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Threshold of Effective Degree SIR Model*

Slim Ibrahim[1,†], Meili Li[2], Junling Ma[1] and Kurtis Manke[1]

Abstract The effective degree SIR model is a precise model for the SIR disease dynamics on a network. The original ODE model is only applicable for a network with finite degree distributions. The new generating function approach rewrites with model as a PDE and allows infinite degree distributions. In this paper, we first prove the existence of a global solution. Then we analyze the linear and nonlinear stability of the disease-free steady state of the PDE effective degree model, and show that the basic reproduction number still determines both the linear and the nonlinear stability. Our method also provides a new tool to study the effective degree SIS model, whose basic reproduction number has been elusive so far.

Keywords Generating function, effective degree model, basic reproduction number, spectral stability, nonlinear stability, steady states

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1. Introduction

Classical compartmental models (see, e.g., [1,7]) assume random mixing, i.e., each pair of individuals has the same rate of contact. Network disease models [8] use contact networks to represent a population and its contacts. Specifically, individuals are represented by nodes and contacts are represented by edges. Such a contact network can model realistic contact patterns in the population such as households and workplaces. It can also be used to study the effectiveness of disease control strategies such as contact tracing and prioritized vaccination.

Early network models are node based, which group individuals by their degrees (the number of contacts) in addition to their infection status (e.g., susceptible, infectious, recovered, etc.) The Sattaros and Vespignani model [11] used such a model to show that, on scale-free networks with an infinite variance of the degree distribution, there exists no disease threshold, i.e., any positive transmission rate

†The corresponding author.
Email address: ibrahims@uvic.ca (S. Ibrahim), stylml@dhu.edu.cn (M. Li), junlingm@uvic.ca (J. Ma), kurtismanke@uvic.ca
1Department of Mathematics and Statistics, University of Victoria, Victoria, BC/V8W 2Y2, Canada
2College of Science, Donghua University, Shanghai, 201620, China
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may cause an outbreak in an infinite population. However, these simple node based models ignore the correlation between the infection status of neighboring nodes, leading to an over-estimate of the disease spread. Lindquist et al. [9] extended these models by grouping the infection status of the nodes and their neighbors, resulting in more precise effective-degree models. Based on effective-degree models, they showed that an SIS type disease (without acquired immunity) is easier to establish than an SIR type model (with acquired immunity), which contrasts the predictions of classical random mixing models that these two types of diseases have identical disease thresholds.

These node based models are systems of ordinary differential equations (ODE) with dimensions dependent on the degree distribution of the network. Model theoretical studies rely on Poisson networks or scale-free networks that have infinite numbers of degrees, resulting in an infinite system. It is a challenge to apply ODE theories to these models. In an earlier paper [6], we have developed a generating function approach to rewrite the effective-degree SIR model to a first-order nonlinear partial differential equation (PDE). We have shown the well-posedness of the PDE model, and shown that, if initially the infection status of the neighbors of a random susceptible node are independent, then the effective-degree PDE model can be simplified to the Volz model [12].

Interestingly, it has been shown (see, e.g., [8]) that, with the same condition on initial conditions, the full SIR model of the pair approximation approach [4, 5] can also be simplified to the Volz model, which can further be simplified to the Miller [10] model. However, if this assumption does not hold, then the Volz-Miller models may not be precise, yet the effective degree models and the pair approximation models may still be applicable. Thus, it is still important to fully understand the effective-degree models.

In this paper, we study the linear and nonlinear stability of the disease-free steady states of the PDE effective-degree SIR model, and show that the disease threshold condition (i.e., the basic reproduction number being unity) that is derived in the finite dimensional effective-degree ODE model is also true for the infinite dimensional model. This disease threshold condition also agrees with that derived from other network models (such as the Volz-Miller models [10, 12].)

In Section 3 we introduce the notation. Section 4 states the main theorems that this paper sets out to prove. Sections 6, 7, and 8 prove the linear stability, nonlinear instability, and nonlinear stability respectively. We provide some discussion and remarks in Section 9.

2. The Effective-Degree PDE model

The Effective-Degree ODE model [9] considers an SIR model on a contact network with degree distribution given by p_k where k is the degree. The susceptibles are infected by neighbouring infectious individuals with a per link transmission rate $\beta \geq 0$, and infectious individuals recover to full immunity at a rate $\gamma > 0$. This model compartmentalizes nodes by both their state and by the number of neighbours that it has in each state.

Denote S_{si} as the fraction of susceptible nodes having s susceptible neighbours and i infected neighbours, where $s+i$ is the effective degree of the node, and denote I and R as the fractions of infected and recovered nodes in the population. You may have noticed two things: 1) we don't keep track of the number of recovered

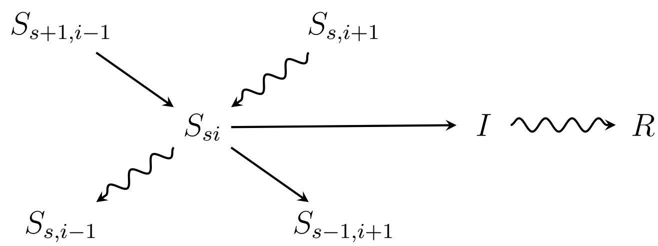

Figure 1. The flow between compartments for the Effective Degree SIR model. The straight arrows represent infection and the curvy arrows represent recovery.

neighbours a susceptible node has, and 2) we don't keep track of the neighbours of infected/recovered nodes at all. By recognizing that once a node enters the recovered state it no longer contributes to the disease dynamics, the model can be simplified by only considering neighbouring nodes that are infected or susceptible. Furthermore, once a central node becomes infected, its neighbours no longer influence its status. The flow chart for the SIR effective degree system is given in Figure 1.

The system of ODEs that governs the SIR effective degree model is thus

$$\begin{aligned}\dot{S}_{si} &= -\beta i S_{si} + \beta \frac{\sum_{s,i} s i S_{si}}{\sum_{s,i} s S_{si}}[(s+1)S_{s+1,i-1} - sS_{si}] \\ &\quad + \gamma[(i+1)S_{s,i+1} - iS_{si}], \qquad (2.1a)\\ S &= \sum_{s,i} S_{si}, \qquad \dot{I} = -\dot{S} - \gamma I, \qquad \dot{R} = \gamma I. \qquad (2.1b)\end{aligned}$$

Given an SIR effective degree model with susceptible fractions of a population given by S_{si}, one can use a generating function effective degree approach by defining

$$S(t,x,y) = \sum_{s,i} x^s y^i S_{si}, \tag{2.2}$$

set on the square $(x,y) \in [0,1] \times [0,1]$ to derive the following closed Partial Differential Equation

$$S_t = -(\beta+\gamma)\big(y - \frac{\gamma}{\beta+\gamma}\big)S_y + \frac{S_{xy}(t,1,1)}{S_x(t,1,1)}\beta(y-x)S_x. \tag{2.3}$$

Biologically, a susceptible node remains susceptible if and only if none of its infectious neighbors transmitted the disease. The probability that an infectious neighbor does not transmit before it recovers is

$$\Gamma = \frac{\gamma}{\beta+\gamma}.$$

The disease dynamics is thus determined by the generating function $S(t,x,z) = S(t,x,\Gamma + z(1-\Gamma))$ where $y = \Gamma + z(1-\Gamma)$ is the probability generating function for a Bernoulli random variable considering an infectious neighbor of a susceptible node that eventually transmits to the node before it recovers. Mathematically, this is equivalent to restricting the domain of the PDE (2.3) to the rectangle $[0,1] \times [\Gamma, 1]$.

The following theorem guarantees that the long term behavior of $S(t,x,z)$ is determined by its behavior when restricted to $x \in [\Gamma, 1]$.

Theorem 2.1. *The square $R=[\Gamma,1]\times[\Gamma,1]$ is negatively invariant for the following system of characteristics that solves the PDE.*

$$\frac{ds}{dt}=0, \tag{2.4a}$$
$$\frac{dy}{dt}=(\beta+\gamma)(y-\Gamma), \tag{2.4b}$$
$$\frac{dx}{dt}=\beta(x-y)\tilde{\phi}'(t), \tag{2.4c}$$

where

$$\tilde{\phi}(t)=\int_0^t \frac{S_{xy}(s,1,1)}{S_x(s,1,1)}\, ds.$$

Proof. For any characteristics with $y(0)>\Gamma$, (2.4b) guarantees that $y(t)>\Gamma$ for all time t. If, in addition, $x(0)<\Gamma$, then (2.4c) guarantees that $dx/dt<0$, and thus the characteristics cannot enter the square R. □

We thus restrict the domain of (2.3) to the square R, and consider the generating function

$$\tilde{S}(t,w,z)=S(t,\Gamma+w(1-\Gamma),\Gamma+z(1-\Gamma))$$

with the additional variable change $x=\Gamma+w(1-\Gamma)$. We abuse the notation by dropping the $\sim$ for simplification. Furthermore, we rescale the time as $t\leftarrow(\beta+\gamma)t$, then the PDE (2.3) becomes

$$S_t=-zS_z+\frac{S_{wz}(t,1,1)}{S_w(t,1,1)}(z-w)S_w, \tag{2.5}$$

with an initial condition

$$S(0,w,z)=S_0(w,z). \tag{2.6}$$

The system is defined on the unit square $(w,z)\in[0,1]\times[0,1]$, which corresponds to $(x,y)\in R$.

In addition, the variables of the original ODE model S_{si} can be computed from St,x,y as

$$S_{si}=\frac{1}{s!i!}\frac{\partial^{s+i}}{\partial x^s\partial y^i}S(t,1,1),$$

and thus they are determined by $S(t,x,y)$ in a small neighborhood of the point $(t,1,1)$.

We do not impose boundary conditions on this PDE due to the special structure of its characteristics. We refer the interested reader to [6] for the development of the model including well-posedness. Using the method of characteristics, solving the characteristic equations

$$\frac{dS}{dt}=0, \tag{2.7a}$$
$$\frac{dz}{dt}=z, \tag{2.7b}$$
$$\frac{dw}{dt}=(w-z)\phi'(t) \tag{2.7c}$$

gives sufficiently regular solutions to (2.5) (see, [6])

$$S(t,w,z)=S(0,w_0(t),z_0(t))=S_0(w_0(t),z_0(t)), \tag{2.8a}$$

$$z_0(t,z) = ze^{-t}, \tag{2.8b}$$

$$w_0(t,w,z) = we^{-\phi(t)} + z\eta(t,\phi(t)), \tag{2.8c}$$

where ϕ and η are defined by

$$\phi(t) = \int_0^t \frac{S_{wz}(s,1,1)}{S_w(s,1,1)}\, ds, \tag{2.9a}$$

$$\eta(t,\phi(t)) = e^{-t} - e^{-\phi(t)} + e^{-t}\int_0^t e^{s-\phi(s)}\, ds. \tag{2.9b}$$

Equation (2.9a) puts a constraint on solutions. Substituting (2.8) into (2.9a) gives the "compatibility condition"

$$\begin{pmatrix} \phi'(t) \\ \eta'(t) \end{pmatrix} = \begin{pmatrix} f(t,\phi,\eta)e^{-t} + h(t,\phi,\eta)\eta \\ f(t,\phi,\eta)e^{-t-\phi} + [h(t,\phi,\eta)e^{-\phi} - 1]\eta \end{pmatrix} \tag{2.10}$$

with

$$f(t,\phi,\eta) = \frac{\partial^2_{w_0 z_0} S_0}{\partial_{w_0} S_0}(w_0, z_0)|_{w=z=1}, \tag{2.11a}$$

$$h(t,\phi,\eta) = \frac{\partial^2_{w_0} S_0}{\partial_{w_0} S_0}(w_0, z_0)|_{w=z=1}. \tag{2.11b}$$

Recall that from [6], a unique solution exists to (2.7), and consequently, the PDE (2.5) has a unique solution. This has been shown by proving the existence and uniqueness of ϕ and η.

In addition, it is clear that $S(t,w,z) = \bar{S}(w)$ solves (2.5). From equation (2.2), we see that $\bar{S}(w)$ corresponds to solutions to the ODE model where $S_{si} = 0$ for all $i \geq 0$, i.e. $I = 0$. Thus $S(t,w,z) = \bar{S}(w)$ represents disease-free equilibrium solutions.

In this paper we will show that the linear stability and the nonlinear instability of the disease-free equilibrium are determined by the disease-threshold condition $\mathcal{R}_0$

$$\mathcal{R}_0 = \frac{\beta}{\beta+\gamma}\frac{\sum_{k=0}^{\infty} k(k-1)p_k}{\sum_{k=0}^{\infty} kp_k}. \tag{2.12}$$

Importantly, this is the same threshold condition that determines the stability of the finite dimensional ODE formulation of the effective degree SIR model.

3. Notation

In this section, we introduce the functional spaces in which we study the stability of disease-free solutions, and recall a few standard definitions about unbounded operators, and their spectrum.

Definition 3.1. Let X be the Banach space of functions of the form

$$S(w,z) = \sum_{m,n} w^m z^n s_{mn} \tag{3.1}$$

endowed with the norm

$$\|S\|_X = \sum_{m,n} (1 + m + m^2)|s_{mn}| < \infty. \tag{3.2}$$

Remark that the space X differs from the space used in the original development of the model [6] by the inclusion of the weight factor m^2. The current space X is a smaller space, so the results of that paper apply here as well. Next, we recall the definition of a bounded operator.

Definition 3.2. A linear operator $L : X \to Y$ between Banach spaces X and Y is said to be bounded if there exists M such that for all $x \in X$ [2, Section 12.5.1.1],

$$\|Lx\|_Y \le M\|x\|_X. \tag{3.3}$$

Next, we recall the definition of the resolvent of an operator.

Definition 3.3. Let X be a Banach space and $L:\ D(L) \subset X \to X$ be a linear operator on X defined on domain $D(L) \subset X$. A complex number λ is said to be in the resolvent set, that is, the complement of the spectrum of a linear operator if the operator

$$L - \lambda I :\ D(L) \to X \tag{3.4}$$

has a bounded inverse (where I is the identity operator). That is, the inverse

$$(L - \lambda I_d)^{-1} : X \to D(L) \tag{3.5}$$

exists, and is bounded [3, Page 566].

Definition 3.4. Let L be the linearized operator of the PDE (2.5) about a disease-free equilibrium $\bar{S}$, i.e.,

$$L[S] = -zS_z + (z - w)\frac{\bar{S}'(w)}{\bar{S}'(1)} S_{wz}(t, 1, 1). \tag{3.6}$$

We define its domain $D(L) \subset X$ that has a finite domain norm

$$\|S\|_{\text{Dom}} = \sum_{m\ge 0,\ n\ge 0} ((1 + m)(1 + n) + m^2)|s_{mn}| < \infty. \tag{3.7}$$

Note that the X and domain norms are equivalent to the ones defined in [6]. Finally, we recall the definition of nonlinear stability.

Definition 3.5. We say that a steady state solution $\bar{S}(w)$ of (2.5) is unstable if there exists an $\epsilon > 0$, such that for all $\delta > 0$, there exists an initial condition $S(0, w, z)$ and a time T such that $\|S(0, w, z) - \bar{S}(w)\|_{\text{Dom}} < \delta$ but $\|S(T, w, z) - \bar{S}(w)\|_X \ge \epsilon$.

4. Main results

In this section we give our main results, with proofs to follow in later sections. The first theorem guarantees that the system (2.5) is well-posed.

Theorem 4.1. *Given an initial condition $S_0(w, z)$ in the domain, i.e., $\|S_0\|_{Dom} < \infty$, there exists a unique solution solution $S(t, w, z) \in X$ for all time t.*

The following theorem shows that the basic reproduction number $\mathcal{R}_0$ determines the linear stability of the disease-free equilibrium $\bar{S}(w)$.

Theorem 4.2. *The spectrum of the linearized operator about the disease-free equilibrium $\bar{S}(w)$ consists of only eigenvalues with negative real part if and only if $\mathcal{R}_0 < 1$, given by equation* (2.12)*. That is, the equilibrium is linearly stable when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

The last theorem shows the sharpness of the reproduction number threshold even for the infinite dimensional system (PDE).

Theorem 4.3. *For $\mathcal{R}_0$ given by equation* (2.12)*, the disease-free equilibrium $\bar{S}(w)$ is*

(i) nonlinearly unstable for $\mathcal{R}_0 > 1$.

(ii) nonlinearly stable for $\mathcal{R}_0 < 1$.

In the following section we set out to prove Theorem 4.1.

5. Well-posedness

To prove Theorem 4.1, note that the solutions to (2.5) is given in implicitly in (2.8). The existence and uniqueness is proved in [6]. In this section, we will show that, if the initial condition $S_0(w,z) = \sum_{m,n} w^m z^n S_{mn}$ is in the domain, i.e.,

$$\|S_0\|_{\mathrm{Dom}} = \sum_{m,n}(1+m)(1+n) + m^2 S_{mn} < \infty,$$

then the solution given in (2.8a)

$$S(t,w,z) = S_0(e^{-\phi(t)}w + \eta(t)z, e^{-t}z) \in X.$$

Note that

$$\begin{aligned}
S_0(e^{-\phi(t)w} + \eta(t)z, e^{-t}z) &= \sum_{mn}(e^{-\phi(t)}w + \eta(t)z)^m e^{-tn} z^n S_{mn} \\
&= \sum_{m,n}\sum_{i=0}^{m}\binom{m}{i} e^{-\phi(t)i}\eta^{m-i}e^{-tn}w^i z^{n+m-i} S_{mn} \\
&= \sum_{i,n}\sum_{m=i}^{\infty}\binom{m}{i} e^{-\phi(t)i}\eta^{m-i}e^{-tn}w^i z^{n+m-i} S_{mn} \\
&= \sum_{i}\sum_{n,m=0}^{\infty}\binom{m+i}{i} e^{-\phi(t)i}\eta^{m}e^{-tn}w^i z^{n+m} S_{m+i,n} \\
&= \sum_{i}\sum_{k}\sum_{n=0}^{k}\binom{k-n+i}{i} e^{-\phi(t)i}\eta^{k-n}e^{-tn}w^i z^{k} S_{k-n+i,n} \\
&= \sum_{i}\sum_{k} w^i z^k e^{-\phi(t)i}\sum_{n=0}^{k}\binom{k-n+i}{i}\eta^{k-n}e^{-tn} S_{k-n+i,n}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \sum_k w^i z^k e^{-\phi(t)i} \sum_{n=0}^{k} \binom{k-n+i}{i} \eta^{k-n} e^{-tn} S_{k-n+i,n} \\
&:= \sum_i \sum_k w^i z^k e^{-\phi(t)i} \tilde{S}_{ik}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|S\|_X &= \sum_{i,k} (1+i+i^2) \tilde{S}_{ik} \\
&= \sum_{i,k=0}^{\infty} (1+i+i^2) e^{-\phi(t)i} \sum_{n=0}^{k} \binom{k-n+i}{i} \eta^{k-n} e^{-tn} S_{k-n+i,n}.
\end{aligned}
$$

We need to prove that this is bounded. Switch the order of k and n,

$$
\|S\|_X = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (1+i+i^2) e^{-\phi(t)i} \sum_{k=n}^{\infty} \binom{k-n+i}{i} \eta^{k-n} e^{-tn} S_{k-n+i,n}.
$$

Let $m = k-n+i$, $k = m+n-i$, and then

$$
\begin{aligned}
\|S\|_X &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} (1+i+i^2) e^{-\phi(t)i} \sum_{m=i}^{\infty} \binom{m}{i} \eta^{m-i} e^{-tn} S_{m,n} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{m} \binom{m}{i} \eta^{m-i} e^{-tn} (1+i+i^2) e^{-\phi(t)i} S_{m,n} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\sum_{i=0}^{m} \binom{m}{i} \eta^{m-i} (1+i+i^2) e^{-\phi(t)i}}{(1+m)(1+n)+m^2} e^{-tn} [(1+m)(1+n)+m^2] S_{m,n}.
\end{aligned}
$$

We give a bound for the fraction above. Note that

$$
\begin{aligned}
\frac{\sum_{i=0}^{m} \binom{m}{i} \eta^{m-i} (1+i+i^2) e^{-\phi(t)i}}{(1+m)(1+n)+m^2} &\le \frac{\sum_{i=0}^{m} \binom{m}{i} \eta^{m-i} e^{-\phi(t)i} (1+m+m^2)}{(1+m)(1+n)+m^2} \\
&\le \sum_{i=0}^{m} \binom{m}{i} \eta^{m-i} e^{-\phi(t)i} \\
&= [e^{-\phi(t)} + \eta(t)]^m.
\end{aligned}
$$

So,

$$
\|S\|_X \le \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (e^{-\phi(t)} + \eta)^m e^{-tn} [(1+m)(1+n)+m^2] S_{m,n}.
$$

Note that

$$
e^{-\phi(t)} + \eta = e^{-t} + e^{-t} \int_0^t e^{s-\phi(s)}\, ds,
$$

which is the solution to the initial value problem

$$
u' = -u + e^{-\phi(t)}, \ u(0) = 1.
$$

Thus $0 < u = e^{-\phi(t)} + \eta \le 1$ because $u'|_{u=1} < 0$. This guarantees that

$$
\|S\|_X \le \|S_0\|_{Dom} < \infty.
$$

That is, $S(t,w,z) \in X$ for all time t. In the following section we set out to prove Theorem 4.2.

6. Linear stability

Let $\bar{S}(w) = S^*(x)$ be an disease-free equilibrium solution of (2.5), i.e.

$$S^*(x) = \sum_k p_k x^k, \tag{6.1}$$

where p_k is the network degree distribution. Let $V = V(t, w, z)$ be a perturbation from the equilibrium. Suppose S solves (2.5) such that

$$S = \bar{S}(w) + V(t, w, z), \tag{6.2}$$

and then the perturbation V satisfies

$$V_t = N[V] + L[V], \tag{6.3}$$

where N is a nonlinear operator, and L is the linearized operator defined in (3.6). To determine the linear stability, we consider the eigenvalue problem of the linear operator:

$$(L - \lambda I_d)V = 0, \tag{6.4}$$

where I_d is the identity operator.

Lemma 6.1. *The set* $\{-k, 0, \frac{\bar{S}''(1)}{\bar{S}'(1)} - 1,$ *for* $k = 1, 2, \dots\}$ *is in the spectrum of the operator* L *defined by equation* (3.6).

Proof. By definition 3.3, the set λ is in the spectrum of L if $(L - \lambda I_d)$ is noninvertible. This is equivalent to showing that non-trivial solutions exist to

$$(L - \lambda I_d)V = 0. \tag{6.5}$$

The eigenvalue problem (6.4) can be treated as an ODE of $V(z)$ with w being a parameter

$$zV_z + \lambda V = (z - w)\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1, 1). \tag{6.6}$$

Using an integrating factor z^λ, (6.6) can be rewritten as

$$\frac{\partial}{\partial z}\left[z^\lambda V\right] = \left(z^\lambda - wz^{\lambda-1}\right)\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1, 1). \tag{6.7}$$

This gives rise to the following cases.

Case 1. If $\lambda = 0$, then (6.7) is

$$V_z = \left(1 - wz^{-1}\right)\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1, 1). \tag{6.8}$$

Integrate with respect to z

$$V(w, z) = \left[z - w\ln(z)\right]\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1, 1) + C(w), \tag{6.9}$$

where $C(w)$ is an undetermined function in X. Because $\ln(z) \notin X$ (as well as $w\ln(z)$), for V to be in X it is required that

$$\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1, 1) = 0. \tag{6.10}$$

Thus, if $\lambda = 0$, then any function that depends only on w is a solution to (6.6). So, $\lambda = 0$ is an eigenvalue.

Case 2. If $\lambda = -1$, equation (6.7) is

$$\frac{d}{dz}\big[z^{-1}V\big] = \big(z^{-1} - wz^{-2}\big)\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1). \tag{6.11}$$

Integrate with respect to z

$$z^{-1}V = \big(\ln(z) + wz^{-1}\big)\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1) + C(w), \tag{6.12}$$

that is,

$$V = \big(z\ln(z) + w\big)\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1) + zC(w), \tag{6.13}$$

where $C(w)$ is an undetermined function in X. Again, $z\ln(z) \notin X$, so for $V \in X$, we require $\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1) = 0$. Hence

$$V_{wz}(1,1) = 0, \tag{6.14}$$

and thus

$$V(w,z) = zC(w). \tag{6.15}$$

So

$$V_{wz}(1,1) = C'(1) = 0. \tag{6.16}$$

Thus the solution to (6.6) for the case $\lambda = -1$ is

$$V(w,z) = zC(w), \tag{6.17}$$

where $C(w) \in X$ and $C'(1) = 0$. So, $\lambda = -1$ is an eigenvalue.

Case 3. If $\lambda \neq 0$ and -1, solve (6.7) by integrating with respect to z

$$z^{\lambda}V = \big(\frac{1}{\lambda+1}z^{\lambda+1} - \frac{1}{\lambda}wz^{\lambda}\big)\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1) + C(w). \tag{6.18}$$

That is,

$$V(w,z) = \big[\frac{1}{\lambda+1}z - \frac{1}{\lambda}w\big]\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1) + C(w)z^{-\lambda}. \tag{6.19}$$

Thus, if $V(w,z) \in X$, then either $\lambda = -2, -3, \ldots$, or $C(w) = 0$.

Subcase 3.1. If $C(w) = 0$, then

$$V(w,z) = \big[\frac{1}{\lambda+1}z - \frac{1}{\lambda}w\big]\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1). \tag{6.20}$$

Note that the right hand side of (6.20) contains its own derivative. So $\partial^2_{wz}V$ of the right hand side must equal $V_{wz}(1,1)$. This compatibility condition determines the eigenvalue λ. Specifically,

$$V_{zw} = \frac{1}{\lambda+1}\frac{\bar{S}''(w)}{\bar{S}'(1)}V_{wz}(1,1). \tag{6.21}$$

At $w = z = 1$,

$$V_{wz}(1,1) = \frac{1}{\lambda+1}\frac{\bar{S}''(1)}{\bar{S}'(1)}V_{wz}(1,1), \tag{6.22}$$

which gives

$$\frac{1}{\lambda+1}\frac{\bar{S}''(1)}{\bar{S}'(1)} = 1. \tag{6.23}$$

Thus,

$$\lambda = \frac{\bar{S}''(1)}{\bar{S}'(1)} - 1 \tag{6.24}$$

is a real eigenvalue with the associated eigenfunction

$$V(w,z) = \big[\frac{1}{\lambda+1}z - \frac{1}{\lambda}w\big]\frac{\bar{S}'(w)}{\bar{S}'(1)}. \tag{6.25}$$

In addition, $\lambda > 0$ is equivalent to

$$\frac{\bar{S}''(1)}{\bar{S}'(1)} > 1. \tag{6.26}$$

Changing variables back to x and y, (6.26) is equivalent to

$$\mathcal{R}_0 = \frac{\beta}{\beta+\gamma}\frac{\bar{S}''(1)}{\bar{S}'(1)} > 1. \tag{6.27}$$

Subcase 3.2. Suppose $C(w) \neq 0$. Then for the solution (6.19) to be in X, it is required that $C(w) \in X$ and $\lambda = -k$ for $k = 2, 3, \ldots$. Thus the solution is

$$V(w,z) = C(w)z^k + \big[\frac{1}{1-k}z + \frac{1}{k}w\big]\frac{\bar{S}'(w)}{\bar{S}'(1)}V_{wz}(1,1). \tag{6.28}$$

Note that (6.28) is still an implicit equation as both sides depend on $V(w,z)$. To see if (6.28) has an analytic solution in X (i.e., it is an eigenfunction of the linear operator L), we take derivatives in z and w to find

$$V_{wz}(w,z) = kC'(w)z^{k-1} + \frac{1}{1-k}\frac{\bar{S}''(w)}{\bar{S}'(1)}V_{wz}(1,1). \tag{6.29}$$

Evaluate at $(w,z) = (1,1)$, and gather like terms

$$\big[1 - \frac{1}{1-k}\frac{\bar{S}''(1)}{\bar{S}'(1)}\big]V_{wz}(1,1) = kC'(1). \tag{6.30}$$

If the coefficient of $V_{wz}(1,1)$ is non-zero, then (6.30) uniquely determines an eigenfunction for arbitrarily analytic $C(w)$. If not, then for all $C(w)$ where $C'(1) = 0$, (6.30) still determines a set of eigenfunctions. That is, $\lambda = -k$ is an eigenvalue. □

Lemma 6.2. *The spectrum of the operator L, defined by equation* (3.6)*, consists of the set of eigenvalues* $\{-k, 0, \frac{\bar{S}''(1)}{\bar{S}'(1)} - 1,$ *for* $k = 1, 2, \ldots\}$ *which are given in Lemma 6.1.*

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Proof. We want to show that, if λ is not an eigenvalue, then $(L-\lambda I_d)$ is invertible, i.e., for all $u=\sum_{m,n} u_{mn} w^m z^n \in X$,

$$(L-\lambda I_d)V = u \tag{6.31}$$

has a unique solution $V \in D(L)$. If in addition the inverse operator is bounded, then the spectrum of L is the discrete set of eigenvalues. As in (6.7), (6.31) can be written as

$$\frac{d}{dz}\big[z^\lambda V\big] - \big(z^\lambda - wz^{\lambda-1}\big)\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1,1) = \sum_{m,n} u_{mn} w^m z^{n+\lambda-1}. \tag{6.32}$$

with an integrating factor z^λ. This equation has a solution

$$V = z^{-\lambda}\int z^{\lambda-1}(z-w)\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1,1)\; dz - \sum_{m,n} u_{mn} w^m z^{-\lambda}\int z^{n+\lambda-1}\; dz. \tag{6.33}$$

Because λ is not an eigenvalue, $\lambda \neq -k$, $(k = 0,1,2,\dots)$ equation (6.33) becomes

$$V = \big[\frac{\bar{S}'(w)}{\bar{S}'(1)} V_{wz}(1,1)\big]\big(\frac{z}{1+\lambda} - \frac{w}{\lambda}\big) + \sum_{m,n}\frac{u_{mn}}{\lambda+n} w^m z^n. \tag{6.34}$$

Note that this is still an implicit equation as both sides depend on V. To see if this equation has a unique solution, we take the derivative ∂^2_{wz} at $(1,1)$, giving

$$V_{wz}(1,1) = \big[1 - \frac{1}{1+\lambda}\frac{\bar{S}''(1)}{\bar{S}'(1)}\big]^{-1}\sum_{m,n} mn\frac{u_{mn}}{\lambda+n}. \tag{6.35}$$

This is uniquely defined because λ is not an eigenvalue, i.e., $\lambda \neq \frac{\bar{S}''(1)}{\bar{S}'(1)} - 1$.

To show that $V \in D(L)$, we see that the first term in (6.34) is in $D(L)$ as long as $\bar{S}'(w) \in D(L)$. For the second term, notice that because α is not a negative integer, $\max_n \big|\frac{1+n}{\alpha+n}\big|$ is well defined. Hence

$$\begin{aligned}\sum_{m,n}((1+m)(1+n)+m^2)\big|\frac{u_{mn}}{\alpha+n}\big| &\le \max_n\big|\frac{1+n}{\alpha+n}\big|\sum_{m,n}(1+m+m^2)\big|u_{mn}\big| \\ &= \max_n\big|\frac{1+n}{\alpha+n}\big|\|u\|_X \\ &< \infty.\end{aligned}$$

This shows that V given by (6.34) is in the domain $D(L)$ and $\|S\|_{\text{Dom}} \le M\|u\|_X$. □

7. Nonlinear instability

In this section we set out to prove Theorem 4.3, part (i). We study the instability by considering a special case that reduces the PDE effective degree model to the Volz model. The Volz model is a system of ODEs that has been shown to be unstable for $\mathcal{R}_0 < 1$. The stability is proven by direct calculation.

7.1. Reduction to the Volz model

Assume a generating function solution to (2.5) of the form

$$S(t,w,z) = \sum_k p_k S_k(t) \left[p_S(t)(w-1) + p_I(t)(z-1) + 1\right]^k , \tag{7.1}$$

where $S_k(t)$ is the proportion of susceptible nodes with degree $k = s+i$ at time t, and $p_S(t)$ and $p_I(t)$ are the probabilities of a susceptible node with a susceptible/infected neighbour at time t, respectively.

Substituting (7.1) into Model (2.5) gives the system of ODEs (after simplification)

$$\theta' = -\theta p_I, \tag{7.2a}$$

$$p_S' = p_I p_S - \frac{\sum_{k=0}^{\infty} k(k-1)p_k\theta^k}{\sum_{k=0}^{\infty} kp_k\theta^k} p_S p_I, \tag{7.2b}$$

$$p_I' = p_I^2 - p_I + \frac{\sum_{k=0}^{\infty} k(k-1)p_k\theta^k}{\sum_{k=0}^{\infty} kp_k\theta^k} p_S p_I. \tag{7.2c}$$

This is equivalent to the Volz model [12], with $S_k = \theta^k$ for $k = 1, 2, \ldots$. See [6] for more details. This system has the disease-free equilibrium $\bar{Y} = (\theta, p_S, p_I) = (1, 1, 0)$. The stability of this equilibrium determined by its only non-zero eigenvalue

$$\lambda_1 = \frac{\sum_k k(k-1)p_k}{\sum_k kp_k} - 1 = \frac{\bar{S}''(1)}{\bar{S}'(1)} - 1 \tag{7.3}$$

with the associated eigenvector

$$\vec{v}_1 = (-1,\, -\lambda_1,\, \lambda_1). \tag{7.4}$$

Noting that $\lambda_1 > 0$ is equivalent to (6.26), i.e., the linear stability of the disease free equilibrium of (2.5) and (7.2) are the same.

7.2. Proof of instability

This section provides a proof of Theorem 4.3, part (i), which states that the disease-free equilibrium $\bar{S}(w)$ is nonlinearly unstable for $\mathcal{R}_0 > 1$, where $\mathcal{R}_0$ is given by equation (2.12).

Proof. Suppose $\mathcal{R}_0 > 1$ and assume an initial condition to the effective degree model (2.5)

$$S_0(w,z) = \sum_k p_k \theta^k(0) \big(p_S(0)(w-1) + p_I(0)(z-1) + 1\big)^k. \tag{7.5}$$

Let $Y(t) = (\theta(t), p_S(t), p_I(t))$ be a solution to the Volz model (7.2), where the initial condition $Y(0)$ is an ε-perturbation from the disease-free equilibrium $\bar{Y}$ along the eigenvector $\vec{v}_1$ given by (7.4), i.e.,

$$Y(0) = \bar{Y} + \varepsilon \vec{v}_1 . \tag{7.6}$$

Substituting (7.6) into (7.5) gives

$$S_0(w,z) = \sum_k p_k (1-\varepsilon)^k \left[(1-\varepsilon\lambda_1)(w-1) + \varepsilon\lambda_1(z-1) + 1\right]^k$$

$$= \bar{S}(w) + \epsilon \bar{S}'(w)(\lambda_1 z - w - \lambda_1 w) + O(\epsilon^2).$$

Since the equilibrium $\bar{Y}$ of the Volz model is unstable when $\lambda_1 > 0$, there exists an $\epsilon > 0$, such that, for all $\delta > 0$, there exists a solution $Y(t)$ with an initial condition $\|Y(0) - \bar{Y}\|_2 < \delta$ that leaves ε neighborhood of $\bar{Y}$ at some time T, i.e., $\|Y(T) - \bar{Y}\|_2 = \epsilon$. Here $\|\cdot\|_2$ is the Euclidean norm. We want to show that the corresponding PDE solution satisfies

$$\|S(T, w, z) - \bar{S}(w)\|_X \geq C\epsilon \tag{7.7}$$

for some constant C, where, from (7.1) with $S_k = \theta^k$,

$$S(T, w, z) = \sum_k p_k \theta(T)^k \big(p_S(T)(w-1) + p_I(T)(z-1) + 1\big)^k. \tag{7.8}$$

Expand $Y(T)$ in a Taylor series with respect to ε about the equilibrium $\bar{Y}$,

$$\begin{aligned} S(T, w, z) &= \bar{S}(w) + \frac{\partial S}{\partial \theta}|_{\bar{Y}}(\theta(T) - 1) + \frac{\partial S}{\partial p_S}|_{\bar{Y}}(p_S(T) - 1) + \\ &\quad \frac{\partial S}{\partial p_I}|_{\bar{Y}} p_I(T) + o(Y - \bar{Y}) \\ &= \bar{S}(w) + \left(\frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial p_S}, \frac{\partial S}{\partial p_I}\right)_{\bar{Y}} \cdot (Y(T) - \bar{Y}) + o(\|Y - \bar{Y}\|_2) \\ &= S(w) + \bar{S}'(w)(w, w-1, z-1) \cdot (Y(T) - \bar{Y}) + o(\|Y - \bar{Y}\|_2). \end{aligned}$$

Then,

$$\begin{aligned} \|S(T, w, z) - \bar{S}(w)\|_X &= \|\bar{S}'(w)(Y(T) - \bar{Y}) \cdot (w, w-1, z-1) + o(\|Y - \bar{Y}\|_2)\|_X \\ &\geq \|\bar{S}'(w)(Y(T) - \bar{Y}) \cdot (w, w-1, z-1)\|_X - \\ &\quad \|o(\|Y(T) - \bar{Y}\|_2)\|_X. \end{aligned}$$

This inequality is true for all w and z, and the dot product $(Y(T) - \bar{Y}) \cdot (w, w-1, z-1) = 0$ for all $(w, z) \in [0,1]^2$ if and only if $Y(T) = \bar{Y}$. Therefore,

$$C = \min_{(w,z) \in [0,1]^2} \frac{Y(T) - \bar{Y}}{\|Y(t) - \bar{Y}\|_2} \cdot (w, w-1, z-1) > 0.$$

Thus

$$\|S(T, w, z) - \bar{S}(w)\|_X \geq C\epsilon - \|o(\|Y(T) - \bar{Y}\|_2)\|_X \geq \frac{C\epsilon}{2}$$

for sufficiently small $Y(T) - \bar{Y}$. □

8. Nonlinear stability

This section sets out to prove Theorem 4.3, part (ii).

Proof. Let $\bar{S}(w)$ be an disease-free equilibrium of our model. Consider a small perturbation $E_0(w, z) \in X$ to the initial condition, i.e., $S_0(w, z) = \bar{S}(w) + E_0(w, z)$. Equation (2.8) allows us to write the solution starting with this initial condition as

$$S(t, w, z) = \bar{S}(w_0(t)) + E_0(w_0(t), z_0(t)).$$

To show the stability, we need to show that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\|S(t,w,z) - \bar{S}(w)\|_X < \varepsilon$ for all $t \geq 0$ as long as $\|E_0(w,z)\|_{\text{Dom}} \leq \delta$.

$$\begin{aligned}\|S(t,w,z) - \bar{S}(w)\|_X &= \|\bar{S}(w_0(t)) + E_0(w_0(t), z_0(t)) - \bar{S}(w)\|_X \\ &\leq \|\bar{S}(w_0(t)) - \bar{S}(w)\|_X + \|E_0(w_0(t), z_0(t))\|_{\text{Dom}}.\end{aligned}$$

We thus will show that $\|\bar{S}(w_0(t)) - \bar{S}(w)\|_X \leq \varepsilon/2$ for some $\delta < \varepsilon/2$. From (2.8c),

$$w_0(t) = we^{-\phi(t)} + z\eta(t),$$

thus, $\bar{S}(w_0(t)) = \bar{S}(we^{-\phi(t)} + z\eta(t))$. Expand this into a Taylor series about t, i.e., $\exists \tilde{t} \in (0,t)$, such that

$$\bar{S}(w_0(t)) = \bar{S}(w) + \bar{S}'(w_0(\tilde{t}))(-w\phi'(\tilde{t})e^{-\phi(\tilde{t})} + z\eta'(\tilde{t})),$$

and thus

$$\begin{aligned}\|\bar{S}(w_0(t)) - \bar{S}(w)\|_X &\leq \|\bar{S}'\|_\infty \| - w\phi'(\tilde{t})e^{-\phi(\tilde{t})} + z\eta'(\tilde{t})\|_X \\ &\leq \|\bar{S}'\|_\infty \sup_{\tilde{t}\in[0,t]} |\phi'(\tilde{t})| + \|\bar{S}'\|_\infty \sup_{\tilde{t}\in[0,t]} |\eta'(\tilde{t})|.\end{aligned}$$

We will show that, for some $\delta > 0$,

$$\sup_t |\phi'(t)| < \frac{\varepsilon}{4\|\bar{S}'\|_\infty}, \ \sup_t |\eta'(t)| < \frac{\varepsilon}{4\|\bar{S}'\|_\infty}.$$

This guarantees the stability. □

Lemma 8.1. *$\exists \delta_1$, such that for $\delta < \delta_1$,*

$$\frac{\partial_1^2 S_0(w_0,z_0)}{\partial_1 S_0(w_0 z_0)}\bigg|_{w=z=1} e^{-\phi} - 1 < \lambda_1/2 < 0,$$

where

$$\lambda_1 = \frac{\bar{S}''(w)}{\bar{S}'(w)}\bigg|_{w=1} - 1.$$

Proof. Let $\rho = e^{-\phi} + \eta$. From (2.9b),

$$\eta = e^{-t} - e^{-\phi} + e^{-t}\int_0^t e^{s-\phi} ds,$$

and thus

$$\rho = e^{-t} + e^{-t}\int_0^t e^{s-\beta\phi} ds > 0.$$

In addition,

$$\rho \leq e^{-t} + e^{-t}\int_0^t e^{s} ds = 1.$$

Thus, $\rho \in [0,1]$. Compute the following via Taylor expansion

$$\frac{\partial_1^2 S_0(w_0,z_0)}{\partial_1 S_0(w_0 z_0)}|_{w=z=1} e^{-\phi}$$

$$\begin{aligned}
&= \left.\frac{\bar{S}''(w_0)+\partial_1^2 E_0(w_0,z_0)}{\bar{S}'(w_0)+\partial_1 E_0(w_0 z_0)}\right|_{w=z=1} e^{-\phi} \\
&= e^{-\phi}\left(\frac{\bar{S}'(w_0)\partial_1^2 E_0(w_0,z_0)-\bar{S}''(w_0)\partial_1 E_0(w_0,z_0)}{\bar{S}'(w_0)^2}+\dots\right)_{w=z=1}+\frac{\bar{S}''(\rho)}{\bar{S}'(\rho)}e^{-\phi} \\
&= \frac{\bar{S}''(\rho)}{\bar{S}'(\rho)}e^{-\phi}+O(\|\partial_1^2 E_0\|_\infty)+O(\|\partial_1 E_0\|_\infty) \\
&\le \frac{\bar{S}''(\rho)}{\bar{S}'(\rho)}\rho+O(\|\partial_1^2 E_0\|_\infty)+O(\|\partial_1 E_0\|_\infty) \\
&\le \frac{\bar{S}''(\rho)}{\bar{S}'(\rho)}\rho+O(\|\partial_1^2 E_0\|_\infty)+O(\|\partial_1 E_0\|_\infty). \qquad (8.1)
\end{aligned}$$

Note that both $O(\|\partial_1^2 E_0\|_\infty)$ and $O(\|\partial_1 E_0\|_\infty)$ are controlled by $\|E_0\|_{\mathrm{Dom}}$, e.g. $\|\partial_1^2 E_0\|_\infty \to 0$ as $\|E_0\|_{\mathrm{Dom}} \to 0$.

We will show that $H(\rho) := \frac{\bar{S}''(\rho)}{\bar{S}'(\rho)}\rho$ is an increasing function of $\rho \in [0,1]$. Let $\bar{S}(\rho) = \sum_{k=0}^\infty \rho^k p_k$ with $p_k > 0$. Then,

$$H(\rho) = \frac{\sum_{k=2}^\infty k(k-1)\rho^{k-1}p_k}{\sum_{k=1}^\infty k\rho^{k-1}p_k},$$

$$H'(\rho) = \frac{\sum_{k,j=1}^\infty k(k-1)^2 j\rho^{k+j-3}p_k p_j - \sum_{k,j=1}^\infty k(k-1)j(j-1)\rho^{k+j-3}p_k p_j}{\left(\sum_{k=1}^\infty k\rho^{k-1}p_k\right)^2}.$$

Note that

$$\begin{aligned}
&\sum_{k,j=1}^\infty k(k-1)^2 j\rho^{k+j-3}p_k p_j - \sum_{k,j=1}^\infty k(k-1)j(j-1)\rho^{k+j-3}p_k p_j \\
&= \frac{1}{2}\sum_{k,j=1}^\infty kjp_k p_j\rho^{k+j-3}[(k-1)^2+(j-1)^2-2(k-1)(j-1)] \\
&= \frac{1}{2}\sum_{k,j=1}^\infty kjp_k p_j\rho^{k+j-3}(k-j)^2 \\
&\ge 0.
\end{aligned}$$

Thus, $H(\rho)$ is increasing. For $q \in [0,1]$,

$$H(\rho) \le H(1) = \frac{\bar{S}''(1)}{\bar{S}'(1)}.$$

Thus,

$$\begin{aligned}
\left.\frac{\partial_1^2 S_0(w_0,z_0)}{\partial_1 S_0(w_0 z_0)}\right|_{w=z=1} e^{-\phi}-1 &< \frac{\bar{S}''(1)}{\bar{S}'(1)}-1+O(\|E_0\|_{\mathrm{Dom}}) \\
&= \lambda_1+O(\|E_0\|_{\mathrm{Dom}}).
\end{aligned}$$

Thus, there exists a $\delta_1 > 0$, such that, for all $\delta < \delta_1$,

$$\left.\frac{\partial_1^2 S_0(w_0,z_0)}{\partial_1 S_0(w_0 z_0)}\right|_{w=z=1} e^{-\phi}-1 < \lambda_1/2.$$

□

Lemma 8.2. *For η' given by* (2.10), $\delta_2 < \delta_1$ *such that*

$$\sup_t \eta'(t) < \frac{\varepsilon}{4\|\bar{S}'(\tilde{w})\|_\infty}, \qquad \sup_t \phi(t) < \frac{\varepsilon}{4\beta\|\bar{S}'(\tilde{w})\|_\infty},$$

for all $\|E_0\|_{Dom} < \delta_2$.

Proof. From equation (2.10),

$$\eta' = \frac{\partial^2_{12} S_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)}\bigg|_{w=z=1} \beta e^{-t} e^{-\phi} + \left(\frac{\partial^2_1 S_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)}\bigg|_{w=z=1} e^{-\phi} - 1 \right) \eta.$$

Lemma 3 gives

$$\eta' \leq \left\| \frac{\partial^2_{12} E_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)} \right\|_\infty e^{-t} + \frac{\lambda_1}{2} \eta. \tag{8.2}$$

From the Comparison Theorem,

$$\begin{aligned} \eta &\leq e^{\lambda_1 t} \left\| \frac{\partial^2_{12} E_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)} \right\|_\infty \int_0^t e^{-(\frac{\lambda_1}{2}+1)s} ds \\ &= \begin{cases} \left\| \frac{\partial^2_{12} E_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)} \right\|_\infty t e^{\lambda_1 t}, & \frac{\lambda_1}{2} = -1 \\ \left\| \frac{\partial^2_{12} E_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)} \right\|_\infty \frac{1}{\frac{\lambda_1}{2}+1} (e^{\frac{\lambda_1}{2} t} - e^{-t}), & \frac{\lambda_1}{2} \neq -1. \end{cases} \end{aligned}$$

In both cases, there exists a $\theta < 0$, such that

$$\eta \leq e^{\theta t} O(\|E_0\|_{\mathrm{Dom}}).$$

Thus, from (8.2),

$$\eta' \sim O(\|E_0\|_{\mathrm{Dom}}).$$

To estimate $\phi'(t)$, again, from (2.10),

$$\begin{aligned} \phi' &= \frac{\partial^2_{12} S_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)}\bigg|_{w=z=1} e^{-t} + \frac{\partial^2_1 S_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)}\bigg|_{w=z=1} \eta \\ &\leq \left\| \frac{\partial^2_{12} E_0(w_0, z_0)}{\partial_1 S_0(w_0 z_0)} \right\|_\infty e^{-t} + \left\| \frac{\partial^2_1 S_0}{\partial_1 S_0} \right\|_\infty O(\|E_0\|_{\mathrm{Dom}}). \end{aligned}$$

Thus,

$$\phi'(t) \sim O(\|E_0\|_{\mathrm{Dom}}).$$

□

9. Concluding remarks

The PDE effective degree SIR model was introduced in [6], which uses a generating function to represent the state variables of the ODE effective degree SIR model in [9]. The PDE model extends the ODE model to infinite degree distributions.

In this paper, we have analyzed the linear and nonlinear stability of the disease-free steady state of the PDE effective degree model. We calculated the spectrum of the linearized system, and showed that there exists only a point spectrum with a zero eigenvalue, an infinite number of negative eigenvalues, and a single eigenvalue

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that may be positive or negative, whose sign is determined by the basic reproduction number $\mathcal{R}_0$ given in [9]. Specifically, the disease-free steady state is linearly unstable if and only if $\mathcal{R}_0 > 1$. We also proved that the linear and nonlinear stability of the disease-free steady state agree for this model.

Thus the basic reproduction number $\mathcal{R}_0 = 1$ is the disease threshold for effective degree SIR models with both finite and infinite degree distributions.

The calculation of the basic reproduction number for the effective degree SIS model as demonstrated in [9] is difficult to pursue for the ODE model. It is only proved that the basic reproduction number for the SIS model is larger than the SIR model, but the value of $\mathcal{R}_0$ remains elusive. Our generating function approach of rewriting the ODE effective degree model into a PDE model provides a new tool for the analysis of the effective degree SIS model.

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