - \eta v & -\frac{i\zeta}{2}uv \end{pmatrix} \\ &\quad - \begin{pmatrix} \frac{i\zeta}{4}(uv_x - vu_x) - \frac{\eta}{2}uv & -\frac{i\zeta}{4}(-u_{xx} + 2u^2v) + \frac{\eta}{2}u_x - i\vartheta u \\ -\frac{i\zeta}{4}(-v_{xx} + 2uv^2) - \frac{\eta}{2}v_x - i\vartheta v & -\frac{i\zeta}{4}(uv_x - vu_x) + \frac{\eta}{2}uv \end{pmatrix}.\end{aligned}\quad (1.2)$$

The Lax pair (1.1) derives the following system:

$$\begin{cases} u_t = -\frac{i\zeta}{4}(u_{xxx} - 6uvu_x) - \frac{\eta}{2}(u_{xx} - 2u^2v) + i\vartheta u_x + 2\iota u, \\ v_t = -\frac{i\zeta}{4}(v_{xxx} - 6uvv_x) + \frac{\eta}{2}(v_{xx} - 2v^2u) + i\vartheta v_x - 2\iota v. \end{cases}\quad (1.3)$$

(I) Taking  $\zeta = -4i$ ,  $\eta = \vartheta = \iota = 0$ , and  $v = -1$ , system (1.3) reduces to the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1.4)$$

(II) Taking  $\zeta = -4i$ ,  $\eta = \vartheta = \iota = 0$ , and  $v = -u$ , system (1.3) reduces to the mKdV equation:

$$u_t + 6u^2u_x + u_{xxx} = 0. \quad (1.5)$$

(III) Taking  $\eta = -2i$ ,  $\zeta = \vartheta = \iota = 0$ , and  $v = \mp \bar{u}$ , system (1.3) reduces to the nonlinear Schrödinger equation:

$$iu_t + u_{xx} \pm 2u^2\bar{u} = 0, \quad (1.6)$$

where superscript bar denotes complex conjugate.

(IV) Taking  $\iota = -2$ ,  $\zeta = \vartheta = \iota = 0$ , and  $q_x = uv = \left(\frac{u_x}{u}\right)_x$ , system (1.3) reduces to the Burger equation

$$q_t = 2qq_x - q_{xx}. \quad (1.7)$$

In addition, taking  $\zeta = -i\alpha$  ( $\alpha > 0$ ),  $\eta = \vartheta = \iota = 0$  and  $v = \bar{u}$ , system (1.3) reduces to the complex mKdV equation:

$$u_t = \frac{\alpha}{4}(-u_{xxx} + 6|u|^2u_x), \quad (1.8)$$

where  $u(x, t)$  is complex-valued function of variate  $(x, t)$ . In [38], Chen and Liu obtained the long-time asymptotics of the mKdV equation in weighted Sobolev spaces. However, the long-time asymptotics of the complex mKdV equation have

not been studied. In this paper, we apply  $\bar{\partial}$ -steepest descent method to analyze long-time asymptotics of the complex mKdV equation with weighted Sobolev initial data  $u(t = 0, x) = u_0(x) \in H^{1,1}(\mathbb{R}) = \{f(x) : f'(x), xf(x) \in L^2(\mathbb{R})\}$ . The significance of our work is that it gives a referenceable example for later generalization of the real equation to complexified equations in the study of the dynamical behaviour of the solutions.

The layout of the paper is as follows. In Section 2, we analyze eigenfunction and spectral function of equation (1.8) to construct the original Riemann-Hilbert problem. In Section 3, by deforming the jump matrix of the original Riemann-Hilbert problem and extending the region, the original Riemann-Hilbert problem is transformed into a model Riemann-Hilbert problem. Then the solution of the model Riemann-Hilbert problem can be expressed by the solution of Weber equation. Finally, we obtain the long-time asymptotics of the Cauchy problem for complex mKdV equation.

## 2. Spectral analysis

In this section, by analyzing the Lax pair, the matrix Jost solutions of complex mKdV equation (1.8) are constructed. Then the Cauchy problem of complex mKdV equation (1.8) turns into the corresponding Riemann-Hilbert problem. The Lax pair of complex mKdV equation is

$$\psi_x = -i\lambda\sigma_3\psi + M\psi, \tag{2.1}$$

$$\psi_t = -i\alpha\lambda^3\sigma_3\psi + N\psi, \tag{2.2}$$

where

$$M = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \tag{2.3}$$

$$N = \alpha\lambda^2 M - i\lambda \begin{pmatrix} \frac{\alpha}{2}uv - \frac{\alpha}{2}u_x \\ \frac{\alpha}{2}v_x - \frac{\alpha}{2}uv \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{4}(uv_x - vu_x) & -\frac{\alpha}{4}(-u_{xx} + 2u^2v) \\ -\frac{\alpha}{4}(-v_{xx} + 2uv^2) & -\frac{\alpha}{4}(uv_x - vu_x) \end{pmatrix}. \tag{2.4}$$

### 2.1. Asymptotics

Lax pair (2.1)-(2.2) has a Jost solution of the following asymptotic form

$$\psi(x, t, \lambda) = e^{-i(\lambda\sigma_3x + \alpha\lambda^3\sigma_3t)}, \quad |x| \rightarrow \infty. \tag{2.5}$$

Therefore, we make the transformation

$$\mu(x, t, \lambda) = \psi(x, t, \lambda)e^{i(\lambda\sigma_3x + \alpha\lambda^3\sigma_3t)}, \tag{2.6}$$

where  $\mu(x, t, \lambda)$  satisfies the following Lax pair

$$\mu_x + i\lambda[\sigma_3, \mu] = M\mu, \tag{2.7}$$

$$\mu_t + i\alpha\lambda^3[\sigma_3, \mu] = N\mu, \tag{2.8}$$

which can be written in the full derivative form

$$d\left(e^{i(\lambda x + \alpha \lambda^3 t)\hat{\sigma}_3}\mu\right) = e^{i(\lambda x + \alpha \lambda^3 t)\hat{\sigma}_3}[(Mdx + Ndt)\mu]. \quad (2.9)$$

Considering the asymptotic expansion

$$\mu = \mu_0 + \frac{\mu_1}{\lambda} + \frac{\mu_2}{\lambda^2} + \frac{\mu_3}{\lambda^3} + o\left(\frac{1}{\lambda^4}\right), \quad \lambda \rightarrow \infty, \quad (2.10)$$

where  $\mu_0$ ,  $\mu_1$ ,  $\mu_2$  and  $\mu_3$  are independent of  $\lambda$ . Substituting (2.10) into (2.7) and comparing the coefficients of  $\lambda$ , we obtain that  $\mu_0$  is a diagonal matrix and

$$\mu_{0,x} + i\sigma_3\mu_1 - i\mu_1\sigma_3 = M\mu_0, \quad (2.11)$$

$$i\sigma_3\mu_0 - i\mu_0\sigma_3 = 0. \quad (2.12)$$

In the same way, substituting (2.10) into (2.8) and comparing the coefficients of  $\lambda$ , we get

$$\mu_{0,t} + i\alpha\sigma_3\mu_3 - i\alpha\mu_3\sigma_3 - \alpha M\mu_2 + \frac{i\alpha}{2}u\bar{u}\sigma_3\mu_1 + \frac{i\alpha}{2}M_x\sigma_3\mu_1 + \frac{\alpha}{4}(u\bar{u}_x - \bar{u}u_x)\sigma_3\mu_0 \quad (2.13)$$

$$+ \frac{\alpha}{4}M_{xx}\mu_0 - \frac{\alpha}{2}M^3\mu_0 = 0,$$

$$i\alpha\sigma_3\mu_2 - i\alpha\mu_2\sigma_3 = \alpha M\mu_1 - \frac{i\alpha}{2}u\bar{u}\sigma_3\mu_0 - \frac{i\alpha}{2}M_x\sigma_3\mu_0, \quad (2.14)$$

$$i\alpha\sigma_3\mu_1 - i\alpha\mu_1\sigma_3 = \alpha M\mu_0, \quad (2.15)$$

$$i\alpha\sigma_3\mu_0 - i\alpha\mu_0\sigma_3 = 0. \quad (2.16)$$

Through a series of calculations, we obtain

$$\mu(x, t, \lambda) \rightarrow I, \quad \lambda \rightarrow \infty, \quad (2.17)$$

$$u(x, t) = 2i \lim_{\lambda \rightarrow \infty} (\lambda\mu)_{12} = 2i(\mu_1)_{12}. \quad (2.18)$$

## 2.2. Analyticity and symmetry

To analyze the eigenfunction  $\mu(x, t, \lambda)$ , we choose two special integral paths

$$(-\infty, t) \rightarrow (x, t) \quad \text{and} \quad (\infty, t) \rightarrow (x, t) \quad (2.19)$$

and acquire two Volterra type integral equations

$$\mu_1(x, t, \lambda) = I + \int_{-\infty}^x e^{i(\lambda y - \lambda x)\hat{\sigma}_3} M(y, t, \lambda)\mu_1(y, t, \lambda)dy, \quad (2.20)$$

$$\mu_2(x, t, \lambda) = I - \int_x^{\infty} e^{i(\lambda y - \lambda x)\hat{\sigma}_3} M(y, t, \lambda)\mu_2(y, t, \lambda)dy. \quad (2.21)$$

From the transformation (2.6), it can be known that  $\mu_1(x, t, \lambda)e^{-i(\lambda\sigma_3x + \alpha\lambda^3\sigma_3t)}$  and  $\mu_2(x, t, \lambda)e^{-i(\lambda\sigma_3x + \alpha\lambda^3\sigma_3t)}$  are the two linear correlation matrix solutions of Lax pair (2.7) and (2.8), so we have

$$\mu_1(x, t, \lambda) = \mu_2(x, t, \lambda)e^{-i\theta(\lambda)\hat{\sigma}_3}S(\lambda), \quad (2.22)$$

where

$$\theta(\lambda) = \lambda \frac{x}{t} + \alpha \lambda^3, \quad S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix},$$

$S(\lambda)$  is irrelevant to  $x$  and  $t$ , and it is called the spectral matrix function. Next, we study the analyticity of  $\mu_1(x, t, \lambda)$ ,  $\mu_2(x, t, \lambda)$  and  $S(\lambda)$ . For the integral equation (2.20), a direct calculation shows that

$$e^{i\lambda(y-x)\hat{\sigma}_3} M(\lambda; \xi, t) = \begin{pmatrix} 0 & ue^{2i\lambda(y-x)} \\ \bar{u}e^{2i\lambda(y-x)} & 0 \end{pmatrix}, \tag{2.23}$$

and

$$2i\lambda(y-x) = 2i(\text{Re}\lambda + i\text{Im}\lambda)(y-x) = 2i\text{Re}\lambda(y-x) - 2\text{Im}\lambda(y-x),$$

so that the first column of  $\mu_1(x, t, \lambda)$  is analytical in the upper half plane  $\mathbb{C}_+$ , the second column of  $\mu_1(x, t, \lambda)$  is analytical in the lower half plane  $\mathbb{C}_-$ , and  $\mu_1(x, t, \lambda)$  can be written as

$$\mu_1 = \begin{pmatrix} \mu_1^{(11)} & \mu_1^{(12)} \\ \mu_1^{(21)} & \mu_1^{(22)} \end{pmatrix} = (\mu_1^+, \mu_1^-). \tag{2.24}$$

Similarly, the first column of  $\mu_2(x, t, \lambda)$  is analytical in the lower half plane  $\mathbb{C}_-$ , the second column of  $\mu_2(x, t, \lambda)$  is analytical in the upper half plane  $\mathbb{C}_+$ , and  $\mu_2(x, t, \lambda)$  can be written as

$$\mu_2 = \begin{pmatrix} \mu_2^{(11)} & \mu_2^{(12)} \\ \mu_2^{(21)} & \mu_2^{(22)} \end{pmatrix} = (\mu_2^-, \mu_2^+). \tag{2.25}$$

**Theorem 2.1.** *The eigenfunctions  $\mu_1(x, t, \lambda)$ ,  $\mu_2(x, t, \lambda)$  and spectral matrix  $S(\lambda)$  have the following symmetry properties*

$$\mu_j(x, t, \lambda) = \overline{\sigma_1 \mu_j(x, t, \bar{\lambda}) \sigma_1}, \quad (j = 1, 2), \tag{2.26}$$

$$S(\lambda) = \sigma_1 \overline{S(\bar{\lambda})} \sigma_1, \tag{2.27}$$

where  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

**Proof.** Through Lax pair (2.7), we have

$$\mu_{j,x}(x, t, \lambda) + i\lambda[\sigma_3, \mu_j(x, t, \lambda)] = M\mu_j(x, t, \lambda). \tag{2.28}$$

Substituting  $\bar{\lambda}$  for  $\lambda$  and taking the conjugate of the left and right sides of equation (2.28) gives

$$\overline{\mu_{j,x}(x, t, \bar{\lambda})} - i\lambda[\sigma_3, \overline{\mu_j(x, t, \bar{\lambda})}] = \overline{M\mu_j(x, t, \bar{\lambda})}. \tag{2.29}$$

Multiplying the left and right sides of equation (2.29) by  $\sigma_1$  leads to

$$[\sigma_1 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_1]_x - i\lambda \sigma_1 \sigma_3 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_1 + i\lambda \sigma_1 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_3 \sigma_1 = \sigma_1 \overline{M\mu_j(x, t, \bar{\lambda})} \sigma_1. \tag{2.30}$$

By calculation, we get

$$[\sigma_1 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_1]_x + i\lambda[\sigma_3, \sigma_1 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_1] = M \sigma_1 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_1. \tag{2.31}$$

Comparing equation (2.7) with equation (2.31), we know that  $\mu_j(x, t, \lambda)$  and  $\sigma_1 \overline{\mu_j(x, t, \bar{\lambda})} \sigma_1$  satisfy the same differential equation and have the same asymptotic property

$$\mu_j(x, t, \lambda), \quad \overline{\sigma_1 \mu_j(x, t, \bar{\lambda})} \sigma_1 \rightarrow I, \quad x \rightarrow \infty. \quad (2.32)$$

Therefore,

$$\mu_j(x, t, \lambda) = \overline{\sigma_1 \mu_j(x, t, \bar{\lambda})} \sigma_1, \quad (2.33)$$

that is

$$\begin{pmatrix} \mu_{11}(\lambda) & \mu_{12}(\lambda) \\ \mu_{21}(\lambda) & \mu_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\mu_{11}(\bar{\lambda})} & \overline{\mu_{12}(\bar{\lambda})} \\ \overline{\mu_{21}(\bar{\lambda})} & \overline{\mu_{22}(\bar{\lambda})} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\mu_{22}(\bar{\lambda})} & \overline{\mu_{21}(\bar{\lambda})} \\ \overline{\mu_{12}(\bar{\lambda})} & \overline{\mu_{11}(\bar{\lambda})} \end{pmatrix}. \quad (2.34)$$

By comparing the two sides of the above equation, we get  $\mu_{11}(\lambda) = \overline{\mu_{22}(\bar{\lambda})}$ , and  $\mu_{12}(\lambda) = \overline{\mu_{21}(\bar{\lambda})}$ . From equation (2.33), we get the symmetry property of spectral matrix  $S(\lambda)$ :

$$\overline{\sigma_1 S(\bar{\lambda})} \sigma_1 = S(\lambda), \quad (2.35)$$

that is  $\overline{s_{11}(\bar{\lambda})} = s_{22}(\lambda)$  and  $\overline{s_{12}(\bar{\lambda})} = s_{21}(\lambda)$ .  $\square$

**Theorem 2.2.** *The eigenfunctions  $\mu_1(x, t, \lambda)$ ,  $\mu_2(x, t, \lambda)$  and spectral matrix function  $S(\lambda)$  also have the following symmetry properties*

$$\sigma_3 \mu_j^H(x, t, \bar{\lambda}) \sigma_3 = \mu_j^{-1}(x, t, \lambda), \quad (2.36)$$

$$\sigma_3 S^H(\bar{\lambda}) \sigma_3 = S^{-1}(\lambda), \quad (2.37)$$

where superscript  $H$  denotes conjugate transpose.

**Proof.** Through direct calculation, we get

$$\begin{aligned} & \sigma_3 \mu_j^H(x, t, \bar{\lambda}) \sigma_3 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_{j,11}(x, t, \bar{\lambda}) & \mu_{j,12}(x, t, \bar{\lambda}) \\ \mu_{j,21}(x, t, \bar{\lambda}) & \mu_{j,22}(x, t, \bar{\lambda}) \end{pmatrix}^H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \overline{\mu_{j,11}(x, t, \bar{\lambda})} & \overline{\mu_{j,21}(x, t, \bar{\lambda})} \\ \overline{\mu_{j,12}(x, t, \bar{\lambda})} & \overline{\mu_{j,22}(x, t, \bar{\lambda})} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_{j,22}(\lambda) & \mu_{j,12}(\lambda) \\ \mu_{j,21}(\lambda) & \mu_{j,11}(\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \mu_{j,22}(\lambda) & -\mu_{j,12}(\lambda) \\ -\mu_{j,21}(\lambda) & \mu_{j,11}(\lambda) \end{pmatrix} \\ &= \mu_j^{-1}(x, t, \lambda). \end{aligned} \quad (2.38)$$

Similarly, we can get

$$\sigma_3 S^H(\bar{\lambda}) \sigma_3 = S^{-1}(\lambda). \quad (2.39)$$

□

From equation (2.22), we obtain that

$$s_{11}(\lambda) = 1 + \int_{-\infty}^{+\infty} u\mu_{1,21}d\xi, s_{21}(\lambda) = \int_{-\infty}^{+\infty} \bar{u}e^{-2i\lambda\xi}\mu_{1,11}d\xi. \tag{2.40}$$

### 3. The construction of an RH problem

Define the reflection coefficient as  $r(\lambda) = \frac{s_{21}(\lambda)}{s_{11}(\lambda)}$ , and a piecewise analytical function  $m(x, t, \lambda)$  that

$$m(x, t, \lambda) = \begin{cases} \left( \frac{\mu_1^+}{s_{11}}, \mu_2^+ \right), & \text{Im}\lambda > 0, \\ \left( \mu_2^-, \frac{\mu_1^-}{s_{22}} \right), & \text{Im}\lambda < 0. \end{cases} \tag{3.1}$$

By applying the analyticities and symmetries of the eigenfunctions and the spectral matrix, the RH problem corresponding to the initial value problem of the complex mKdV equation can be obtained.

**RH problem 1 :**

- $m_{\pm}(x, t, \lambda)$  is analytical in  $\mathbb{C}_{\pm}$ ,
  - $m_+(x, t, \lambda) = m_-(x, t, \lambda)v(x, t, \lambda)$ ,
  - $m_{\pm}(x, t, \lambda) \rightarrow I, \text{ as } \lambda \rightarrow \infty$ ,
- (3.2)

where the jump matrix is

$$v(x, t, \lambda) = \begin{pmatrix} 1 - |r(\lambda)|^2 & -e^{-2it\theta}\overline{r(\lambda)} \\ e^{2it\theta}r(\lambda) & 1 \end{pmatrix}. \tag{3.3}$$

This is an RH problem defined on the real axis, as shown in figure 1 and the solution  $u(x, t)$  to the initial value problem of the complex mKdV equation can be expressed as the RH problem above

$$u(x, t) = 2i \lim_{\lambda \rightarrow \infty} (\lambda\mu(x, t, \lambda))_{12} = 2i \lim_{\lambda \rightarrow \infty} (\lambda m(x, t, \lambda))_{12} = 2i \lim_{\lambda \rightarrow \infty} (m_1(x, t, \lambda))_{12}. \tag{3.4}$$



Figure 1. The oriented contour of  $m(\lambda)$

### 4. Triangular decomposition of jump matrix

We write the oscillating term of the jump matrix as

$$e^{it\theta(\lambda)} = e^{t\varphi(\lambda)}, \varphi(\lambda) = i\theta(\lambda),$$

to obtain two steady state phase points  $\pm\lambda_0 = \pm\sqrt{-\frac{x}{3\alpha t}}$ . Since

$$\theta(\lambda) = \alpha ((\lambda + \lambda_0)^3 - 3\lambda_0(\lambda + \lambda_0)^2 + 2\lambda_0^3), \tag{4.1}$$

and

$$\theta(\lambda) = \alpha ((\lambda - \lambda_0)^3 + 3\lambda_0(\lambda - \lambda_0)^2 - 2\lambda_0^3), \quad (4.2)$$

the complex plane can be divided into two types of regions on the basis of the exponential decay of  $e^{i\theta}$ , see figure 2. The jump matrix  $v(x, t, \lambda)$  has the lower/upper triangular decomposition

$$v(x, t, \lambda) = \begin{pmatrix} 1 - \overline{r(\bar{\lambda})}e^{-2it\theta(\lambda)} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda)e^{2it\theta(\lambda)} & 1 \end{pmatrix}, \quad (\lambda \rightarrow \infty) \quad (4.3)$$

and the upper/diagonal/lower decomposition

$$\begin{aligned} & v(x, t, \lambda) \\ &= \begin{pmatrix} 1 & 0 \\ e^{2it\theta(\lambda)} \frac{r(\lambda)}{1-r(\lambda)r(\bar{\lambda})} & 1 \end{pmatrix} \begin{pmatrix} 1 - r(\lambda)\overline{r(\bar{\lambda})} & 0 \\ 0 & \frac{1}{1-r(\lambda)r(\bar{\lambda})} \end{pmatrix} \begin{pmatrix} 1 - e^{-2it\theta(\lambda)} \frac{\overline{r(\bar{\lambda})}}{1-r(\lambda)r(\bar{\lambda})} & \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (4.4)$$

$$\lambda \in (-\lambda_0, \lambda_0).$$

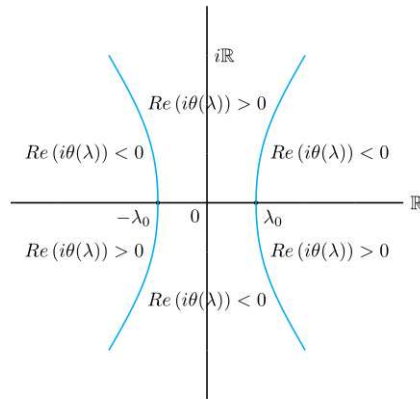


Figure 2. Symbol distribution map of  $Re(i\theta(\lambda))$ .

In order to remove the intermediate diagonal matrix in the decomposition for  $\lambda \in (-\lambda_0, \lambda_0)$ , we introduce a scalar RH problem:

- $\delta(\lambda)$  is analytical in  $\mathbb{C} \setminus \mathbb{R}$ ,
- $\delta_+(\lambda) = \delta_-(\lambda) \left(1 - r(\lambda)\overline{r(\bar{\lambda})}\right)$ ,  $\lambda \in (-\lambda_0, \lambda_0)$ ,
- $\delta_+(\lambda) = \delta_-(\lambda)$ ,  $\lambda \rightarrow \infty$ ,
- $\delta(\lambda) \rightarrow 1$ ,  $\lambda \rightarrow \infty$ .

According to Plemelj formula, we get the unique solution of the above RH problem:

$$\delta(\lambda) = \exp \left( \frac{1}{2\pi i} \int_{-\lambda_0}^{\lambda_0} \frac{\log(1 - |r(\xi)|^2)}{\xi - \lambda} d\xi \right) = \exp \left( i \int_{-\lambda_0}^{\lambda_0} \frac{\nu(\xi)}{\xi - \lambda} d\xi \right), \quad (4.6)$$



where

$$\nu(\xi) = -\frac{1}{2\pi} \log(1 - |r(\xi)|^2).$$

Assuming that  $r(\lambda) \in L^\infty \cap L^2$  and  $\|r(\lambda)\|_{L^\infty} \leq \rho < 1$ ,  $\delta(\lambda)$  has the following properties:

- (1)  $\delta(\lambda)$  is analytic in  $\mathbb{C} \setminus (-\lambda_0, \lambda_0)$ ;
- (2)  $\delta(\lambda)\overline{\delta(\bar{\lambda})} = 1$ ,  $\|\delta_\pm - 1\|_{L^2} \leq \frac{c\|r\|_{L^2}}{1-\rho}$ ;
- (3)  $(1 - \rho^2)^{\frac{1}{2}} \leq |\delta(\lambda)| \leq (1 - \rho^2)^{-\frac{1}{2}}$ .

Further, we rewrite  $\delta(\lambda)$  as

$$\begin{aligned} & \delta(\lambda) \\ &= \exp\left(i \int_{-\lambda_0}^0 \frac{\nu(\xi)}{\xi - \lambda} d\xi + i \int_0^{\lambda_0} \frac{\nu(\xi)}{\xi - \lambda} d\xi\right) \\ &= \exp\left(i \int_{-\lambda_0}^0 \frac{\nu(\xi) - \chi_2(\xi)\nu(-\lambda_0)(\xi + \lambda_0 - 1)}{\xi - \lambda} d\xi + i\nu(-\lambda_0) \int_{-\lambda_0}^{-\lambda_0+1} \frac{\xi + \lambda_0 - 1}{\xi - \lambda} d\xi\right) \\ & \quad \cdot \exp\left(i \int_0^{\lambda_0} \frac{\nu(\xi) - \chi_1(\xi)\nu(\lambda_0)(\xi - \lambda_0 + 1)}{\xi - \lambda} d\xi + i\nu(\lambda_0) \int_{\lambda_0-1}^{\lambda_0} \frac{\xi - \lambda_0 + 1}{\xi - \lambda} d\xi\right) \\ &= \exp(i\beta_2(\lambda, -\lambda_0) + i\nu(-\lambda_0) + i\nu(-\lambda_0)[(\lambda + \lambda_0 - 1) \log(\lambda + \lambda_0 - 1) - (\lambda + \lambda_0) \log(\lambda + \lambda_0)]) \cdot \exp(i\nu(-\lambda_0) \log(\lambda + \lambda_0) + i\nu(\lambda_0) \log(\lambda - \lambda_0)) \cdot \exp(i\beta_1(\lambda, \lambda_0) + i\nu(\lambda_0) + i\nu(\lambda_0)[(\lambda - \lambda_0) \log(\lambda - \lambda_0) - (\lambda - \lambda_0 + 1) \log(\lambda - \lambda_0) + 1]) \\ &= \exp(i\nu(-\lambda_0) + i\beta_2(\lambda, -\lambda_0)) (\lambda + \lambda_0)^{i\nu(-\lambda_0)} \exp(i\nu(\lambda_0) + i\beta_1(\lambda, \lambda_0)) (\lambda - \lambda_0)^{i\nu(\lambda_0)} \\ & \quad \cdot \exp(i\nu(-\lambda_0)[(\lambda + \lambda_0 - 1) \log(\lambda + \lambda_0 - 1) - (\lambda + \lambda_0) \log(\lambda + \lambda_0)]) \\ & \quad \cdot \exp(i\nu(\lambda_0)[(\lambda - \lambda_0) \log(\lambda - \lambda_0) - (\lambda - \lambda_0 + 1) \log(\lambda - \lambda_0 + 1)]), \end{aligned} \tag{4.7}$$

where  $\chi_1(\xi)$  is an eigenfunction defined on the  $(-\lambda_0, -\lambda_0 + 1)$ ,  $\chi_2(\xi)$  is an eigenfunction defined on the  $(\lambda_0 - 1, \lambda_0)$ , and

$$\begin{aligned} \beta_1(\lambda, \lambda_0) &= \int_0^{\lambda_0} \frac{\nu(\xi) - \chi_1(\xi)\nu(\lambda_0)(\xi - \lambda_0 + 1)}{\xi - \lambda} d\xi, \\ \beta_2(\lambda, -\lambda_0) &= \int_{-\lambda_0}^0 \frac{\nu(\xi) - \chi_2(\xi)\nu(-\lambda_0)(\xi + \lambda_0 + 1)}{\xi - \lambda} d\xi. \end{aligned}$$

Letting  $\Sigma^{(1)} = \mathbb{R}$ , we make a transformation

$$m^{(1)}(\lambda) = m(\lambda)\delta(\lambda)^{-\sigma_3}, \tag{4.8}$$

and  $m^{(1)}(\lambda)$  satisfies the following RH problem.

**RH problem 2 :**

- $m^{(1)}(x, t, \lambda)$  is analytical in  $\mathbb{C} \setminus \Sigma^{(1)}$ ,
  - $m_+^{(1)}(x, t, \lambda) = m_-^{(1)}(x, t, \lambda)v^{(1)}(x, t, \lambda)$ ,  $\lambda \in \Sigma^{(1)}$ ,
  - $m^{(1)}(x, t, \lambda) \rightarrow I$ , as  $\lambda \rightarrow \infty$ ;
- $$\tag{4.9}$$

for  $\lambda \in (-\lambda_0, \lambda_0)$ ,

$$v^{(1)}(x, t, \lambda) = \begin{pmatrix} 1 & 0 \\ \frac{r(\lambda)}{1-r(\lambda)r(\bar{\lambda})} \delta_-^{-2} e^{2it\theta(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} 1 - \frac{\overline{r(\bar{\lambda})}}{1-r(\lambda)r(\bar{\lambda})} \delta_+^2 e^{-2it\theta(\lambda)} \\ 0 \end{pmatrix}; \quad (4.10)$$

for  $\lambda \rightarrow \infty$ ,

$$v^{(1)}(x, t, \lambda) = \begin{pmatrix} 1 - \overline{r(\bar{\lambda})} \delta_-^2 e^{-2it\theta(\lambda)} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda) \delta_-^{-2} e^{2it\theta(\lambda)} & 1 \end{pmatrix}. \quad (4.11)$$

Due to  $\delta(\lambda) \rightarrow I$ ,  $\lambda \rightarrow \infty$ , the relation between the solution of the complex mKdV equation and the solution of the corresponding RH problem is

$$u(x, t) = 2i \lim_{\lambda \rightarrow \infty} (\lambda m^{(1)}(\lambda) \delta^{\sigma_3})_{12} = 2i \lim_{\lambda \rightarrow \infty} (\lambda m^{(1)}(\lambda))_{12}. \quad (4.12)$$

## 5. A hybrid $\bar{\partial}$ -problem

We extended the scattering data by the  $\bar{\partial}$ -steepest descent method:

- (1)  $r(\lambda)$  extends to the regions  $\Omega_{11}$  and  $\Omega_{21}$ ;
- (2)  $r(\bar{\lambda})$  extends to the regions  $\Omega_{16}$  and  $\Omega_{26}$ ;
- (3)  $\frac{r(\bar{\lambda})}{1-r(\lambda)r(\bar{\lambda})}$  extends to the regions  $\Omega_{13}$  and  $\Omega_{23}$ ;
- (4)  $\frac{r(\bar{\lambda})}{1-r(\lambda)r(\bar{\lambda})}$  extends to the regions  $\Omega_{14}$  and  $\Omega_{24}$ .

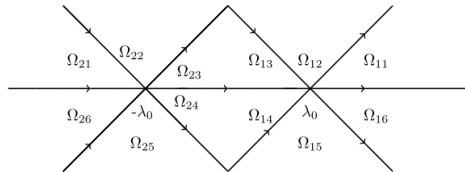


Figure 3. Regions  $\Omega_{ij}$ .

**Proposition 5.1.** *There exist functions  $R_{nj}(\lambda) \rightarrow C$ ,  $n = 1, 2$ ,  $j = 1, 3, 4, 6$  satisfy the following boundary conditions*

$$R_{11}(\lambda) = \begin{cases} r(\lambda), & \lambda \in (\lambda_0, \infty), \\ f_{11} = \hat{r}_{10}(\lambda - \lambda_0)^{-2i\nu(\lambda_0)} \delta^2, & \lambda \in \Sigma_{11}, \end{cases} \quad (5.1)$$

$$R_{13}(\lambda) = \begin{cases} -\frac{\overline{r(\bar{\lambda})}}{1-|\overline{r(\bar{\lambda})}|^2}, & \lambda \in (0, \lambda_0), \\ f_{13} = -\frac{\hat{r}_{10}}{1-|\hat{r}_{10}|^2}(\lambda - \lambda_0)^{2i\nu(\lambda_0)} \delta^{-2}, & \lambda \in \Sigma_{12}, \end{cases} \quad (5.2)$$

$$R_{14}(\lambda) = \begin{cases} \frac{r(\lambda)}{1-|r(\lambda)|^2}, & \lambda \in (0, \lambda_0), \\ f_{14} = \frac{\hat{r}_{10}}{1-|\hat{r}_{10}|^2}(\lambda - \lambda_0)^{-2i\nu(\lambda_0)} \delta^2, & \lambda \in \Sigma_{13}, \end{cases} \quad (5.3)$$

$$R_{16}(\lambda) = \begin{cases} -\overline{r(\bar{\lambda})}, & \lambda \in (\lambda_0, \infty), \\ f_{16} = -\hat{r}_{10}(\lambda - \lambda_0)^{2i\nu(\lambda_0)}\delta^{-2}, & \lambda \in \Sigma_{14}, \end{cases} \quad (5.4)$$

$$R_{21}(\lambda) = \begin{cases} r(\lambda), & \lambda \in (-\infty, -\lambda_0), \\ f_{21} = \hat{r}_{20}(\lambda + \lambda_0)^{-2i\nu(-\lambda_0)}\delta^2, & \lambda \in \Sigma_{21}, \end{cases} \quad (5.5)$$

$$R_{23}(\lambda) = \begin{cases} -\frac{\overline{r(\bar{\lambda})}}{1-|\overline{r(\bar{\lambda})}|^2}, & \lambda \in (-\lambda_0, 0), \\ f_{23} = -\frac{\hat{r}_{20}}{1-|\hat{r}_{20}|^2}(\lambda + \lambda_0)^{2i\nu(-\lambda_0)}\delta^{-2}, & \lambda \in \Sigma_{22}, \end{cases} \quad (5.6)$$

$$R_{24}(\lambda) = \begin{cases} \frac{r(\lambda)}{1-|r(\lambda)|^2}, & \lambda \in (-\lambda_0, 0), \\ f_{24} = \frac{\hat{r}_{20}}{1-|\hat{r}_{20}|^2}(\lambda + \lambda_0)^{-2i\nu(-\lambda_0)}\delta^2, & \lambda \in \Sigma_{23}, \end{cases} \quad (5.7)$$

$$R_{26}(\lambda) = \begin{cases} -\overline{r(\bar{\lambda})}, & \lambda \in (-\infty, \lambda_0), \\ f_{26} = -\hat{r}_{20}(\lambda + \lambda_0)^{2i\nu(-\lambda_0)}\delta^{-2}, & \lambda \in \Sigma_{24}, \end{cases} \quad (5.8)$$

where  $\hat{r}_{10} = r(\lambda_0)e^{-2i\nu(\lambda_0)-2\beta_1(\lambda_0,\lambda_0)}$ ,  $\hat{r}_{20} = r(-\lambda_0)e^{-2i\nu(-\lambda_0)-2\beta_2(-\lambda_0,-\lambda_0)}$  and  $R_{n_j}(\lambda)$  have the following estimations

$$|\bar{\partial}R_{n_j}(\lambda)| \leq c_1|\lambda - \lambda_0|^{-\frac{1}{2}} + c_2|r'(Re(\lambda))|, \quad (5.9)$$

$$|R_{n_j}(\lambda)| \leq c_1 \sin^2(arg\lambda) + c_1(Re\lambda)^{-\frac{1}{2}}. \quad (5.10)$$

Define the contour

$$\Sigma^{(2)} = \Sigma_{11} \cup \Sigma_{12} \cup \Sigma_{13} \cup \Sigma_{14} \cup \Sigma_{21} \cup \Sigma_{22} \cup \Sigma_{23} \cup \Sigma_{24},$$

$$R^{(2)}(\lambda) = \begin{cases} \left( \begin{pmatrix} 1 & 0 \\ R_{11}e^{2it\theta(\lambda)}\delta^{-2} & 1 \end{pmatrix} \right)^{-1}, & \lambda \in \Omega_{11}, \\ \left( \begin{pmatrix} 1 & R_{13}e^{-2it\theta(\lambda)}\delta^2 \\ 0 & 1 \end{pmatrix} \right)^{-1}, & \lambda \in \Omega_{13}, \\ \left( \begin{pmatrix} 1 & 0 \\ R_{14}e^{2it\theta(\lambda)}\delta^{-2} & 1 \end{pmatrix} \right), & \lambda \in \Omega_{14}, \\ \left( \begin{pmatrix} 1 & R_{16}e^{-2it\theta(\lambda)}\delta^2 \\ 0 & 1 \end{pmatrix} \right), & \lambda \in \Omega_{16}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_{12} \cup \Omega_{15}; \end{cases} \quad (5.11)$$

and

$$R^{(2)}(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R_{21}e^{2it\theta(\lambda)}\delta^{-2} & 1 \end{pmatrix}^{-1}, & \lambda \in \Omega_{21}, \\ \begin{pmatrix} 1 & R_{23}e^{-2it\theta(\lambda)}\delta^2 \\ 0 & 1 \end{pmatrix}^{-1}, & \lambda \in \Omega_{23}, \\ \begin{pmatrix} 1 & 0 \\ R_{24}e^{2it\theta(\lambda)}\delta^{-2} & 1 \end{pmatrix}, & \lambda \in \Omega_{24}, \\ \begin{pmatrix} 1 & R_{26}e^{-2it\theta(\lambda)}\delta^2 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_{26}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_{22} \cup \Omega_{25}. \end{cases} \quad (5.12)$$

Due to the boundedness of  $\delta(\lambda)$  and  $R_{nj}(\lambda)$ , and the exponential decay of  $e^{\pm 2it\theta}$ , we have

$$R^{(2)}(\lambda) \sim I, \quad t \rightarrow \infty.$$

We make a transformation

$$m^{(2)}(\lambda) = m^{(1)}(\lambda)R^{(2)}(\lambda), \quad (5.13)$$

then the RH problem on  $\Sigma^{(1)}$  becomes the RH problem on  $\Sigma^{(2)}$ .

**RH problem 3 :**

- $m^{(2)}(x, t, \lambda)$  is continuous in  $\mathbb{C} \setminus \Sigma^{(2)}$ ,
- $m_+^{(2)}(x, t, \lambda) = m_-^{(2)}(x, t, \lambda)v^{(2)}(x, t, \lambda), \quad \lambda \in \Sigma^{(2)},$  (5.14)
- $m_+^{(2)}(x, t, \lambda) \rightarrow I, \quad \text{as } \lambda \rightarrow \infty,$

the relation between the solution of the complex mKdV equation and the solution of corresponding RH problem is

$$u(x, t) = 2i \lim_{\lambda \rightarrow \infty} (\lambda m^{(2)}(x, t, \lambda))_{12}. \quad (5.15)$$

The exact expression for the jump matrix  $v^{(2)}(\lambda)$  is

$$v^{(2)}(\lambda) = (R_-^2(\lambda))^{-1}v^{(1)}(\lambda)R_+^2(\lambda), \quad (5.16)$$

that is

$$v^{(2)}(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R_{n1}e^{2it\theta(\lambda)}\delta^{-2} & 1 \end{pmatrix}, & \lambda \in \Sigma_{n1}, \\ \begin{pmatrix} 1 & R_{n3}e^{-2it\theta(\lambda)}\delta^2 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Sigma_{n2}, \\ \begin{pmatrix} 1 & 0 \\ R_{n4}e^{2it\theta(\lambda)}\delta^{-2} & 1 \end{pmatrix}, & \lambda \in \Sigma_{n3}, \\ \begin{pmatrix} 1 & R_{n6}e^{-2it\theta(\lambda)}\delta^2 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Sigma_{n4}. \end{cases} \quad (5.17)$$

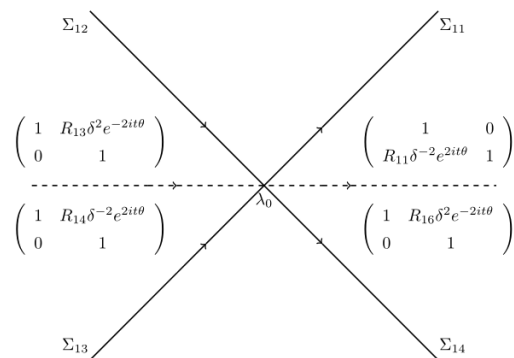


Figure 4. The jump matrix  $v^{(2)}(\lambda)$ .

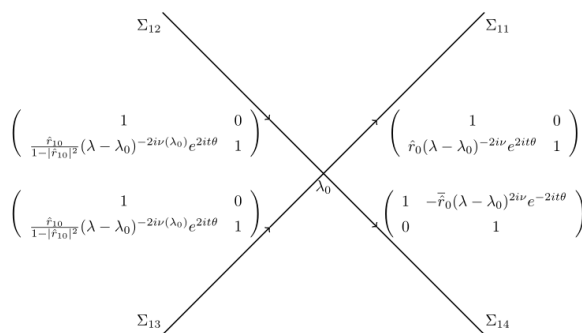


Figure 5. The jump matrix  $v^{(2)}(\lambda)$

In order for  $v^{(2)}(\lambda)$  to match the jump matrix  $v^{(pc)}(\lambda)$  of the parabolic cylindrical RH problem, we make the scale transformation

$$r_{10} = \hat{r}_{10} e^{i\nu \log(12\alpha t \lambda_0) - 4i\alpha t \lambda_0^3}, \tag{5.18}$$

then the jump matrix  $v^{(pc)}[\sqrt{12\alpha t \lambda_0}(\lambda - \lambda_0)]$  corresponding to  $m^{(pc)}[\sqrt{12\alpha t \lambda_0}(\lambda - \lambda_0)]$  is consistent with  $v^{(2)}(\lambda)$ . We make the transformation

$$r_{20} = \hat{r}_{20} e^{i\nu \log(-12\alpha t \lambda_0) + 4i\alpha t \lambda_0^3}, \tag{5.19}$$

the jump matrix  $v^{(pc)}[\sqrt{-12\alpha t \lambda_0}(\lambda + \lambda_0)]$  corresponding to  $m^{(pc)}[\sqrt{12\alpha t \lambda_0}(\lambda + \lambda_0)]$  is consistent with  $v^{(2)}(\lambda)$ . Therefore, we infer that

$$v^{(2)}(\lambda) = (R_-^{(2)})^{-1} v^{(1)}(\lambda) R_+^{(2)} = v^{(pc)}[\sqrt{12\alpha t \lambda_0}(\lambda - \lambda_0)]. \tag{5.20}$$

On the boundary  $(\lambda_0, \infty)$ , we have

$$v^{(pc)}[\sqrt{12\alpha t \lambda_0}(\lambda - \lambda_0)] = I, v^{(1)}(\lambda) = \begin{pmatrix} 1 & -\overline{r(\lambda)} \delta_-^2 e^{-2it\theta(\lambda)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda) \delta_-^{-2} e^{2it\theta(\lambda)} & 1 \end{pmatrix}.$$

Therefore, equation (5.20) derives

$$v^{(2)}(\lambda)$$

$$\begin{aligned}
&= (R^{(2)}(\lambda)|_{\Omega_{16}})^{-1} \begin{pmatrix} 1 - \overline{r(\bar{\lambda})}\delta_-^2 e^{-2it\theta(\lambda)} & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda)\delta_-^2 e^{2it\theta(\lambda)} & 1 \end{pmatrix} R^{(2)}(\lambda)|_{\Omega_{11}} \\
&= I.
\end{aligned} \tag{5.21}$$

We define

$$R^{(2)}(\lambda) = \begin{pmatrix} 1 & 0 \\ r\delta^{-2}e^{2it\theta} & 1 \end{pmatrix}^{-1}, \quad \lambda \in \Omega_{11}; \quad R^{(2)}(\lambda) = \begin{pmatrix} 1 - \bar{r}\delta^2 e^{-2it\theta} & \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \Omega_{16}, \tag{5.22}$$

where  $R_{11}(\lambda)$  and  $R_{16}(\lambda)$  on the line  $(\lambda_0, \infty)$  have the boundary values

$$R_{11}(\lambda) = r(\lambda), \quad R_{16}(\lambda) = \bar{r}(\lambda), \quad \lambda \in (\lambda_0, \infty). \tag{5.23}$$

On  $\Sigma_{11}$ , we have

$$v^1(\lambda) = I, \quad v^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda - \lambda_0)] = e^{-it\theta\hat{\sigma}_3}(\lambda - \lambda_0)^{i\nu\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \hat{r}_0 & 1 \end{pmatrix}.$$

By using (5.23), (5.16) can be written as

$$v^{(2)}(\lambda) = \begin{pmatrix} 1 & 0 \\ R_{11}\delta^{-2}e^{2it\theta} & 1 \end{pmatrix} \cdot I \cdot R^{(2)}(\lambda)|_{\Omega_2} = \begin{pmatrix} 1 & 0 \\ \hat{r}_{10}e^{2it\theta}(\lambda - \lambda_0)^{-2i\nu} & 1 \end{pmatrix}. \tag{5.24}$$

We take

$$R^{(2)}(\lambda) = I, \quad \lambda \in \Omega_2,$$

Comparing the elements at position 21 of the jump matrix (5.24), we get

$$R_{11}(\lambda)\delta^{-2}e^{2it\theta} = f_1\delta^{-2}e^{2it\theta} = \hat{r}_{10}e^{2it\theta}(\lambda - \lambda_0)^{-2i\nu}. \tag{5.25}$$

Therefore, the boundary value of  $R_{11}(\lambda)$  on  $\Sigma_{11}$  is

$$R_{11}(\lambda) = \hat{r}_{10}(\lambda - \lambda_0)^{-2i\nu}\delta^2, \quad \lambda \in \Sigma_{11}. \tag{5.26}$$

The proofs for the other regions are similar.

From the literature [39], we get

$$M_{\lambda_0}^{PC}(\xi) = I + \frac{M_1^{PC}(\lambda_0)}{i\xi} + O(\xi^{-2}), \tag{5.27}$$

where

$$M_1^{PC}(\lambda_0) = \begin{pmatrix} 0 & \beta_{12}(\hat{r}_{10}) \\ \beta_{21}(\hat{r}_{10}) & 0 \end{pmatrix} \tag{5.28}$$

with

$$\beta_{12}(\hat{r}_{10}) = \frac{-\sqrt{2\pi}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu(\lambda_0)}{2}}}{\hat{r}_{10}\Gamma(-i\nu(\lambda_0))}. \tag{5.29}$$

Similarly, through the literature [39], we obtain

$$M_{-\lambda_0}^{PC}(\xi) = I + \frac{M_1^{PC}(-\lambda_0)}{i\xi} + O(\xi^{-2}), \tag{5.30}$$

where

$$M_1^{PC}(-\lambda_0) = \begin{pmatrix} 0 & \beta_{12}(\hat{r}_{20}) \\ \beta_{21}(\hat{r}_{20}) & 0 \end{pmatrix} \tag{5.31}$$

with

$$\beta_{12}(\hat{r}_{10}) = \frac{-\sqrt{2\pi}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu(-\lambda_0)}{2}}}{\hat{r}_{20}\Gamma(-i\nu(-\lambda_0))}. \tag{5.32}$$

Since  $m^{(1)}(\lambda)$  is analytical in the regions  $\Omega_{nj}, n = 1, 2, j = 1, 3, 4, 6$ , and  $(R^{(2)}(\lambda))^{-1}\bar{\partial}R^{(2)}(\lambda) = \bar{\partial}R^{(2)}(\lambda)$ , it follows that

$$\bar{\partial}m^{(2)}(\lambda) = m^{(1)}(\lambda)\bar{\partial}R^{(2)}(\lambda) = m^{(2)}(\lambda)(R^{(2)}(\lambda))^{-1}\bar{\partial}R^{(2)}(\lambda) = m^{(2)}(\lambda)\bar{\partial}R^{(2)}(\lambda), \tag{5.33}$$

where

$$\bar{\partial}R^{(2)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 0 \\ -\bar{\partial}R_{n1}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix}, & \lambda \in \Omega_{n1}, \\ \begin{pmatrix} 0 & -\bar{\partial}R_{n3}e^{-2it\theta}\delta^2 \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_{n3}, \\ \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_{n4}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix}, & \lambda \in \Omega_{n4}, \\ \begin{pmatrix} 0 & \bar{\partial}R_{n6}e^{-2it\theta}\delta^2 \\ 0 & 0 \end{pmatrix}, & \lambda \in \Omega_{n6}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \lambda \in \Omega_{n2} \cup \Omega_{n5}. \end{cases} \tag{5.34}$$

$n = 1, 2$ .

## 6. Pure $\bar{\partial}$ -problem and asymptotics of its solutions

We define

$$E(\lambda) = m^{(2)}(\lambda)[m^{(pc)}(\sqrt{12\alpha t\lambda_0}(\lambda - \lambda_0))], \tag{6.1}$$

$E(\lambda)$  is continuous in  $\mathbb{C}$  without jumping. Through (5.14) and (6.1), on  $\Sigma_{1j}, j = 1, 2, 3, 4$ , have

$$\begin{aligned} E_-^{-1}(\lambda)E_+(\lambda) &= m_-^{(pc)}(m_-^{(2)})^{-1}m_+^{(2)}(m_+^{(pc)})^{-1} \\ &= m_-^{(pc)}v^{(2)}(m_-^{(pc)}v^{(pc)})^{-1} \\ &= m_-^{(pc)}v^{(2)}(m_-^{(pc)}v^{(2)})^{-1} \\ &= I. \end{aligned} \tag{6.2}$$

Therefore, we get a pure  $\bar{\partial}$ -problem

- $E(\lambda)$  is continuous in  $\mathbb{C}$ ,
  - $\bar{\partial}E(\lambda) = E(\lambda)W(\lambda), \lambda \in \mathbb{C}$ ,
  - $E(\lambda) \sim I, \lambda \rightarrow \infty$ ,
- $$\tag{6.3}$$

where

$$W(\lambda) = \begin{cases} m^{(pc)}(\lambda) \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_{11}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{11}, \\ m^{(pc)}(\lambda) \begin{pmatrix} 0 & -\bar{\partial}R_{13}e^{-2it\theta}\delta^2 \\ 0 & 1 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{13}, \\ m^{(pc)}(\lambda) \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_{14}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{14}, \\ m^{(pc)}(\lambda) \begin{pmatrix} 0 & -\bar{\partial}R_{16}e^{-2it\theta}\delta^2 \\ 0 & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{16}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \lambda \in \Omega_{12} \cup \Omega_{15}. \end{cases} \quad (6.4)$$

$\bar{\partial}$ -problem (6.3) is equivalent to the following integral equation

$$E(\lambda) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{E(s)W(s)}{s-\lambda} dA(s), \quad (6.5)$$

where  $dA(s)$  is the Lebesgue measure on the real plane. Equation (6.5) can also be expressed as an operator

$$(1 - S)E(\lambda) = I, \quad (6.6)$$

where  $S$  is a Cauchy operator

$$S[f](\lambda) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W(s)}{s-\lambda} dA(s). \quad (6.7)$$

**Proposition 6.1.** *For sufficiently large  $t$ , the operator  $S$  is a small norm,  $(1-S)^{-1}$  exists, and*

$$\|S\|_{L^\infty \rightarrow L^\infty} \leq ct^{-\frac{1}{4}}. \quad (6.8)$$

From equation (6.7), further expand  $E(\lambda)$  as

$$E(\lambda) = I + \frac{E_1(\lambda)}{\lambda} + O(\lambda^{-2}), \quad (6.9)$$

where

$$E_1(\lambda) = \frac{1}{\pi} \iint_{\Omega_{11}} E(s)W(s) dA(s),$$

and satisfies the following estimation

$$|E_1(\lambda)| \leq ct^{-\frac{3}{4}}. \quad (6.10)$$

In addition, we define

$$T(\lambda) = m^{(2)}(\lambda)[m^{(pc)}(\sqrt{-12\alpha t\lambda_0(\lambda + \lambda_0)})], \quad (6.11)$$

and  $T(\lambda)$  is continuous in  $\mathbb{C}$  without jumping. Through (5.14) and (6.11), on  $\Sigma_{2j}, j = 1, 2, 3, 4$ , we have

$$T_-^{-1}(\lambda)T_+(\lambda) = m_-^{(pc)}(m_-^{(2)})^{-1}m_+^{(2)}(m_+^{(pc)})^{-1}$$



$$\begin{aligned}
 &= m_-^{(pc)} v^{(2)} (m_-^{(pc)} v^{(pc)})^{-1} \\
 &= m_-^{(pc)} v^{(2)} (m_-^{(pc)} v^{(2)})^{-1} \\
 &= I.
 \end{aligned} \tag{6.12}$$

Therefore, we get a pure  $\bar{\partial}$ -problem

$$\begin{aligned}
 &\bullet T(\lambda) \text{ is continuous in } \mathbb{C}, \\
 &\bullet \bar{\partial}T(\lambda) = T(\lambda)M(\lambda), \quad \lambda \in \mathbb{C}, \\
 &\bullet T(\lambda) \sim I, \quad \lambda \rightarrow \infty,
 \end{aligned} \tag{6.13}$$

where

$$M(\lambda) = \begin{cases} m^{(pc)}(\lambda) \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_{21}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{21}, \\ m^{(pc)}(\lambda) \begin{pmatrix} 0 & -\bar{\partial}R_{23}e^{-2it\theta}\delta^2 \\ 0 & 1 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{23}, \\ m^{(pc)}(\lambda) \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_{24}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{24}, \\ m^{(pc)}(\lambda) \begin{pmatrix} 0 & -\bar{\partial}R_{26}e^{-2it\theta}\delta^2 \\ 0 & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, & \lambda \in \Omega_{26}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \lambda \in \Omega_{22} \cup \Omega_{25}. \end{cases} \tag{6.14}$$

$\bar{\partial}$ -problem (6.13) is equivalent to the following integral equation

$$T(\lambda) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{EW}{s - \lambda} \, dA(s), \tag{6.15}$$

where  $dA(s)$  is the Lebesgue measure on the real plane. Equation (6.11) can also be represented as an operator

$$(1 - S)T(\lambda) = I, \tag{6.16}$$

where  $S$  is a Cauchy operator

$$S[f](\lambda) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W(s)}{s - \lambda} \, dA(s). \tag{6.17}$$

From (6.17), we further expand  $T(\lambda)$  as

$$T(\lambda) = I + \frac{T_1(\lambda)}{\lambda} + O(\lambda^{-2}), \tag{6.18}$$

where

$$T_1(\lambda) = \frac{1}{\pi} \iint_{\Omega_{11}} TW \, dA(s),$$

and satisfies the following estimation

$$|T_1(\lambda)| \leq ct^{-\frac{3}{4}}. \tag{6.19}$$

Recalling the series of transformations we have made (4.8), (5.13), (6.1) and (6.11), we can reverse these transformations

$$m(\lambda) \leftrightarrow m^{(1)}(\lambda) \leftrightarrow m^{(2)}(\lambda) \leftrightarrow (E(\lambda) + T(\lambda)), \quad (6.20)$$

and have

$$m(\lambda) = E(\lambda)M_{\lambda_0}^{PC}(R^{(2)}(\lambda))^{-1}\delta(\lambda)^{\sigma_3} + T(\lambda)M_{-\lambda_0}^{PC}(R^{(2)}(\lambda))^{-1}\delta(\lambda)^{\sigma_3}. \quad (6.21)$$

Particularly, considering  $\lambda \rightarrow \infty$  in  $\lambda \in \Omega_{12}, \Omega_{15}, \Omega_{22}, \Omega_{25}$ ,  $R^{(2)} = I$ , then we have

$$\begin{aligned} m(\lambda) &= \left( I + \frac{E_1(\lambda)}{\lambda} + \dots \right) \left( I + \frac{(M_{\lambda_0}^{PC})}{\sqrt{12\alpha t \lambda_0} + \dots} \right) \left( I + \frac{\Delta_1}{\lambda} + \dots \right) \\ &= \left( I + \frac{T_1(\lambda)}{\lambda} + \dots \right) \left( I + \frac{(M_{-\lambda_0}^{PC})}{\sqrt{-12\alpha t \lambda_0} + \dots} \right) \left( I + \frac{\Delta_1}{\lambda} + \dots \right). \end{aligned} \quad (6.22)$$

There is also

$$m_1(\lambda) = E_1(\lambda) + T_1(\lambda) + \frac{(M_{\lambda_0}^{PC})_{12}}{\sqrt{12\alpha t \lambda_0}} + 2i \frac{(M_{-\lambda_0}^{PC})_{12}}{\sqrt{-12\alpha t \lambda_0}} + \Delta_1, \quad (6.23)$$

where  $\Delta_1 = \begin{pmatrix} \delta_1 & 0 \\ 0 & -\delta_1 \end{pmatrix}$ .

Therefore, we get

$$u(x, t) = 2i \frac{(M_{\lambda_0}^{PC})_{12}}{\sqrt{12\alpha t \lambda_0}} + 2i \frac{(M_{-\lambda_0}^{PC})_{12}}{\sqrt{-12\alpha t \lambda_0}} + O(t^{\frac{3}{4}}), \quad (6.24)$$

where  $M_{\lambda_0}^{PC} = -i\beta_{12}(\hat{r}_{10})$ ,  $M_{-\lambda_0}^{PC} = -i\beta_{12}(\hat{r}_{20})$ .

## Appendix A

Here we describe the solution to the parabolic cylindrical model problem introduced by [39], which has been widely used to study the long-time asymptotics of integrable systems in the literature [21, 40]. Define the contour

$$\Sigma^{pc} = \cup_{j=1}^4 \Sigma_j, \Sigma_j = \left\{ \zeta = \mathbb{R}^+ e^{\frac{i(2j-1)\pi}{4}}, j = 1, 2, 3, 4 \right\}, \quad (6.25)$$

we have the following parabolic cylinder model problem.

**RH Problem A.1 :**

- $M^{(pc)}(\xi)$  is analytic in  $\mathbb{C} \setminus \Sigma^{pc}$ , (6.26)

- $M_+^{(pc)}(\xi) = M_-^{(pc)}(\xi)V^{(pc)}(\xi)$ ,  $\xi \in \Sigma^{pc}$ , (6.27)

- $M^{(pc)}(\xi) = I + \frac{M_1}{\xi} + O(\xi^2)$ ,  $\xi \rightarrow \infty$ . (6.28)

where

$$v^{(pc)}(\xi) = \begin{cases} \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ r_0 & 1 \end{pmatrix}, & \xi \in \Sigma_1, \\ \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & \frac{-\bar{r}_0}{1-|r_0|^2} \\ 0 & 1 \end{pmatrix}, & \xi \in \Sigma_2, \\ \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \frac{r_0}{1-|r_0|^2} & 1 \end{pmatrix}, & \xi \in \Sigma_3, \\ \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & -\bar{r}_0 \\ 0 & 1 \end{pmatrix}, & \xi \in \Sigma_4. \end{cases} \tag{6.29}$$

(See figure 6)

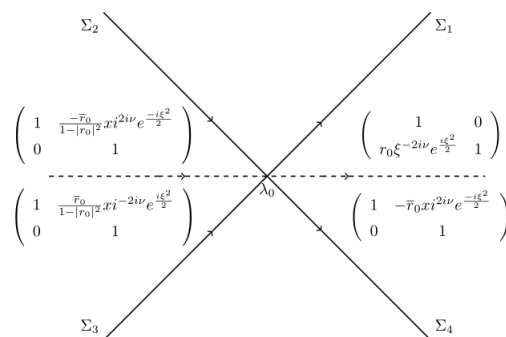


Figure 6. The jump matrix  $v^{(pc)}(\xi)$

We know that the RH Problem A.1 admits the solution

$$M^{pc}(\zeta, r_0) = I + \frac{M_1^{pc}(r_0)}{i\zeta} + \mathcal{O}(\zeta^{-2}) \tag{6.30}$$

where

$$M_1^{pc}(r_0) = \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix}$$

with  $\beta_{12}$  and  $\beta_{21}$  which are two complex constants

$$\beta_{12} = \frac{\sqrt{2\pi}e^{i\pi/4}e^{-\pi\nu/2}}{r_0\Gamma(-i\nu)}, \quad \beta_{21} = -\frac{\sqrt{2\pi}e^{-i\pi/4}e^{-\pi\nu/2}}{\varepsilon_n \bar{r}_0\Gamma(i\nu)}.$$

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## References

- [1] J. Yang, P. Yu and X. Sun, *On the independent perturbation parameters and the number of limit cycles of a type of Lienard system*, Journal of Mathematical Analysis and Applications, 2018, 464(1), 679–692.
- [2] K. Dysthe, H. E. Krogstad, P. Müller, *Oceanic rogue waves*, Annual Review of Fluid Mechanics, 2008, 40, 287–310.
- [3] S. K. El-Labany, W. M. Moslem, E. I. El-Awady et al., *Nonlinear dynamics associated with rotating magnetized electron-positron-ion plasmas*, Physics Letters A, 2010, 375(2), 159–164.
- [4] A. Chabchoub, N. Hoffmann, M. Onorato, et al., *Super rogue waves: observation of a higher-order breather in water waves*, Physical Review X, 2012, 2(1), 011015.
- [5] M. Alam, M. Rahman, R. Islam et al., *Application of the new extended  $(G'/G)$ -expansion method to find exact solutions for nonlinear partial differential equation*, Computational Methods for Differential Equations, 2015, 3(1), 59–69.
- [6] A. Yokus, H. M. Baskonus, T. A. Sulaiman et al., *Numerical simulation and solutions of the two-component second order KdV evolutionary system*, Numerical Methods for Partial Differential Equations, 2018, 34(1), 211–227.
- [7] C. M. Khalique, I. E. Mhlanga, *Travelling waves and conservation laws of a  $(2+1)$ -dimensional coupling system with Korteweg-de Vries equation*, Applied Mathematics and Nonlinear Sciences, 2018, 3(1), 241–254.
- [8] Y. Zhang, J. Mei, X. Zhang, et al., *Symmetry properties and explicit solutions of some nonlinear differential and fractional equations*, Applied Mathematics and Computation, 2018, 408–418.
- [9] Y. Zhang, Q. Liu, Z. Qiao, et al., *Fifth-order b-family Novikov (FObFN) model with pseudo-peakons and multi-peakons*, Modern Physics Letters B, 2019, 33(18).
- [10] C.S. Gardner, J.M. Green, M.D. Kruskal, R.M. Miura, *Method for solving the Korteweg-de Vries equation*, Physical Review Letters, 1967,19, 1095–1097.
- [11] R.F. Bikbaev, *Asymptotic behavior as  $t \rightarrow \infty$  of the solution to the Cauchy problem for the Landau-Lifshitz equation*, Teoreticheskaya i Matematicheskaya Fizika, 1988, 77(2), 163–170.
- [12] A.S. Fokas, A.R. Its, *Soliton generation for initial-boundary-value problems*. Physical Review Letters, 1992, 68(21), 3117–3120.
- [13] E. K. Sklyanin, *Method of the inverse scattering problem and the nonlinear quantum Schrödinger equation*, Soviet Physics Doklady, 1979, 24.
- [14] A. Abdeljabbar, W. Ma, A. Yildirim, *Determinant solutions to a  $(3+1)$ -dimensional generalized KP equation with variable coefficients*, Chinese Annals of Mathematics, Series B, 2012, 33(5), 641–650.
- [15] J. Chen, Y. Chen, B. Feng, et al., *Rational solutions to two- and one-dimensional multicomponent Yajima-Oikawa systems*, Physics Letters A, 2015, 379(24-25), 1510–1519.
- [16] W. X. Ma, *Soliton solutions by means of Hirota bilinear forms*, Partial Differential Equations in Applied Mathematics, 2021, 5.

- [17] J. Zhang, L. Wang, C. Liu, *Superregular breathers, characteristics of nonlinear stage of modulation instability induced by higher-order effects*, Proceedings of the Royal Society A Mathematical Physical and Engineering Sciences, 2017, 473(2199), 20160681.
- [18] X. Wang, Y. Li, Y. Chen, *Generalized Darboux transformation and localized waves in coupled Hirota equations*, Wave Motion, 2014, 51(7), 1149–1160.
- [19] J. Plemelj, *Riemannsche funktionenscharen mit gegebener monodromiegruppe*, Monatshefte für Mathematik und Physik, 1908, 19, 211–245
- [20] D. V. Anosov, A. A. Bolibruch, *The Riemann-Hilbert Problem*, Wiesbaden: Springer, 1994
- [21] P. A. Deift, X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Annals of Mathematics, 1993, 137, 295–368.
- [22] P. A. Deift, X. Zhou, *Long-time asymptotics for integrable nonlinear wave equation*, in "Important developments in soliton theory", Springer Ser, Nonlinear Dynamics, Springer, Berlin, 1993, 181–204.
- [23] S. Cuccagna, R. Jenkins, *On the Asymptotic Stability of  $N$ -Soliton Solutions of the Defocusing Nonlinear Schrödinger Equation*. Communications in Mathematical Physics, 2016, 343(3), 921–969.
- [24] R. Jenkins, J. Liu, P. Perry, et al., *Soliton resolution for the derivative nonlinear Schrödinger equation*. Communications in Mathematical Physics, 2018, 363(3), 1003–1049.
- [25] J. Xu, E. Fan, et al., *Long-time Asymptotic for the Derivative Nonlinear Schrödinger Equation with Step-like Initial Value*, Mathematical Physics, Analysis & Geometry, 2013, 16(3), 253–288.
- [26] J. Xu, E. Fan, *Long-time asymptotics for the Fokas-Lenells equation with decaying initial value problem: Without solitons*, Journal of Differential Equations, 2015, 259(3), 1098–1148.
- [27] Q. Cheng, E. Fan, *Long-time asymptotics for a mixed nonlinear Schrödinger equation with the Schwartz initial data*, Journal of Mathematical Analysis and Applications, 2020, 489(2), 124188.
- [28] Q. Cheng, E. Fan, *Long-time asymptotics for the focusing Fokas-Lenells equation in the solitonic region of space-time*. Journal of Differential Equations, 2022, 309, 883-948.
- [29] X. Geng, M. Chen, K. Wang, *Long-time asymptotics of the coupled modified Korteweg-de Vries equation*, Journal of Geometry and Physics, 2019, 142, 151–167.
- [30] X. Geng, K. Wang, M. Chen, *Long-time asymptotics for the spin-1 Gross-Pitaevskii equation*, Communications in Mathematical Physics, 2021, 382(1), 585-611.
- [31] W. X. Ma, *Riemann-Hilbert problems and  $N$ -soliton solutions for a coupled mKdV system*, Journal of Geometry and Physics, 2018, 132, 45–54.
- [32] W. X. Ma, *Long-time asymptotics of a three-component coupled nonlinear Schrödinger system*, Journal of Geometry and Physics, 2020, 153, 103669.

- [33] W. X. Ma, *Reduced nonlocal integrable mKdV equations of type  $(-\lambda, \lambda)$  and their exact soliton solutions*, Communications in Theoretical Physics, 2022, 74, 065002.
- [34] F. d. Wang, W. X. Ma, *A  $\bar{\partial}$ -Steepest descent method for oscillatory Riemann-CHilbert problems*, Journal of Nonlinear Science, 2022, 32, 10.
- [35] W. X. Ma, *Type  $(-\lambda, -\lambda^*)$  reduced nonlocal integrable mKdV equations and their soliton solutions*, Applied Mathematics Letters, 2022, 131, 108074.
- [36] W. X. Ma, *Soliton hierarchies and soliton solutions of type  $(-\lambda^*, -\lambda)$  reduced nonlocal nonlinear Schrödinger equations of arbitrary even order*, Partial Differential Equations in Applied Mathematics, 2023, 7, 100515.
- [37] Y. Li, *Solitons and integrable systems*, Shanghai Science & Technology Education Press, 1999.
- [38] G. Chen, J. Liu. *Long-time asymptotics of the modified KdV equation in weighted Sobolev spaces*, Forum of Mathematics, Sigma, 2022, 10, 1–52.
- [39] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, fourth ed., Cambridge University Press, Cambridge, 1927.
- [40] A. Its, *Asymptotic behavior of the solutions to the nonlinear Schrödinger equation, and isomonodromic deformations of systems of linear differential equations*, Dokl. Akad. Nauk SSSR 261 (1981) 14-18.