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Exponential Stability of Positive Conformable BAM Neural Networks with Communication Delays

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Abstract In this paper, we consider a class of nonlinear differential equations with delays described by conformable fractional derivative. This type of differential equations can be used to describe dynamics of various practical models including biological and artificial neural networks with heterogeneous time-varying delays. By novel comparison techniques via fractional differential and integral inequalities, we prove under assumptions involving the order-preserving property of nonlinear vector fields that, with nonnegative initial states and inputs, the system state trajectories are always nonnegative for all time. This feature, called positivity, induces a special character, namely the monotonicity of the system. We then derive tractable conditions in terms of linear programming and prove, by utilizing the Brouwer's fixed point theorem and comparisons induced by the monotonicity, that the system possesses a unique positive equilibrium point which attracts exponentially all state trajectories. An application to the exponential stability of fractional linear time-delay systems is also discussed. Numerical examples with simulations are given to illustrate the theoretical results.

Keywords Conformable derivative, time-delay systems, BAM neural networks, positive equilibrium, M-matrix

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1. Introduction

Consider a class of nonlinear fractional differential delay equations of the form

$$ {}^{c}D^{\alpha}_{t_0}\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = -D_{\beta,\gamma}\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} + \begin{pmatrix} Af(y(t))\\ Cg(x(t))\end{pmatrix} + \begin{pmatrix} Bf(y_\sigma(t))\\ Dg(x_\tau(t))\end{pmatrix} + \begin{pmatrix} I\\ J\end{pmatrix}, \tag{1.1}$$

where ${}^{c}D^{\alpha}_{t_0}$ represents the conformable fractional derivative (CFD). More details on CFD and the description of system (1.1) will be presented in the next section. System (1.1) can be used to describe dynamics of various practical models such as fractional Hopfield-type neural networks or bidirectional associative memory (BAM)

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neural networks [1,2]. System (1.1) also encompasses important classes of time-delay systems like fractional linear systems with delays.

The theory of fractional calculus is one of the most active research areas in the past few years due to its demonstrated applications in numerous practical models such as data analysis, intelligent control, associative memory or optimization [3]. It has been recognized that functional calculus and fractional differential equations (FrDEs) can more adequately describe many physical phenomena compared to integer-order models. Several different approaches to fundamental concepts like fractional derivatives (FrDs) or fractional integrals (FrIs) have been developed in the senses of the Riemann-Liouville, Caputo or Grünwald-Letnikov [3, 4]. For example, the approach of Riemann-Liouville is constructed based on iterating the integral operator n times combining with the Cauchy formula, while the approach of Grünwald-Letnikov is based on iterating the derivative n times combining with the use of Gamma function in the binomial coefficients. The concepts of FrDs formulated in this direction are quite complicated in applications and have some common drawbacks. For instance, some basic properties of usual derivatives like product rule or chain rule are not preserved for FrDs. In addition to this, the monotonicity of a function f typically cannot be determined by FrDs of f in certain meanings.

To overcome some drawbacks of existing FrDs, the authors of [5] proposed a new well-behaved simple derivative called the conformable fractional derivative (CFD). Basic results in calculus of functions subject to CFD were also developed in [6, 7]. Recently, conformable fractional-order systems have also attracted considerable research attention and a number of interesting results involving various aspects of analysis and control of dynamical systems described by conformable fractional-order differential equations with or without delays have been published. For a few references, we refer the reader to recent works [8–15].

Positive systems form a particular class of dynamical systems, whose states and outputs starting from nonnegative inputs are always nonnegative. This type of systems is widely used to describe dynamics of many practical models in a variety of disciplines from biology, ecology and epidemiology, chemistry, pharmacokinetics to air traffic flow networks, control engineering, telecommunication and chemical-physical processes [16]. In the past few decades, the theory of positive systems has been intensively studied for various kinds of linear systems and nonlinear systems in integer-order models (see, e.g., [17–19] and the references therein). However, this area is still considerably less well-developed for fractional nonlinear systems, in particular, for models arising in artificial and biological neural networks.

The research topic of fractional differential equations and fractional neural networks (FrNNs) has received growing attention in recent years [20–22]. Some important issues in analysis such as stability, passivity, disspativity or identification and H_∞ control have also been extensively studied and developed for neural network models with delays (see, e.g., [2, 20, 23–26] and the references therein). However, there are only a few works concerning stability of conformable FrDEs. In particular, the positivity characterization and the existence, uniqueness and exponential stability of conformable delay systems in the form of (1.1) have not been studied. This motivates our present study.

In this paper, we consider a class of nonlinear delay differential equations described by conformable fractional derivative as presented in Eq. (1.1). This type of differential equations can be used to describe dynamics of various models in practice. By novel comparison techniques via fractional differential and integral

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inequalities, we analyze the positivity of system (1.1) and derive tractable conditions in terms of linear programming (LP) to ensure that the system possesses a unique positive equilibrium point which attracts exponentially all state trajectories. The decay rate is determined by some fractional exponential function of the form $E_\alpha(-\lambda, t-t_0) = e^{-\lambda \frac{(t-t_0)^\alpha}{\alpha}}$, where $0<\alpha<1$ is the fractional-order and λ is some positive scalar representing the fractional exponential decay rate (FEDR). For linear time-delay systems, the maximum allowable FEDR can be determined by an LP-based procedure. Numerical examples with simulations are given to illustrate the obtained analysis results.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $\|x\|_\infty = \max_{1\le i\le n}|x_i|$ and $\mathbf{1}_n \in \mathbb{R}^n$ is the vector with all entries equal one. $\mathbb{R}^{m\times n}$ is the set of $m\times n$-matrices. $\sigma(A)$ is the set of eigenvalues of matrix $A\in\mathbb{R}^{n\times n}$ and $\rho(A) = \max\{|\lambda| : \lambda\in\sigma(A)\}$ is its spectral radius. $v = \mathrm{vec}(v_1, v_2, \ldots, v_k)$ denotes the augmented vector formulated by stacking components of $v_1, v_2, \ldots, v_k$. For any vectors $x=(x_i)\in\mathbb{R}^n$ and $y=(y_i)\in\mathbb{R}^n$, we write $x\preceq y$ if $x_i\le y_i$ and $x\prec y$ if $x_i<y_i$ for all $i\in[n]\triangleq\{1,2,\ldots,n\}$. $\mathbb{R}^n_+ = \{x\in\mathbb{R}^n : x\succeq 0\}$ and $|x| = (|x_i|)\in\mathbb{R}^n_+$ for $x=(x_i)\in\mathbb{R}^n$. A matrix $A=(a_{ij})\in\mathbb{R}^{m\times n}$ is nonnegative, $A\succeq 0$, if $a_{ij}\ge 0$ for all i,j and A is an M-matrix if $a_{ij}\le 0$ for all $i\ne j$. $C([a,b],\mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued continuous functions on $[a,b]$ endowed with the *supremum* norm $\|\phi\|_{\mathcal{C}} = \sup_{a\le t\le b}\|\phi(t)\|$ for a $\phi\in C([a,b],\mathbb{R}^n)$.

2. Preliminaries of fractional calculus

Definition 2.1. The (left) conformable fractional derivative of order $0<\alpha<1$ starting from a of a function $f:[t_0,+\infty)\to\mathbb{R}^n$ is defined by

$$ {}^cD^\alpha_{t_0} f(t) = \lim_{\epsilon\to 0}\frac{f\left(t+\epsilon(t-t_0)^{1-\alpha}\right)-f(t)}{\epsilon},\ t>t_0. $$

If ${}^cD^\alpha_{t_0}f(t)$ exists for $t\in(t_0,t_0+\delta)$ with some $\delta>0$ and exists finite limits $\lim_{t\to t_0^+}{}^cD^\alpha_{t_0}f(t)$, then we define ${}^cD^\alpha_{t_0}f(t_0) = \lim_{t\to t_0^+}{}^cD^\alpha_{t_0}f(t)$.

Definition 2.2. A function $f:[t_0,+\infty)\to\mathbb{R}^n$ is said to be α-differentiable on the interval $(t_0,+\infty)$ if the derivative ${}^cD^\alpha_{t_0}f(t)$ exists for all $t\in(t_0,+\infty)$. We denote by $C_\alpha([t_0,+\infty),\mathbb{R}^n)$ the set of α-differentiable functions $x(t)$ on $(t_0,+\infty)$ and ${}^cD^\alpha_{t_0}x(t)$ is continuous on $(t_0,+\infty)$.

Remark 2.1. If a function f is differentiable on $(t_0,+\infty)$, then

$$ {}^cD^\alpha_{t_0}f(t) = (t-t_0)^{1-\alpha}f'(t). $$

Thus, for a constant function, ${}^cD^\alpha_{t_0}f(t)=0$. Conversely, if ${}^cD^\alpha_{t_0}f(t)=0$ for $t\in(t_0,t_0+\delta)$ then it can be shown by using the conformable fractional mean value theorem [5] that $f(t)=0$ for all $t\in(t_0,t_0+\delta)$ [6].

Proposition 2.1. *If the functions f, g are α-differentiable at $t>t_0$, then we have*

(i) ${}^cD^\alpha_{t_0}(\lambda f(t)+\mu g(t)) = \lambda\,{}^cD^\alpha_{t_0}f(t)+\mu\,{}^cD^\alpha_{t_0}g(t)$ *for* $\lambda,\mu\in\mathbb{R}$.

(ii) ${}^cD^\alpha_{t_0}(f(t)g(t)) = g(t)\,{}^cD^\alpha_{t_0}f(t)+f(t)\,{}^cD^\alpha_{t_0}g(t)$.

(iii) ${}^cD^\alpha_{t_0}\left(\dfrac{f(t)}{g(t)}\right) = \dfrac{g(t)\,{}^cD^\alpha_{t_0}f(t)-f(t)\,{}^cD^\alpha_{t_0}g(t)}{g^2(t)}$.

Proposition 2.2 (Chain rule [6]). *If the functions f and g are α-differentiable on $(t_0,+\infty)$, then $h(t)=f(g(t))$ is also α-differentiable for $t\in(t_0,+\infty)$ with $g(t)\neq 0$, and we have*

$$^{c}D^{\alpha}_{t_0}h(t)={}^{c}D^{\alpha}_{t_0}f(g(t))g(t)^{\alpha-1}\,{}^{c}D^{\alpha}_{t_0}g(t),\quad t\in(t_0,+\infty).$$

Remark 2.2. As an application of the chain rule given in (2.2), if $\varphi:[t_0,+\infty)\to\mathbb{R}$ is an α-differentiable function, then $f(t)=\varphi^2(t)$ is also α-differentiable on $(t_0,+\infty)$ and we have

$$^{c}D^{\alpha}_{t_0}\left(\varphi^2(t)\right)=2\varphi(t)\,{}^{c}D^{\alpha}_{t_0}\varphi(t),\ t>t_0.$$

Let $\xi:[t_0,+\infty)\to\mathbb{R}^n$ be an α-differentiable function and $\|\xi(t)\|^2=\xi^\top(t)\xi(t)$. Then, we have

$$^{c}D^{\alpha}_{t_0}\left(\|\xi(t)\|^2\right)=2\xi^\top(t)\,{}^{c}D^{\alpha}_{t_0}\xi(t),\ t>t_0.$$

Moreover, for any symmetric positive definite matrix $P\in\mathbb{S}^n_+$, the following derivative rule holds

$$^{c}D^{\alpha}_{t_0}\left(\xi^\top(t)P\xi(t)\right)=\xi^\top(t)P\,{}^{c}D^{\alpha}_{t_0}\xi(t)+{}^{c}D^{\alpha}_{t_0}\xi^\top(t)P\xi(t),\ t>t_0.$$

The following monotone property can be derived with the help of conformable fractional mean value theorem proven in [5].

Lemma 2.1. *Let f be a continuous, α-differentiable function on $(t_0,+\infty)$. If ${}^{c}D^{\alpha}_{t_0}f(t)\geq 0$ (respectively ${}^{c}D^{\alpha}_{t_0}f(t)\leq 0$) for all $t\in(t_0,+\infty)$, then the function f is increasing (decreasing) on $[t_0,+\infty)$.*

Definition 2.3 (see [27]). The conformable integral of order $0<\alpha<1$ of a function f starting from a is given by

$$I^{\alpha}_{a}f(t)=\int_a^t\frac{f(s)}{(s-a)^{1-\alpha}}ds,$$

if the concerned integral is well defined.

The following properties that are similar to fundamental theorems of calculus can be obtained [7]. Let $f:[a,+\infty)\to\mathbb{R}^n$ be a continuous function. Then, we have

$$I^{\alpha}_{a}\left({}^{c}D^{\alpha}_{a}f(t)\right)=f(t)-f(a).$$

If, in addition, the function f is α-differentiable function on $(a,+\infty)$, then

$$^{c}D^{\alpha}_{a}\left(I^{\alpha}_{a}f(t)\right)=f(t).$$

Finally, the following auxiliary result will be used in our next derivation. For given $\alpha\in(0,1)$ and $\lambda\in\mathbb{R}$, the fractional conformable exponential function $E_\alpha(\lambda,s)$, $s\geq 0$, is defined as

$$E_\alpha(\lambda,s)=\exp\left(\lambda\frac{s^\alpha}{\alpha}\right).$$

Lemma 2.2. *For a given $\alpha\in(0,1)$, the solution $x\in C([t_0,+\infty),\mathbb{R})$ of the following problem*

$$\begin{aligned}{}^{c}D^{\alpha}_{t_0}x(t)+\mu x(t)&=q(t),\ t>t_0,\\ x(t_0)&=x_0,\end{aligned}$$

can be represented in the form

$$x(t) = \exp\left(-\mu\frac{(t-t_0)^\alpha}{\alpha}\right)\left[x_0 + \int_{t_0}^{t}\exp\left(\mu\frac{(s-t_0)^\alpha}{\alpha}\right)\frac{q(s)}{(s-t_0)^{1-\alpha}}ds\right]$$
$$= E_\alpha(-\mu, t-t_0)\left(x_0 + \int_{t_0}^{t}\frac{E_\alpha(\mu, s-t_0)q(s)}{(s-t_0)^{1-\alpha}}ds\right),\ t > t_0,$$

where μ is some real constant.

Proof. The proof can be deduced using a type of fractional constant variation formula proposed in [27]. We omit it here. □

2.1. Model description

Consider system (1.1). This system can be written in the componentwise form as follows

$$\begin{aligned} {}^cD^\alpha_{t_0}x_i(t) &= -\beta_i x_i(t) + \sum_{j=1}^{m}a_{ij}f_j(y_j(t)) + \sum_{j=1}^{m}b_{ij}f_j(y_j(t-\sigma_j(t))) + I_i, \\ {}^cD^\alpha_{t_0}y_j(t) &= -\gamma_j y_j(t) + \sum_{i=1}^{n}c_{ji}g_i(x_i(t)) + \sum_{i=1}^{n}d_{ji}g_i(x_i(t-\tau_i(t))) + J_j, \end{aligned} \tag{2.1}$$

where β_i, $i \in [n]$, and γ_j, $j \in [m]$ are given positive scalars; $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ji})$ and $D = (d_{ji})$ are known real matrices of appropriate dimensions; delays $\tau_i(t) \in [0, \overline{\tau}]$, $\sigma_j(t) \in [0, \overline{\sigma}]$ are continuous functions, where $\overline{\sigma}$ and $\overline{\tau}$ are known positive constants. System (1.1) is typically used to describe dynamics of neural networks in bidirectional associative memory (BAM) model. For more detail biology description of system (1.1), we refer the reader to [18].

For a given $t_0 \geq 0$, the initial condition of system (1.1) is specified as

$$x_{t_0} = x_0 \in C([-\overline{\tau}, 0], \mathbb{R}^n), \quad y_{t_0} = y_0 \in C([-\overline{\sigma}, 0], \mathbb{R}^m), \tag{2.2}$$

that is,

$$\begin{aligned} x_{t_0}(s) &= x(t_0+s) = x_0(s),\ s \in [-\overline{\tau}, 0], \\ y_{t_0}(\theta) &= y(t_0+\theta) = y_0(\theta),\ \theta \in [-\overline{\sigma}, 0]. \end{aligned}$$

2.2. Preliminaries

Assumption (A): We assume that nonlinear functions (which represent neuron activation functions) $f_j(\cdot)$ and $g_i(\cdot)$ are continuous; $f_j(0) = 0$, $g_i(0) = 0$, and there exist positive scalars l^f_j, l^g_i such that

$$0 \leq \frac{f_j(u) - f_j(v)}{u - v} \leq l^f_j, \quad 0 \leq \frac{g_i(u) - g_i(v)}{u - v} \leq l^g_i, \tag{2.3}$$

for all $u, v \in \mathbb{R}$, $u \neq v$.

Proposition 2.3. *Let Assumption (A) hold. Then, for each initial condition (x_0, y_0), the problem governed by system (1.1) and (2.2) possesses a unique solution $\chi(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, which is continuous in t on $[t_0, +\infty)$.*

Proof. We define the function space

$$\mathscr{F}_d = \left\{ \phi = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} : x_0 \in C([-\overline{\tau}, 0], \mathbb{R}^n), y_0 \in C([-\overline{\sigma}, 0], \mathbb{R}^m) \right\},$$

and a function $G : [t_0, +\infty) \times \mathscr{F}_d \to \mathbb{R}^{n+m}$, $(t, \phi) \mapsto G(t, \phi)$, given by

$$G(t, \phi) = \begin{bmatrix} -D_\beta x_0(0) + Af(y_0(0)) + Bf(y_0(-\sigma(t))) + I \\ -D_\gamma y_0(0) + Cg(x_0(0)) + Dg(x_0(-\tau(t))) + J \end{bmatrix}.$$

where

$$\begin{aligned} D_\beta &= \operatorname{diag}\{\beta_1, \beta_2, \dots, \beta_n\}, \quad D_\gamma = \operatorname{diag}\{\gamma_1, \gamma_2, \dots, \gamma_m\}, \\ f(y(t)) &= \operatorname{vec}(f_j(y_j(t))), \quad f(y(t - \sigma(t))) = \operatorname{vec}(f_j(y_j(t - \sigma_j(t)))), \\ g(x(t)) &= \operatorname{vec}(g_i(x_i(t))), \quad g(x(t - \tau(t)) = \operatorname{vec}(g_i(x_i(t - \tau(t)))), \\ I &= (I_i) \in \mathbb{R}^n, \quad J = (J_j) \in \mathbb{R}^m. \end{aligned}$$

Then, the problem (1.1) and (2.2) can be written in the from of functional differential equation

$$\begin{aligned} {}^cD^\alpha_{t_0} \chi(t) &= -D_{\beta,\gamma}\chi(t) + G(t, \chi_t), \ t > t_0, \\ \chi_{t_0} &= \phi \in \mathscr{F}_d, \end{aligned} \tag{2.4}$$

where $D_{\beta,\gamma} = \operatorname{diag}\{D_\beta, D_\gamma\}$, $\chi_t = \operatorname{vec}(x_t, y_t) \in \mathscr{F}_d$ and $x_t \in C([-\overline{\tau}, 0), \mathbb{R}^n)$, $y_t \in C([-\overline{\sigma}, 0), \mathbb{R}^m)$ are defined as $x_t(s) = x(t + s)$, $s \in [-\overline{\tau}, 0]$, $y_t(\theta) = y(t + \theta)$, $\theta \in [\overline{\sigma}, 0]$.

By assumption (2.3), the function $G(t, \phi)$ is continuous and satisfies Lipschitz condition with respect to ϕ. By the fundamental results of functional differential equations, and similar to the method of [11], the existence and uniqueness of a global solution $\chi(t)$ of the problem (1.1)-(2.2) can be obtained. In addition, such a solution is absolutely continuous in t on $[t_0, +\infty)$. The proof is completed. □

Similar to [18], we say that the solution $\chi(t)$ is a positive solution of system (2.2) is $\chi(t) \succeq 0$ for all $t \geq t_0$. Thus, to characterize the positivity of system (2.2), we define the following admissible set of initial conditions

$$\mathscr{F}_d^+ = \left\{ \phi = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} : x_0 \in C([-\overline{\tau}, 0], \mathbb{R}^n_+), \ y_0 \in C([-\overline{\sigma}, 0], \mathbb{R}^m_+) \right\}.$$

Definition 2.4. System (1.1) is said to be *positive* if for any initial function $\phi \in \mathscr{F}_d^+$ and nonnegative input vectors $I \succeq 0$, $J \succeq 0$, the corresponding solution $\chi(t) = \operatorname{vec}(x(t), y(t))$ of system (1.1) is positive.

The concept of equilibrium points (EPs) of system (1.1) is defined as usual, which is also given the following definition.

Definition 2.5. A (constant) vector $\chi_* = \operatorname{vec}(x_*, y_*)$, where $x_* \in \mathbb{R}^n$ and $y_* \in \mathbb{R}^m$, is said to be an EP of system (1.1) if it satisfies the following algebraic system

$$\begin{cases} -D_\beta x_* + (A + B)f(y_*) + I = 0, \\ -D_\gamma y_* + (C + D)g(x_*) + J = 0. \end{cases} \tag{2.5}$$

Moreover, χ^* is called a positive EP if $\chi_* \succeq 0$.

Definition 2.6. An EP $\chi_* = \mathrm{vec}(x_*, y_*)$ of system (1.1) is said to be *globally fractional exponentially stable* (GFES) if there exist positive scalars C, which is independent of initial conditions, and λ such that any solution $\chi(t) = \mathrm{vec}(x(t), y(t))$ of system (1.1) satisfies the following inequality

$$\|\chi(t) - \chi_*\|_\infty \le C\|\phi - \chi_*\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0), \quad t \ge t_0.$$

2.3. Auxiliary results

In this section, we recall some results in nonlinear analysis and nonnegative matrix theory that will be useful for our next derivation in this paper. First, let $f : V \to V$ be a mapping from a metric space V into itself. A point $x \in V$ is called a fixed point of f if it is unchanged under the effect of f, that is, $f(x) = x$. The following result is a form of Brouwer's fixed point theorem [28].

Proposition 2.4. *Suppose that V is a nonempty, convex and compact subset of $\mathbb{R}^n$ and $f : V \to V$ is a continuous mapping. Then, f possesses at least a fixed point in V.*

Next, we list some properties of nonsingular M-matrices [29]. A matrix $P = (p_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix if it can be expressed in the form $P = sI_n - Q$, where $Q = (Q_{ij}) \succeq 0$ and $s \ge \rho(Q)$. In addition, P is a nonsingular M-matrix if and only if $s > \rho(Q)$.

Proposition 2.5 ([29]). *Let $P \in \mathbb{R}^{n\times n}$ be an M-matrix. The following statements are equivalent.*

(i) *P is a nonsingular M-matrix.*

(ii) *All principal minors of P are positive.*

(iii) *P is inverse-positive, that is, the inverse P^{-1} exists and $P^{-1} \succeq 0$.*

(iv) *There exists a vector $\nu \in \mathbb{R}^n$, $\nu \succ 0$, such that $P\nu \succ 0$.*

It follows from Proposition 2.5 that if $H = (h_{ij}) \in \mathbb{R}^{n\times n}$ is a nonnegative matrix whose spectral radius satisfies $\rho(H) < 1$, then $(I_n - H)^{-1} \succeq 0$ and there exists a positive vector $\xi = (\xi_i)$ such that $(I_n - H)\xi \succ 0$. Therefore,

$$\sum_{j=1}^{n} h_{ij}\xi_j < \xi_i, \quad \forall i \in [n].$$

3. Main results

3.1. Positive solutions

In this section, we will prove that, with nonnegative initial states and weighted coefficients, and under the assumption given in (2.3), any solution $\chi(t) = \mathrm{vec}(x(t), y(t))$ of system (1.1) is positive. First, by extending Lemma 2.3 in [30], we obtain the following result.

Lemma 3.1. *For a given function $r(\cdot) \in C([t_0, +\infty), \mathbb{R}^+)$ and a real number p, the corresponding solution of the problem*

$$\begin{aligned} {}^{c}D^{\alpha}_{t_0} x(t) &= -px(t) + r(t),\ t \ge t_0, \\ x(t_0) &= x_0, \end{aligned} \tag{3.1}$$

is nonnegative for all $t \geq t_0$ provided that $x_0 \geq 0$.

Proof. It suffices to prove for the case $x_0 > 0$ that $x(t) > 0$ for $t \geq t_0$. On the contrary, assume that there exists a $t_r > t_0$ such that $x(t_r) = 0$ and $x(t) > 0$ for $t \in [t_0, t_r)$. For any $t > t_0$, we have

$$\begin{aligned}\int_{t_0}^{t} \frac{ds}{(s-t_0)^{1-\alpha}} &= \lim_{\epsilon \to 0^+} \int_{t_0+\epsilon}^{t} \frac{ds}{(s-t_0)^{1-\alpha}} \\ &= \lim_{\epsilon \to 0^+} \frac{(t-t_0)^{\alpha} - \epsilon^{\alpha}}{\alpha} = \frac{(t-t_0)^{\alpha}}{\alpha} > 0.\end{aligned}$$

Thus, for a continuous function $\varphi(s) \geq 0$, the integral $\int_{t_0}^{t} \frac{\varphi(s)ds}{(s-t_0)^{1-\alpha}}$ is well-defined and nonnegative for $t \in (t_0, t_r)$. By Lemma 2.2, it follows from (3.1) that

$$\begin{aligned}x(t) &= E_{\alpha}(-p, t-t_0)\left(x_0 + \int_{t_0}^{t} \frac{E_{\alpha}(p, s-t_0) r(s)}{(s-t_0)^{1-\alpha}} ds\right) \\ &\geq E_{\alpha}(-p, t-t_0) x_0, \quad t \in (t_0, t_r). \end{aligned} \tag{3.2}$$

Let $t \uparrow t_r$, and the last inequality in (3.2) gives

$$0 = x(t_r) \geq x_0 e^{-p\frac{(t_r-t_0)^{\alpha}}{\alpha}} > 0.$$

This contradiction shows that $x(t) > 0$ for $t \in [t_0, +\infty)$. The proof is completed. □

We now prove the positivity of system (1.1) as presented in the following theorem.

Theorem 3.1. *Let Assumption (A) hold and assume that the weight coefficient matrices A, B, C and D are nonnegative (equivalently, the augmented matrix*

$$W = \left(\begin{array}{c:c} A & B \\ \hdashline C^{\top} & D^{\top} \end{array}\right)$$

is nonnegative). Then, system (1.1) *is positive. More precisely, for any initial function $\phi \in \mathscr{F}_d^+$ and nonnegative input vectors $I \succeq 0$, $J \succeq 0$, the corresponding solution $\chi(t) = \mathrm{vec}(x(t), y(t))$ is nonnegative, $\chi(t) \succeq 0$, for all $t \geq t_0$.*

Proof. Let $\chi(t) = \mathrm{vec}(x(t), y(t))$ be a solution of system (1.1) with initial condition $\phi \in \mathscr{F}_d^+$ and input vector $\mathscr{J} = \mathrm{vec}(I, J) \in \mathbb{R}_+^{n+m}$. We first recall here that a vector field $F : \mathbb{R}_+^n \to \mathbb{R}^n$ is said to be order-preserving (see [16]) on $\mathbb{R}_+^n$ if for any $y^1, y^2 \in \mathbb{R}_+^n$, $y^1 \preceq y^2$, then $F(y^1) \preceq F(y^2)$. By assumption (2.3), if $y^1 = (y_j^1)$, $y^2 = (y_j^2) \in \mathbb{R}_+^n$, $y^1 \preceq y^2$, then we have $f_j(y_j^1) \leq f_j(y_j^2)$ for all $j \in [m]$. Thus,

$$\sum_{j=1}^{m} a_{ij} f_j(y_j^1) \leq \sum_{j=1}^{m} a_{ij} f_j(y_j^2),\ i \in [n].$$

Similarly, we also have $\sum_{j=1}^{m} b_{ij} f_j(y_j^1) \leq \sum_{j=1}^{m} b_{ij} f_j(y_j^2)$, $i \in [n]$. This shows that the vector fields $F_A(y) = Af(y)$ and $F_B(y) = Bf(y)$ are order-preserving on $\mathbb{R}_+^m$. Consequently, if $y(t) \succ 0$, $t \in [-\overline{\sigma}, t_1)$, for some $t_1 > t_0$, then by taking into account the ith components of the vector fields $F_A(y)$ and $F_B(y)$, we have

$$r_i(t) \triangleq (Af(y(t)))_i + (Bf(y_{\sigma}(t)))_i + I_i$$

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$$= \sum_{j=1}^{m} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t-\sigma_j(t))) + I_i \geq 0, \quad t \in [t_0, t_1).$$

It follows from (2.1) that

$${}^{c}D_{t_0}^{\alpha} x_i(t) = -\beta_i x_i(t) + r_i(t),\ t \geq t_0,\ i \in [n],$$

and $r_i(t) \geq 0$ for $t \geq t_0$. By Lemma 3.1, we have $x_i(t) \geq 0$ for all $t \in [t_0, t_1)$, $i \in [n]$. Therefore, $\chi(t) \succeq 0$ for $t \in [t_0, t_1)$.

We now prove that $\chi(t) \succeq 0$ for all $t \geq t_0$. For this, by the idea of modifying initial conditions, let $\chi_\epsilon(t) = \mathrm{vec}(x_\epsilon(t), y_\epsilon(t))$ be the solution of system (1.1) with initial condition $\phi_\epsilon = \phi + \epsilon \mathbf{1}_{n+m}$, where $\epsilon > 0$ is sufficiently small. Due to the continuity dependence, there exists a $t_1 > t_0$ such that $\chi_\epsilon(t) \succ 0$ for $t \in [t_0, t_1)$. We will show that $y_\epsilon(t) \succ 0$ for all $t \geq t_0$. In contrast, suppose that there exists a $\hat{t} > t_0$ and an index $j \in [m]$ such that

$$y_{j\epsilon}(\hat{t}) = 0 \text{ and } y_{j\epsilon}(t) > 0, \quad t \in [t_0, \hat{t}),$$

and $y_\epsilon(t) \succeq 0$ for $t \in [t_0, \hat{t}]$. Then, we have $x_\epsilon(t) \succeq 0$ for $t \in [-\overline{\tau}, \hat{t}]$. Similar to the first part of the proof, we have

$$\begin{aligned}\hat{r}_j(t) &\triangleq (Cg(x_\epsilon(t)))_j + (Dg(x_{\tau\epsilon}(t)))_j \\ &= \sum_{i=1}^{n} c_{ji} g_i(x_{i\epsilon}(t)) + \sum_{i=1}^{n} d_{ji} g_i(x_{i\epsilon}(t-\tau_i(t))) + J_j \geq 0, \quad t \in [t_0, \hat{t}].\end{aligned}$$

By similar lines in the proof of Lemma 3.1, we also obtain

$$\begin{aligned}y_{j\epsilon}(t) &= E_\alpha(-\gamma_j, t-t_0)\left(y_{0j}(0) + \epsilon + \int_{t_0}^{t} \frac{E_\alpha(\gamma_j, s-t_0)\hat{r}_j(s)}{(s-t_0)^{1-\alpha}} ds\right) \\ &\geq \epsilon E_\alpha(-\gamma_j, t-t_0), \quad t \in [t_0, \hat{t}).\end{aligned} \tag{3.3}$$

Let $t \uparrow \hat{t}$, from (3.3), we obtain

$$y_{j\epsilon}(\hat{t}) \geq \epsilon e^{-\gamma_j \frac{(\hat{t}-t_0)^\alpha}{\alpha}} > 0,$$

which yields a contradiction. By this we can conclude that $y_\epsilon(t) \succ 0$ and thus $x_\epsilon(t) \succeq 0$ for $t \geq t_0$. Let $\epsilon \downarrow 0$ we obtain

$$\chi(t) = \lim_{\epsilon \downarrow 0} \chi_\epsilon(t) \succeq 0$$

for all $t \in [t_0, +\infty)$. The proof is completed. □

3.2. Positive equilibria

In this section, by utilizing the Brouwer's fixed point theorem, we derive conditions for the existence of positive EP of system (1.1). First, it can be verified from (2.5) that a vector $\chi_* = \mathrm{vec}(x_*, y_*) \in \mathbb{R}^{n+m}$ is an EP of system (1.1) if and only if it satisfies the algebraic system

$$\begin{cases} D_\beta^{-1}\left((A+B)f(y_*) + I\right) = x_*, \\ D_\gamma^{-1}\left((C+D)g(x_*) + J\right) = y_*. \end{cases} \tag{3.4}$$

From (3.4), we define the functions $s_i(y)$, $\tilde{s}_j(x)$ and a mapping $\mathscr{S}:\mathbb{R}^{n+m}\to\mathbb{R}^{n+m}$ as

$$s_i(y)=\frac{1}{\beta_i}\left[\sum_{j=1}^{m}(a_{ij}+b_{ij})f_j(y_j)+I_i\right],\quad i\in[n],$$
$$\tilde{s}_j(x)=\frac{1}{\gamma_j}\left[\sum_{i=1}^{n}(c_{ji}+d_{ji})g_i(x_i)+J_j\right],\quad j\in[m],$$

and

$$\begin{aligned}\mathscr{S}(\chi)&=\Big(s_1(y)\ s_2(y)\ \cdots\ s_n(y)\ \tilde{s}_1(x)\ \tilde{s}_1(x)\ \cdots\ \tilde{s}_m(x)\Big)^\top\\&=\begin{pmatrix}D_\beta^{-1}\big((A+B)f(y)+I\big)\\D_\gamma^{-1}\big((C+D)g(x)+J\big)\end{pmatrix},\end{aligned}\tag{3.5}$$

where $\chi=\mathrm{vec}(x,y)$, $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$.

Regarding (3.4) and (3.5), a vector $\chi_*\in\mathbb{R}^{n+m}$ is an EP of system (1.1) if and only if it is a fixed point of the mapping $\mathscr{S}$, that is, $\mathscr{S}(\chi_*)=\chi_*$. Based on the Brouwer's fixed point theorem, we have the following result.

Theorem 3.2. *Let Assumption (A) hold and assume that*

$$\rho(\mathscr{H})<1,\quad \mathscr{H}=\left(\begin{array}{c|c}0_n&\mathscr{H}_{12}\\\hline\mathscr{H}_{21}&0_m\end{array}\right),\tag{3.6}$$

where the matrices $\mathscr{H}_{12}=(h^f_{ij})\in\mathbb{R}^{n\times m}$, $\mathscr{H}_{21}=(h^g_{ji})\in\mathbb{R}^{m\times n}$ are defined by entries

$$h^f_{ij}=\frac{1}{\beta_i}\left(|a_{ij}|+|b_{ij}|\right)l^f_j,\quad h^g_{ji}=\frac{1}{\gamma_j}\left(|c_{ji}|+|d_{ji}|\right)l^g_i,\ i\in[n],\ j\in[m].$$

Then, for a given input vector $\mathscr{J}=\mathrm{vec}(I,J)\in\mathbb{R}^{n+m}$, system (1.1) has at least one EP.

Proof. By Proposition 2.5, it follows from (3.6) that $I_{n+m}-\mathscr{H}$ is a nonsingular M-matrix. Thus, there exists a vector $\xi=\mathrm{vec}(\xi_1,\xi_2)\in\mathbb{R}^{n+m}$, $\xi\succ 0$, such that $(I_{n+m}-\mathscr{H})\xi\succ 0$. In addition, the inverse matrix $(I_{n+m}-\mathscr{H})^{-1}$ exists and is nonnegative.

For given input vectors $I\in\mathbb{R}^n$ and $J\in\mathbb{R}^m$, we define the vectors

$$\zeta_1=(D_\beta)^{-1}|I|\in\mathbb{R}^n_+,\quad \zeta_2=(D_\gamma)^{-1}|J|\in\mathbb{R}^m_+$$

and

$$\begin{pmatrix}\rho_1\\\rho_2\end{pmatrix}=(I_{n+m}-\mathscr{H})^{-1}\begin{pmatrix}\zeta_1\\\zeta_2\end{pmatrix}+\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}.\tag{3.7}$$

It is clear that $\rho=\mathrm{vec}(\rho_1,\rho_2)\succ 0$ and, from (3.7), we have

$$\begin{pmatrix}\rho_1\\\rho_2\end{pmatrix}=\mathscr{H}\begin{pmatrix}\rho_1\\\rho_2\end{pmatrix}+\begin{pmatrix}\zeta_1\\\zeta_2\end{pmatrix}+(I_{n+m}-\mathscr{H})\xi.\tag{3.8}$$

Revealed by the decomposition (3.8), we define the convex compact subset $\mathscr{B} \subset \mathbb{R}^{n+m}$ as follows

$$\mathscr{B} = \left\{\chi = \operatorname{vec}(x,y)\big| x \in \mathbb{R}^n, y \in \mathbb{R}^m, |x| \preceq \rho_1 \text{ and } |y| \preceq \rho_2\right\}.$$

For any $\chi = \operatorname{vec}(x,y) \in \mathscr{B}$, where $x = (x_i) \in \mathbb{R}^n$ and $y = (y_j) \in \mathbb{R}^m$, we have $|x_i| \le \rho_{1i}$ and $|y_j| \le \rho_{2j}$. Thus, together with condition (2.3), we have the following estimate

$$\begin{aligned} |s_i(y)| &= \left|\frac{1}{\beta_i}\left(\sum_{j=1}^m (a_{ij}+b_{ij})f_j(y_j) + I_i\right)\right| \\ &\le \frac{1}{\beta_i}\left(\sum_{j=1}^m (|a_{ij}|+|b_{ij}|)|f_j(y_j)| + |I_i|\right) \\ &\le \frac{1}{\beta_i}\left(\sum_{j=1}^m l_j^f(|a_{ij}|+|b_{ij}|)|y_j| + |I_i|\right) \\ &\le \sum_{j=1}^m h_{ij}^f \rho_{2j} + \zeta_{1i}, \quad i \in [n]. \end{aligned} \tag{3.9}$$

Similarly, we also have

$$|\tilde{s}_j(x)| \le \sum_{i=1}^n h_{ji}^g \rho_{1i} + \zeta_{2j}, \quad j \in [m]. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$|\mathscr{S}(\chi)| \preceq \mathscr{H}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} + \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \preceq \begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}. \tag{3.11}$$

This shows that $\mathscr{S}(\chi) \in \mathscr{B}$ for any $\chi \in \mathscr{B}$. In other words, $\mathscr{S}$ is a continuous mapping that maps the convex compact set $\mathscr{B}$ into itself. By the Brouwer's fixed point theorem [28], $\mathscr{S}$ possesses at least a fixed point $\chi_* \in \mathscr{B}$, which is an EP of system (1.1). The proof is completed. □

Remark 3.1. If

$$W = \left(\begin{array}{c:c} A & B \\ \hdashline C^\top & D^\top \end{array}\right) \succeq 0 \text{ and } \mathscr{J} = \begin{pmatrix} I \\ J \end{pmatrix} \succeq 0,$$

then, according to condition (2.3), we have

$$\begin{aligned} s_i &= \frac{1}{\beta_i}\left(\sum_{j=1}^m (a_{ij}+b_{ij})f_j(y_j) + I_i\right) \ge 0, \\ \tilde{s}_j &= \frac{1}{\gamma_j}\left(\sum_{i=1}^n (c_{ji}+d_{ji})g_i(x_i) + J_j\right) \ge 0, \end{aligned}$$

for any $x = (x_i) \in \mathbb{R}_+^n$ and $y = (y_i) \in \mathbb{R}_+^m$. Thus, $\mathscr{S}(\chi) \succeq 0$ for any $\chi \in \mathbb{R}_+^{n+m}$. This shows that the mapping $\mathscr{S}$ keeps the convect cone $\mathscr{B}_+ = \mathscr{B} \cap \mathbb{R}_+^{n+m}$ invariant. Once again, the Brouwer's fixed point theorem ensures that the mapping $\mathscr{S}$ possesses at least a positive fixed point $\chi_*^+ \in \mathscr{B}_+$, which is a positive EP of system (1.1). We summarize this result in the following corollary.

Corollary 3.1. *Let Assumption (A) hold. Assume that the matrix* $W = \left(\begin{array}{c|c} A & B \\ \hline C^\top & D^\top \end{array}\right)$ *is nonnegative and condition* (3.6) *is satisfied. Then, for a given nonnegative input vector* $\mathscr{J} = \mathrm{vec}(I,J)$*, system* (1.1) *has at least a positive EP* $\chi_*^+ \in \mathbb{R}_+^{n+m}$.

Remark 3.2. Let $\widetilde{\mathscr{H}} = \mathscr{H}_{12}\mathscr{H}_{21} = (\widetilde{\mathscr{H}_{ij}}) \in \mathbb{R}^{n\times n}$, where

$$\widetilde{\mathscr{H}_{ij}} = \frac{1}{\beta_i}\sum_{k=1}^{m}\frac{1}{\gamma_k}\left(|a_{ik}|+|b_{ik}|\right)\left(|c_{kj}|+|d_{kj}|\right) l_k^f l_j^g,\ i,j\in[n].$$

By the Schur identities (see [18, Remark 4] for more details), we have

$$\begin{aligned}\det(\lambda I_{n+m} - \mathscr{H}) &= \det\left(\begin{array}{c|c} \lambda I_n & -\mathscr{H}_{12} \\ \hline -\mathscr{H}_{21} & \lambda I_m \end{array}\right) \\ &= \lambda^m \det\left(\lambda I_n - \frac{1}{\lambda}\widetilde{\mathscr{H}}\right) \\ &= \lambda^{m-n}\det\left(\lambda^2 I_n - \widetilde{\mathscr{H}}\right) \end{aligned} \tag{3.12}$$

for any $\lambda\in\mathbb{C}$, $\lambda\neq 0$. Therefore, $\lambda\in\sigma(\mathscr{H})\backslash\{0\}$ if and only if $\mu=\lambda^2\in\sigma(\widetilde{\mathscr{H}})\backslash\{0\}$ and, consequently, $\rho(\mathscr{H})<1$ if and only if $\rho(\widetilde{\mathscr{H}})<1$. By this, condition (3.6) holds if and only if $\rho(\widetilde{\mathscr{H}})<1$. Similarly, we can also conclude that condition (3.6) holds if and only if $\rho(\widehat{\mathscr{H}})<1$, where $\widehat{\mathscr{H}}=\mathscr{H}_{21}\mathscr{H}_{12}=(\widehat{\mathscr{H}_{ij}})\in\mathbb{R}^{m\times m}$ and

$$\widehat{\mathscr{H}_{ij}} = \frac{1}{\gamma_i}\sum_{k=1}^{n}\frac{1}{\beta_k}\left(|c_{ik}|+|d_{ik}|\right)\left(|a_{kj}|+|b_{kj}|\right) l_k^g l_j^f,\ i,j\in[m].$$

Corollary 3.2. *Let condition* (2.3) *hold and assume that the weight coefficient matrices* A*,* B*,* C*, and* D *are nonnegative. If* $\rho(\widetilde{\mathscr{H}})<1$ *or* $\rho(\widehat{\mathscr{H}})<1$*, then system* (1.1) *has at least one positive EP* $\chi_*^+\in\mathbb{R}_+^{n+m}$ *for any nonnegative input vector* $\mathscr{J}=\mathrm{vec}(I,J)$.

3.3. Fractional exponential stability

In this section, we focus on the (fractional) exponential stability of the unique positive EP of system (1.1). The main result of this section is presented as in the following theorem.

Theorem 3.3. *Let Assumption (A) hold. Assume that the weight coefficient matrices are nonnegative and one of the three following conditions is satisfied*

(i) *The spectral radius of the matrix* $\mathscr{H}$ *satisfies* $\rho(\mathscr{H})<1$*;*

(ii) *there exists a vector* $\eta=(\eta_i)\in\mathbb{R}^n$*,* $\eta\succ 0$*, such that*

$$\frac{1}{\beta_i}\sum_{k=1}^{m}\frac{l_k^f}{\gamma_k}\left(a_{ik}+b_{ik}\right)\sum_{j=1}^{n}\left(c_{kj}+d_{kj}\right) l_j^g\eta_j<\eta_i,\quad i\in[n];$$

(iii) *there exists a vector* $\zeta=(\zeta_i)\in\mathbb{R}^m$*,* $\zeta\succ 0$*, such that*

$$\frac{1}{\gamma_i}\sum_{k=1}^{n}\frac{l_k^g}{\beta_k}\left(c_{ik}+d_{ik}\right)\sum_{j=1}^{m}\left(a_{kj}+b_{kj}\right) l_j^f\zeta_j<\zeta_i,\quad i\in[m].$$

Then, for any nonnegative input vector $\mathscr{J} = \mathrm{vec}(I,J) \in \mathbb{R}_+^{n+m}$, *system* (1.1) *has a unique positive EP* $\chi_*^+ \in \mathbb{R}_+^{n+m}$, *which is GFES for any delays* $\tau_i(t) \in [0,\overline{\tau}]$, $\sigma_j(t) \in [0,\overline{\sigma}]$.

Proof. As discussed in Remark 3.2, the three conditions (i), (ii), and (iii) are equivalent. Thus, in what follows, we only consider condition $\rho\big(\mathscr{H}\big) < 1$. The rest of the proof will be divided into some steps.

(a) *The existence of positive EP:* By Corollary 3.1, for any input vector $\mathscr{J} = \mathrm{vec}(I,J) \in \mathbb{R}_+^{n+m}$, system (1.1) possesses a positive EP $\chi_*^+ = \mathrm{vec}(x_*,y_*) \in \mathbb{R}_+^{n+m}$, which satisfies the following algebraic system

$$\begin{cases} -D_\beta x_* + (A+B)f(y_*) + I = 0, \\ -D_\gamma y_* + (C+D)g(x_*) + J = 0. \end{cases} \tag{3.13}$$

Then, we have

$$\begin{aligned} {}^cD_{t_0}^{\alpha}(x(t)-x_*) = &\ -D_\beta(x(t)-x_*) + A\left[f(y(t)) - f(y_*)\right] \\ &+ B\left[f(y_\sigma(t)) - f(y_*)\right], \\ {}^cD_{t_0}^{\alpha}(y(t)-y_*) = &\ -D_\gamma(y(t)-y_*) + A\left[g(x(t)) - g(x_*)\right] \\ &+ D\left[g(x_\tau(t)) - g(x_*)\right]. \end{aligned} \tag{3.14}$$

(b) *Upper differential inequalities:* We define the following functions

$$\mu(t) = |x(t) - x^*| = (\mu_i(t)) \text{ and } \eta(t) = |y(t) - y^*| = (\eta_j(t)),\ t \geq t_0.$$

Let

$${}^cD_a^{\alpha+} f(t) = \limsup_{\epsilon\to 0^+} \frac{f\left(t + \epsilon(t-a)^{1-\alpha}\right) - f(t)}{\epsilon}$$

denote the fractional upper-right Dini derivative of order α, starting from a of a function f. It follows from (3.14) that

$$\begin{aligned} {}^cD_{t_0}^{\alpha+}\mu_i(t) &= \mathrm{sgn}(x_i(t) - x_{*i})\, {}^cD_{t_0}^{\alpha}\big(x_i(t) - x_i^*\big) \\ &= -\beta_i\, \mathrm{sgn}\left(x_i(t) - x_{*i}\right)\big(x_i(t) - x_{i*}\big) \\ &\quad + \sum_{j=1}^m a_{ij}\, \mathrm{sgn}\left(x_i(t) - x_i^*\right)\Big(f_j(y_j(t)) - f_j(y_{j*})\Big) \\ &\quad + \sum_{j=1}^m b_{ij}\, \mathrm{sgn}\left(x_i(t) - x_i^*\right)\Big(f_j\big(y_j(t-\sigma_j(t))\big) - f_j(y_{j*})\Big) \\ &\leq -\beta_i\mu_i(t) + \sum_{j=1}^m l_j^f\Big(a_{ij}\eta_j(t) + b_{ij}\eta_j(t-\sigma_j(t))\Big). \end{aligned} \tag{3.15}$$

Similarly, we have

$${}^cD_{t_0}^{\alpha+}\eta_j(t) \leq -\gamma_j\eta_j(t) + \sum_{i=1}^n l_i^g\Big(c_{ji}\mu_i(t) + d_{ji}\mu_i(t-\tau_i(t))\Big). \tag{3.16}$$

(c) *Frational exponential domination:* It is noticed at first that condition (i), $\rho(\mathscr{H}) < 1$, holds if and only if $I_{n+m} - \mathscr{H}$ is a nonsingular M-matrix. By Proposition 2.5, there exists a vector $\xi = \mathrm{vec}(\xi_1, \xi_2) \succ 0$, where $\xi_1 = (\xi_{1i}) \in \mathbb{R}^n$ and $\xi_2 = (\xi_{2j}) \in \mathbb{R}^m$, such that

$$\left(\begin{array}{c|c} I_n & -\mathscr{H}_{12} \\ \hline -\mathscr{H}_{21} & I_m \end{array}\right) \left(\begin{array}{c} \xi_1 \\ \hline \xi_2 \end{array}\right) \succ 0.$$

Thus, for sufficiently small $\epsilon > 0$, we have

$$\begin{cases} \mathscr{H}_{12}\xi_2 \prec \xi_1 - \epsilon \mathbf{1}_n, \\ \mathscr{H}_{21}\xi_1 \prec \xi_2 - \epsilon \mathbf{1}_m. \end{cases} \tag{3.17}$$

Let $\hat{\varepsilon} = \epsilon \min_{i,j}\{\beta_i, \gamma_j\} > 0$. It follows from (3.17) that

$$\begin{aligned} &-\beta_i \xi_{1i} + \sum_{j=1}^{m} l_j^f (a_{ij} + b_{ij})\xi_{2j} < -\hat{\epsilon},\ i \in [n], \\ &-\gamma_j \xi_{2j} + \sum_{k=1}^{n} l_k^g (c_{jk} + d_{jk})\xi_{1k} < -\hat{\epsilon},\ j \in [m]. \end{aligned} \tag{3.18}$$

We now define functions $\varphi_i, \psi_j : [0, +\infty) \to \mathbb{R}$, $i \in [n]$, $j \in [m]$, by

$$\begin{aligned} \varphi_i(\lambda) &= (\lambda - \beta_i)\xi_{1i} + \sum_{j=1}^{m} l_j^f \left(a_{ij} + b_{ij} E_\alpha(\lambda, \overline{\sigma})\right) \xi_{2j}, \\ \psi_j(\lambda) &= (\lambda - \gamma_j)\xi_{2j} + \sum_{i=1}^{n} l_i^g \left(c_{ji} + d_{ji} E_\alpha(\lambda, \overline{\tau})\right) \xi_{1i}. \end{aligned}$$

Clearly, the functions $\varphi_i(\cdot)$, $\psi_j(\cdot)$ are continuous and strictly increasing, $\varphi_i(0) < -\hat{\epsilon}$, $\psi_j(0) < -\hat{\epsilon}$ and $\varphi_i(\lambda) \to +\infty$, $\psi_i(\lambda) \to +\infty$ as $\lambda \to +\infty$. Thus, there exist positive constants $\tilde{\lambda}_i$, $\hat{\lambda}_j$ such that $\varphi_i(\tilde{\lambda}_i) = 0$, $\psi_j(\hat{\lambda}_j) = 0$ and $\varphi_i(\lambda) < 0$ for $\lambda \in [0, \tilde{\lambda}_i)$, $\psi_i(\lambda) < 0$ for $\lambda \in [0, \hat{\lambda}_j)$.

Let $\lambda_\star = \min\left\{\tilde{\lambda}_i, \hat{\lambda}_j : i \in [n], j \in [m]\right\}$. Then, $\varphi_i(\lambda_\star) \le 0$, $\psi_j(\lambda_\star) \le 0$ and, for any $\lambda \in (0, \lambda_\star)$, we have

$$\begin{aligned} &-\beta_i \xi_{1i} + \sum_{j=1}^{m} l_j^f \left(a_{ij} + b_{ij} E_\alpha(\lambda, \overline{\sigma})\right) \xi_{2j} < -\lambda \xi_{1i}, \\ &-\gamma_j \xi_{2j} + \sum_{i=1}^{n} l_i^g \left(c_{ji} + d_{ji} E_\alpha(\lambda, \overline{\tau})\right) \xi_{1i} < -\lambda \xi_{2j}. \end{aligned} \tag{3.19}$$

For a fixed $\lambda \in (0, \lambda^\star)$, we define the following comparative functions

$$\begin{aligned} &\hat{\mu}(t) = C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0)\xi_1, \\ &\hat{\eta}(t) = C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0)\xi_2, \quad t \ge t_0, \end{aligned} \tag{3.20}$$

where $C(\xi) = 1/\min_{i,j}\{\xi_{1i}, \xi_{2j}\}$.

Let

$$\Upsilon(t) = \begin{pmatrix} \mu(t) \\ \eta(t) \end{pmatrix}, \quad \hat{\Upsilon}(t) = \begin{pmatrix} \hat{\mu}(t) \\ \hat{\eta}(t) \end{pmatrix} = C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

By direct calculation using the properties of conformable fractional derivative, we get

$$ {}^{c}D_{t_0}^{\alpha+}\hat{\Upsilon}(t) = -\lambda\hat{\Upsilon}(t), \quad t > t_0. $$

If $t \in [t_0, +\infty) \cap \{t \geq t_0 : t - \tau_i(t) \leq t_0\}$, then

$$ \begin{aligned} \hat{\mu}_i(t - \tau_i(t)) &= \hat{\mu}_i(t_0) = C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}}\xi_{1i} \\ &\leq E_\alpha(\lambda, \overline{\tau})\hat{\mu}_i(t). \end{aligned} $$

If $t - \tau_i(t) > t_0$, then we have

$$ \begin{aligned} (t - \tau_i(t) - t_0)^\alpha + \overline{\tau}^\alpha &\geq (t - \tau_i(t) - t_0)^\alpha + \tau_i^\alpha(t) \\ &\geq (t - t_0)^\alpha. \end{aligned} $$

Therefore,

$$ \begin{aligned} \hat{\mu}_i(t - \tau_i(t)) &= C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - \tau_i(t) - t_0)\xi_{1i} \\ &\leq C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0)E_\alpha(\lambda, \overline{\tau})\xi_{1i} \\ &= E_\alpha(\lambda, \overline{\tau})\hat{\mu}_i(t). \end{aligned} \tag{3.21} $$

Similarly, for any $t \geq t_0$, we also have

$$ \hat{\eta}_j(t - \sigma_j(t)) \leq E_\alpha(\lambda, \overline{\sigma})\hat{\eta}_j(t). \tag{3.22} $$

We now show that $\Upsilon(t) \preceq \hat{\Upsilon}(t)$ for all $t \geq t_0$. To this end, we notice at first that $C(\xi)\xi \succeq \mathbf{1}_{n+m}$ and

$$ \Upsilon(t_0) \preceq C(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}}\xi = \hat{\Upsilon}(t_0). $$

Thus, for given $q > 1$, we have $\Upsilon(t_0) \prec q\hat{\Upsilon}(t_0)$. If $\Upsilon(t) \prec q\hat{\Upsilon}(t)$ does not hold for all $t > t_0$, then there exists an index k and a $\overline{t} > t_0$ such that

$$ \Upsilon_k(\overline{t}) = q\hat{\Upsilon}_k(\overline{t}), \quad \Upsilon_k(t) < q\hat{\Upsilon}_k(t),\ t \in [t_0, \overline{t}), \tag{3.23} $$

and $\Upsilon(t) \preceq q\hat{\Upsilon}(t)$ for all $t \in [t_0, \overline{t}]$.

If $1 \leq k \leq n$, then from (3.15), (3.18) and (3.22), for $t \in [t_0, \overline{t})$, we have

$$ \begin{aligned} {}^{c}D_{t_0}^{\alpha+}\Upsilon_k(t) &\leq -\beta_k\Upsilon_k(t) + \sum_{j=1}^{m} l_j^f \left(a_{kj}\eta_j(t) + b_{kj}\eta_j(t - \sigma_j(t))\right) \\ &\leq -\beta_k\Upsilon_k(t) + q\sum_{j=1}^{m} l_j^f \left(a_{kj}\hat{\eta}_j(t) + b_{kj}\hat{\eta}_j(t - \sigma_j(t))\right) \\ &\leq -\beta_k\Upsilon_k(t) + q\sum_{j=1}^{m} l_j^f \left(a_{kj} + b_{kj}E_\alpha(\lambda, \overline{\sigma})\right)\hat{\eta}_j(t) \\ &= -\beta_k\Upsilon_k(t) + qC(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0) \\ &\quad \times \sum_{j=1}^{m} l_j^f \big(a_{kj} + b_{kj}E_\alpha(\lambda, \overline{\sigma})\big)\xi_{2j} \\ &\leq -\beta_k\Upsilon_k(t) + qC(\xi)\|\phi - \chi_*^+\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0)\left(-\lambda + \beta_k\right)\xi_{1k} \\ &= -\beta_k\left(\Upsilon_k(t) - q\hat{\Upsilon}_k(t)\right) - q\lambda\hat{\Upsilon}_k(t) \\ &= -\beta_k\left(\Upsilon_k(t) - q\hat{\Upsilon}_k(t)\right) + q\,{}^{c}D_{t_0}^{\alpha+}\hat{\Upsilon}_k(t). \end{aligned} \tag{3.24} $$

The last inequality in (3.24) implies that

$$ {}^{c}D_{t_0}^{\alpha+}\left(\Upsilon_k(t)-q\hat{\Upsilon}_k(t)\right)\le-\beta_k\left(\Upsilon_k(t)-q\hat{\Upsilon}_k(t)\right),\quad t\in[t_0,\bar{t}). \tag{3.25} $$

By Lemmas 2.1 and 2.2, it follows from (3.25) that

$$ \Upsilon_k(t)-q\hat{\Upsilon}_k(t)\le\left(\Upsilon_k(t_0)-q\hat{\Upsilon}_k(t_0)\right)E_\alpha(-\beta_k,t-t_0),\quad t\in[t_0,\bar{t}). \tag{3.26} $$

Let $t\uparrow\bar{t}$, from (3.23) and (3.26), we readily obtain

$$ 0=\Upsilon_k(\bar{t})-q\hat{\Upsilon}_k(\bar{t})\le\left(\Upsilon_k(t_0)-q\hat{\Upsilon}_k(t_0)\right)E_\alpha(-\beta_k,\bar{t}-t_0)<0. $$

This contradiction shows that $\Upsilon(t)\prec q\hat{\Upsilon}(t)$ for all $t\ge t_0$. Letting $q\downarrow 1$ we get $\Upsilon(t)\preceq\hat{\Upsilon}(t)$, $t\ge t_0$. The case $1\le k\le m$ can be proved by similar arguments. Finally, from (3.20), we have

$$ \begin{aligned}\|\chi(t)-\chi_*^+\|_\infty&=\|\Upsilon(t)\|_\infty\\&\le C(\xi)\|\xi\|_\infty\|\phi-\chi_*^+\|_{\mathcal{C}}E_\alpha(-\lambda,t-t_0),\quad t\ge t_0,\end{aligned} $$

by which we can conclude the fractional exponential stability of the EP χ_*.
(d) *Uniqueness of the positive EP:* Assume that $\hat{\chi}_*^+=\mathrm{vec}(\hat{x}_*,\hat{y}_*)$ is also a positive EP of system (1.1). Then, $\hat{\chi}_*^+$ can be regarded as a stationary solution of (1.1) with constant initial condition $\phi=\hat{\chi}_*^+$. By *Step* (c), we have

$$ \|\hat{\chi}_*^+-\chi_*^+\|_\infty\le C(\xi)\|\xi\|_\infty\|\hat{\chi}_*^+-\chi_*^+\|_\infty E_\alpha(-\lambda,t-t_0)\to 0 $$

as $t\to+\infty$. Thus, $\hat{\chi}_*^+=\chi_*^+$. This indicates the uniqueness of χ_*^+. The proof is completed. □

Remark 3.3. The result of Theorem 3.3 encompasses those of Theorem 3 in [18] and Theorem 1 in [31] as some special cases. More specifically, since system (1.1) describes a class of fractional BAM neural networks, which include the nominal model of BAM neural networks [18] and Hopfield neural networks [31], the results of [18, 31] are critical cases of Theorem 3.3 and can be revoked by significantly simplifying the derivation process of Theorem 3.3.

3.4. An application to fractional linear systems with delay

Consider the following fractional linear system with delay

$$ \begin{aligned}&{}^{c}D_{t_0}^{\alpha}x(t)=Ax(t)+A_dx(t-\tau(t)),\ t>t_0,\\&x(t_0+s)=\phi(s),\ s\in[-\overline{\tau},0],\end{aligned} \tag{3.27} $$

where A, $A_d\in\mathbb{R}^{n\times n}$ are given real matrices, and $\tau(t)\in[0,\overline{\tau}]$ is the time-varying delay.

Similar to the proof of Theorem 3.3, it can be verified from (3.27) that any solution $x(t,t_0,\phi)$ of system (3.27) satisfies

$$ |x(t,t_0,\phi)|\preceq\hat{x}(t,t_0,|\phi|),\quad t\ge t_0, \tag{3.28} $$

where $\hat{x}(t, t_0, |\phi|)$ is the corresponding solution of the following problem

$$\begin{aligned} &{}^cD^{\alpha}_{t_0}\hat{x}(t) = A^D\hat{x}(t) + |A_d|\hat{x}(t-\tau(t)),\ t > t_0,\\ &\hat{x}(t_0+s) = |\phi(s)|,\ s \in [-\overline{\tau}, 0], \end{aligned} \tag{3.29}$$

where the matrix $A^D = (a^D_{ij})$ is determined as

$$a^D_{ij} = \begin{cases} a_{ii} \text{ if } i = j,\\ |a_{ij}| \text{ if } i \neq j. \end{cases}$$

Clearly, $\mathcal{M} = A^D + |A_d|$ is a Metzler matrix. Thus, if the matrix $\mathcal{M}$ is Hurwitz, then system (3.29), and hence system (3.27), are GFES. We summarize this result in the following proposition.

Proposition 3.1. *Assume that one of the following equivalent conditions is satisfied.*

(i) *The matrix $\mathcal{M} = A^D + |A_d|$ is Hurwitz.*

(ii) *There exists a vector $\xi \in \mathbb{R}^n$, $\xi \succ 0$, such that*

$$\left(A^D + |A_d|\right)\xi \prec 0.$$

Then, system (3.27) is GFES. More precisely, there exists a positive scalar δ such that any solution $x(t, t_0, \phi)$ of system (3.27) satisfies the fractional exponential estimate

$$\|x(t, t_0, \phi)\|_\infty \le \delta\|\phi\|_{\mathcal{C}} E_\alpha(-\lambda, t - t_0), \quad t \ge t_0,$$

where the maximum allowable decay rate $\lambda > 0$ can be determined by the following generic procedure

$$\begin{aligned} &\texttt{maximize} \quad \lambda > 0 \quad \texttt{s.t.}\\ &a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|\frac{\xi_j}{\xi_i} + \sum_{j=1}^{n} |a_{dij}|E_\alpha(\lambda, \overline{\tau})\frac{\xi_j}{\xi_i} \le 0,\ i \in [n]. \end{aligned}$$

4. Simulations

Example 4.1. Consider system (1.1) with Sigmoidal-Boltzmann nonlinear functions $f_j(y_j) = \mathcal{S}_{\theta^f_j}(y_j)$, $g_i(x_i) = \mathcal{S}_{\theta^g_i}(x_i)$, where θ^f_j, θ^g_i $(i, j = 1, 2, 3)$ are given positive scalars and weighted sigmoid function $\mathcal{S}_\theta(x)$ is defined as

$$\mathcal{S}_\theta(x) = \frac{1 - \exp\left(-\frac{x}{\theta}\right)}{1 + \exp\left(-\frac{x}{\theta}\right)}, \quad \theta > 0, x \in \mathbb{R}. \tag{4.1}$$

It can be verified by simple calculation from (4.1) that

$$\mathcal{S}'_\theta(x) = \frac{2\exp\left(-\frac{x}{\theta}\right)}{\theta\left(1 + \exp\left(-\frac{x}{\theta}\right)\right)^2} > 0$$

and

$$\sup_{t\in(0,+\infty)} \frac{t}{(1+t)^2} = \frac{1}{4}.$$

Thus, Assumption (A) is satisfied with $l_j^f = \frac{1}{2\theta_j^f}$ and $l_i^g = \frac{1}{2\theta_i^g}$. To illustrate the obtained theoretical results, we specify the system parameters as follows

$$\begin{aligned}
&\theta_j^f = 1.75,\ \theta_i^g = 1.6\ (i,j=1,2,3),\\
&D_\beta = \beta I_3,\ D_\gamma = \gamma I_3,\ \beta = 2.05,\ \gamma = 1.86,\\
&A = \begin{bmatrix} 2.05\ 1.12\ 0.68\\ 0.85\ 1.21\ 0.66\\ 1.12\ 1.35\ 0.98 \end{bmatrix}, \quad B = \begin{bmatrix} 0.52\ 1.18\ 1.35\\ 1.46\ 0.95\ 0.88\\ 0.64\ 0.78\ 1.23 \end{bmatrix},\\
&C = \begin{bmatrix} 0.62\ 1.38\ 1.22\\ 0.86\ 0.57\ 1.15\\ 1.25\ 0.68\ 0.94 \end{bmatrix}, \quad D = \begin{bmatrix} 1.15\ 0.56\ 0.72\\ 0.45\ 1.23\ 0.92\\ 1.14\ 0.87\ 0.79 \end{bmatrix}.
\end{aligned}$$

Then, we have

$$\mathscr{H}_{12} = \begin{bmatrix} 0.3582\ 0.3206\ 0.2829\\ 0.3220\ 0.3010\ 0.2146\\ 0.2453\ 0.2969\ 0.3080 \end{bmatrix}, \quad \mathscr{H}_{21} = \begin{bmatrix} 0.2974\ 0.2201\ 0.4015\\ 0.3259\ 0.3024\ 0.2604\\ 0.3259\ 0.3478\ 0.2907 \end{bmatrix}$$

and

$$\widetilde{H} = \mathscr{H}_{12}\mathscr{H}_{21} = \begin{bmatrix} 0.3032\ 0.2742\ 0.3095\\ 0.2638\ 0.2365\ 0.2701\\ 0.2701\ 0.2509\ 0.2653 \end{bmatrix}.$$

Clearly,

$$\left(I_3 - \widetilde{H}\right)^{-1} = \begin{bmatrix} 2.6492\ 1.4993\ 1.6673\\ 1.4331\ 2.3009\ 1.4496\\ 1.4634\ 1.3370\ 2.4692 \end{bmatrix} \succ 0.$$

By Proposition 2.5, $I_3 - \widetilde{H}$ is a nonsingular M-matrix. Thus, the derived conditions in Theorem 3.3 are fulfilled. By Theorem 3.3, for a given input vector $\mathscr{J} = \mathrm{vec}(I, J)$, where $I, J \in \mathbb{R}_+^3$, system (1.1) has a unique positive EP χ_*, which is GFES for any bounded time-varying delays $\tau_i(t)$, $\sigma_j(t)$.

For given input vectors

$$I = \begin{pmatrix} 0.5\\ 1.0\\ 0.75 \end{pmatrix} \text{ and } J = \begin{pmatrix} 1.2\\ 0.8\\ 1.5 \end{pmatrix},$$

by solving system (2.5) with Matlab Symbolic Toolbox, we obtain a unique positive EP $\chi_* = \mathrm{vec}(x_*, y_*)$, where

$$x_* = \begin{pmatrix} 2.2255\\ 2.2051\\ 2.1205 \end{pmatrix} \text{ and } y_* = \begin{pmatrix} 2.4457\\ 2.0775\\ 2.6167 \end{pmatrix}.$$

The behavior of nonlinear functions $f_j(y_j)$ and $g_i(x_i)$ is presented in Fig. 1. Simulation results for state trajectories $x_i(t)$ and $y_j(t)$ with various values of order

$\alpha \in (0,1)$, initial conditions $x_0(s) = (2.0, 1.5, 2.5)^\top$, $y_0(s) = (2.0, 1.5, 3.0)^\top$, $s \in [-5, 0]$ and time-varying delays $\tau_i(t) = 1 + 4|\sin(10\pi t)|$, $\sigma_j(t) = 5|\cos(5\pi t)|$ are presented in Fig. 2 and Fig. 3, respectively. Clearly, the conducted state trajectories of the system are positive and converge to the unique positive EP χ_*, which validates the obtained theoretical results.

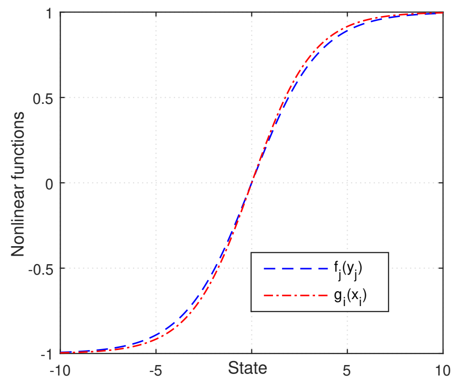
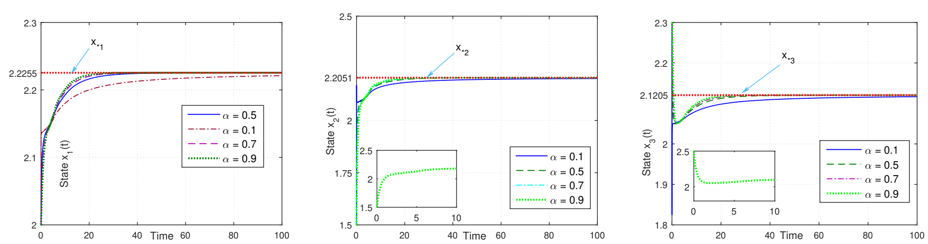
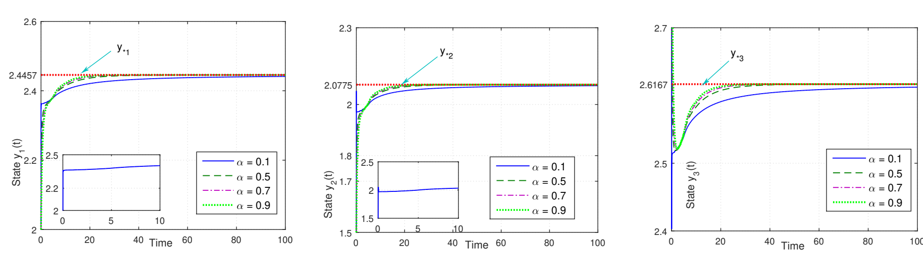


Figure 1. Behavior of nonlinear functions



(a) x_1 (b) x_2 (c) x_3

Figure 2. State trajectories $x(t)$ with delays $\tau_i(t) = 1 + 4|\sin(10\pi t)|$ and $\sigma_j(t) = 5|\cos(5\pi t)|$



(a) y_1 (b) y_2 (c) y_3

Figure 3. State trajectories $y(t)$ with delays $\tau_i(t) = 1 + 4|\sin(10\pi t)|$ and $\sigma_j(t) = 5|\cos(5\pi t)|$

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Example 4.2. Consider system (3.27) with the system matrices

$$A = \begin{pmatrix} -1.0 & 0.5 \\ 0.5 & -1.2 \end{pmatrix}, \quad A_d = \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.3 \end{pmatrix}.$$

It is easy to verify that the matrix

$$\mathcal{M} = A^D + |A_d| = \begin{pmatrix} -0.8 & 0.6 \\ 0.5 & -0.9 \end{pmatrix}$$

is Metzler and Hurwitz. By Proposition 3.1, system (3.27) is GFES for bounded delay $\tau(t) \in [0, \overline{\tau}]$. We now apply Proposition 3.1 to determine the decay rate λ. By direct computation, it is found that the condition

$$\begin{cases} -\xi_1 + 0.5\xi_2 + (0.2\xi_1 + 0.1\xi_2)\, E_\alpha(\lambda, \overline{\tau}) \leq 0 \\ 0.5\xi_1 - 1.2\xi_2 + 0.3\xi_2 E_\alpha(\lambda, \overline{\tau}) \leq 0 \end{cases}$$

is feasible for a vector $\xi = (\xi_1, \xi_2)^\top \succ 0$ if and only if $E_\alpha(\lambda, \overline{\tau}) \leq 2.0287$. By this, the critical decay rate $\bar{\lambda}$ can be obtained as

$$\bar{\lambda} = \frac{\alpha}{\overline{\tau}^\alpha} \ln(2.0287). \tag{4.2}$$

Figure 4 demonstrates the correspondence of delay rate $\bar{\lambda}$ to the distribution of order $\alpha \in (0, 1)$ and upper bound of delay $\overline{\tau}$. It can be seen from Fig. 4 that the decay rate decreases when the delay goes larger. In addition, for a fixed range of delay, there exists an order α such that the decay rate $\bar{\lambda}$ attains its maximal value.

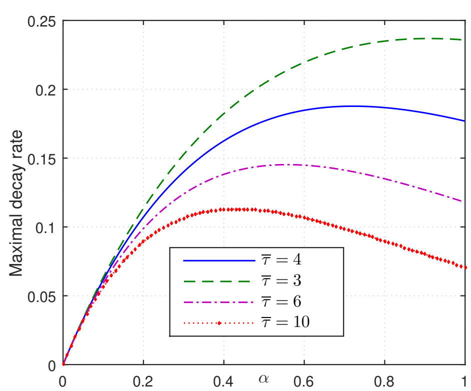

Figure 4. Decay rate $\bar{\lambda}$

5. Conclusion

In this paper, the positivity of solutions and fractional exponential stability of positive steady states have been studied for a class of nonlinear time-delay differential equations in a model of conformable fractional BAM neural networks. Based on some newly derived comparison techniques via fractional differential and integral inequalities, tractable conditions in terms of linear programming problems have been formulated to ensure that, for each nonnegative input vector, the system possesses a unique positive equilibrium point, which is fractional exponentially stable for any bounded delays. As an application, fractional exponential estimate for linear time-delay systems has also been discussed. The efficacy of the obtained results has been demonstrated by numerical numerical examples with simulations.

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