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On the Uniqueness of Limit Cycles in Codimension Two Bifurcations

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Abstract In this paper we present a survey on the uniqueness of limit cycles bifurcating from a center, homoclinic loop or a heterclinic loop in planar systems, introducing five bifurcation theorems, and then apply the theorems to the study of codimension two bifurcations, obtaining a complete analysis on the uniqueness of limit cycles.

Keywords Uniqueness, limit cycle, bifurcation, codimension two

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1. Introduction

Consider a 2-dimensional system of the form

$$\begin{aligned}\dot{x} &= f(x, y, \varepsilon),\\ \dot{y} &= g(x, y, \varepsilon),\end{aligned} \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2,\ \varepsilon \in \mathbb{R}^2,\ f, g \in C^\infty$. Suppose

$$\frac{\partial(f, g)}{\partial(x, y)}\Big|_{x=y=\varepsilon=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.2}$$

Then from [5] we know that under certain conditions (1.1) is equivalent to

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \mu_1 + \mu_2 y + x^2 + xyP(x, \mu) + y^2 Q(x, y, \mu)\end{aligned} \tag{1.3}$$

for $|x| + |y| + |\varepsilon|$ small, where $\mu = (\mu_1, \mu_2)$, $P, Q \in C^\infty$ with $P(0, 0) = 1$.

If (1.1) is centrally symmetric with respect to the origin, then under (1.2) and certain conditions (1.1) is equivalent to

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \mu_1 x + \mu_2 y \pm x^3 + x^2 yP(x, \mu) + y^2 Q(x, y, \mu)\end{aligned} \tag{1.4}$$

for $|x| + |y| + |\varepsilon|$ small, where $\mu = (\mu_1, \mu_2)$, $P, Q \in C^\infty$ with

$$P(-x, \mu) = P(x, \mu) = -1 + O(|\mu|),\ Q(-x, -y, \mu) = -Q(x, y, \mu) = O(|\mu|).$$

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For a C^∞ 3-dimensional system of the form

$$\dot{x} = f(x, \varepsilon), \ x \in \mathbb{R}^3, \tag{1.5}$$

where $\varepsilon \in \mathbb{R}^2$, and

$$\frac{\partial f}{\partial x}(0,0) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the truncated normal form system for (1.5) leads to a 2-dimensional system of the form

$$\begin{aligned} \dot{x} &= \varepsilon_1 x + axy + d_1 x^3 + d_2 xy^2, \\ \dot{y} &= \varepsilon_2 + bx^2 + cy^2 + d_3 x^2 y + d_4 y^3, \end{aligned} \tag{1.6}$$

where $(x, y) \in \mathbb{R}^2$ with $x \geq 0$ and $x + |y|$ small. See [5, 23].

Similarly, under some nonresonant conditions one can get a 2-dimensional system of the form

$$\begin{aligned} \dot{x} &= x(\varepsilon_1 + p_1 x^2 + p_2 y^2 + q_1 x^4 + q_2 x^2 y^2 + q_3 y^4), \\ \dot{y} &= y(\varepsilon_2 + p_3 x^2 + p_4 y^2 + q_4 x^4 + q_5 x^2 y^2 + q_6 y^4) \end{aligned} \tag{1.7}$$

with $x \geq 0, y \geq 0$ small from a C^∞ 4-dimensional system

$$\dot{x} = f(x, \varepsilon_1, \varepsilon_2), \ x \in \mathbb{R}^4,$$

where

$$\frac{\partial f}{\partial x}(0,0,0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}, \ \omega \neq 0.$$

System (1.3) was studied by Bogdanov [1, 2] and Takens [30]. System (1.4) was studied by Khorozov [25] and Carr [3]. Systems (1.6) and (1.7) were studied by Zoladek [34,35] and others. One can also find detailed studies and related references for systems (1.3), (1.4), (1.6) and (1.7) in the books [5, 13, 28].

In this paper we present a survey on the uniqueness of limit cycles bifurcating from a center, homoclinic loop or a heterclinic loop in planar systems by giving five bifurcation theorems, and then apply them to the study of systems (1.3), (1.4), (1.6) and (1.7) which are absent in many references.

In section 2 we list some general results on the number of limit cycles in Hopf, homoclinic and hetercinic bifurcations. In section 3 we provide an introduction to the study of limit cycles of (1.3), (1.4), (1.6) and (1.7) and an application of the main results in section 2 to the systems.

2. Preliminary theorems

In this section we list five theorems obtained by the author of this paper and his coauthors in different papers. However, all of these theorems have been rewritten using different notations from the original ones for easy understanding and applications.

Consider a near-Hamiltonian system of the form

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x,y,\varepsilon,\delta) = P(x,y,\varepsilon,\delta), \\ \dot{y} &= -H_x + \varepsilon g(x,y,\varepsilon,\delta) = Q(x,y,\varepsilon,\delta), \end{aligned} \tag{2.1}$$

where H, f, g are C^∞ functions, $\delta \in \mathbb{R}^m, m \geq 1$, and ε a small parameter. Let the equation $H(x,y) = h$ define a closed orbit L_h of $(2.1)_{\varepsilon=0}$ for $h \in J$, where $J = (\alpha, \beta)$. Then (2.1) has the following first order Melnikov function

$$M(h,\delta) = \oint_{L_h} g dx - f dy\Big|_{\varepsilon=0}, \ h \in J. \tag{2.2}$$

For Hopf bifurcation, we suppose that the limit of L_h as $h \to \alpha$ is an elementary center, denoted by $C = (x_c, y_c)$. Without loss of generality, we assume that

$$\frac{\partial(P,Q)}{\partial(x,y)}\Big|_C = \begin{pmatrix} a(\varepsilon,\delta) & -b(\varepsilon,\delta) \\ b(\varepsilon,\delta) & a(\varepsilon,\delta) \end{pmatrix}. \tag{2.3}$$

Then introducing the polar coordinates

$$(x,y) = (x_c, y_c) + r(\cos\theta, \sin\theta)$$

we can obtain from (2.1) a C^∞ $2\pi-$periodic equation below

$$\frac{dr}{d\theta} = R(\theta, r, \varepsilon, \delta).$$

Let $r(\theta, r_0, \varepsilon, \delta)$ denote the solution of this equation satisfying $r(0) = r_0$, and let

$$\Delta(r_0, \varepsilon, \delta) = r(2\pi, r_0, \varepsilon, \delta) - r_0.$$

Then for $|r_0|$ small we have

$$\Delta(r_0, \varepsilon, \delta) = \varepsilon \sum_{i\geq 1} \Delta_i(\varepsilon,\delta) r_0^i, \ \Delta_1(0,\delta) = 2\pi \frac{\partial a}{\partial \varepsilon}(0,\delta)/b(0,\delta).$$

The following theorem was proved in [10, 23].

Theorem 2.1. *(Hopf Bifurcation) Let* (2.3) *hold. Then*

(a) If

$$\Delta_{2k+1}(0,\delta_0) \neq 0, \ \Delta_i(0,\delta_0) = 0, \ i = 1, \cdots, 2k$$

for some $\delta_0 \in \mathbb{R}^m$ and $k \geq 1$, then there exist $\varepsilon_0 > 0$ and a neighborhood U of the point C such that (2.1) *has at most k limit cycles in U for $|\varepsilon| + |\delta - \delta_0| \leq \varepsilon_0$.*

(b) If (2.1) *is analytic, $H(C) = \alpha = 0$, and $M(h,\delta_0) \not\equiv 0$ for some $\delta_0 \in \mathbb{R}^m$, then for some $k \geq 1$ and $N_k \neq 0$*

$$M(h,\delta_0) = N_k h^{k+1} + O(h^{k+3/2}).$$

In this case, $\Delta_i(0,\delta_0) = 0$, $i = 1, \cdots, 2k$, and $\Delta_{2k+1}(0,\delta_0) = K_k N_k$ for some constant $K_k \neq 0$.

(c) If

$$\frac{\partial a}{\partial \varepsilon}(0,\delta_0)=0,\ \Delta_3(0,\delta_0)\neq 0$$

for some $\delta_0 \in \mathbb{R}^m$*, then there exist* $\varepsilon_0 > 0$ *and a neighborhood* U *of the point* C *such that* (2.1) *has a unique limit cycle in* U *if and only if* $\mu a(\varepsilon,\delta) < 0$ *for* $|\varepsilon| + |\delta-\delta_0| \le \varepsilon_0$*, where* $\mu = \mathrm{sgn}(\mathrm{b}(0,\delta_0)\Delta_3(0,\delta_0))$.

In (ε,δ)-plane, the set defined by $a(\varepsilon,\delta)=0$ for $|\varepsilon|>0$ is called a Hopf bifurcation curve. In 2000, Han [12] proved that M in (2.2) is C^∞ at $h=0$, and has the expansion below

$$M(h,\delta)=b_1(\delta)h+b_2(\delta)h^2+\cdots.$$

The conditions $\frac{\partial a}{\partial \varepsilon}(0,\delta_0)=0,\ \Delta_3(0,\delta_0)\neq 0$ in Theorem 2.1 are equivalent to $b_1(\delta_0)=0,\ b_2(\delta_0)\neq 0$. A bifurcation result similar to conclusion (c) of Theorem 2.1 can be found in Theorem 2.5 in Chapter 3 of [5].

One can find computing formulas of $b_1(\delta),\ b_2(\delta)$ and $b_3(\delta)$ in [14] and general computing methods for more coefficients in [21]. Some new advances on Hopf bifurcations were obtained in [26].

Now we assume that $(2.1)_{\varepsilon=0}$ has a homoclinic loop L defined by the equation $H(x,y)=\beta$ with a hyperbolic saddle S satisfying $H(S)=\beta$. Without loss of generality, we assume that

$$\frac{\partial(P,Q)}{\partial(x,y)}\Big|_S=\begin{pmatrix}\lambda_1(\varepsilon,\delta) & 0\\ 0 & -\lambda_2(\varepsilon,\delta)\end{pmatrix}. \tag{2.4}$$

Introduce

$$c_0(\delta)=M(\beta,\delta),\ d_1(\delta)=\frac{\partial}{\partial\varepsilon}(\lambda_1-\lambda_2)\Big|_{\varepsilon=0}. \tag{2.5}$$

It is direct that $d_1(\delta)=(f_x+g_y)\big|_{(x,y)=S,\varepsilon=0}$. Luo et al [27] obtained the following result.

Theorem 2.2. *(Homolinic Bifurcation) Let* (2.4) *and* (2.5) *hold with* $\delta\in\mathbb{R}$*. Then*

(1) If $c_0(\delta_0)\neq 0$*, then* (2.1) *has no limit cycles near* L *for* $|\varepsilon|+|\delta-\delta_0|$ *small.*

(2) If $c_0(\delta_0)=0,\ d_1(\delta_0)\neq 0$*, then* (2.1) *has at most one limit cycle near* L *for* $|\varepsilon|+|\delta-\delta_0|$ *small. If* $c_0'(\delta_0)\neq 0$ *in addition, then there exists a unique smooth function* $\delta^*(\varepsilon)=\delta_0+O(\varepsilon)$ *such that for* $|\varepsilon|+|\delta-\delta_0|$ *small* (2.1) *has a homoclinic loop (a limit cycle, resp.) near* L *if and only if* $\delta=\delta^*(\varepsilon)$($\mu(\delta-\delta^*(\varepsilon))>0$*,resp.), where*

$$\mu=\mathrm{sgn}(c_0'(\delta_0)d_1(\delta_0)).$$

Moreover, the limit cycle or the homoclinic loop is stable (unstable) if $\varepsilon d_1(\delta_0)<0(>0)$ *when it exists.*

The curve $\delta=\delta^*(\varepsilon)$ on (ε,δ) plane is called a homoclinic bifurcation curve of (2.1).

Roussarie [29] proved that

$$M(h,\delta)=\sum_{k\ge 0}[c_{2k}(\delta)(h-\beta)^k+c_{2k+1}(\delta)(h-\beta)^{k+1}\ln|h-\beta|],\ 0<\beta-h\ll 1$$

for analytic system (2.1) and that there exist at most k limit cycles near L for $|\varepsilon|+|\delta-\delta_0|$ small if

$$c_k(\delta_0)\neq 0,\ c_j(\delta_0)=0,\ j=1,\cdots,k-1.$$

Next we consider limit cycle bifurcation near a double homoclinic loop. Suppose that $(2.1)_{\varepsilon=0}$ has a double homoclinic loop $L=L_1\bigcup L_2$ passing through a hyperbolic saddle point at the origin. We can suppose that L is of type figure eight and that $(2.1)_{\varepsilon=0}$ has a family of closed orbits L_h defined by $H(x,y)=h$ for $0<h\ll 1$ satisfying that $L_h\to L$ as $h\to 0$. This kind of L_h is called a large closed orbit.

Introduce

$$C_0(\delta)=M_1(\delta)+M_2(\delta),\ D_1(\delta)=(f_x+g_y)|_{x=y=\varepsilon=0}, \tag{2.6}$$

where

$$M_j(\delta)=\oint_{L_j} gdx-fdy\Big|_{\varepsilon=0},\ j=1,2.$$

Han et al [18] proved the following theorem.

Theorem 2.3. *Let* (2.6) *hold. Then*

(1) If $C_0(\delta_0)\neq 0$ for some $\delta_0\in\mathbb{R}^m$, then (2.1) *has no large limit cycles near L for $|\varepsilon|+|\delta-\delta_0|$ small.*

(2) If $C_0(\delta_0)=0$, $D_1(\delta_0)\neq 0$, then (2.1) *has at most one large limit cycle near L for $|\varepsilon|+|\delta-\delta_0|$ small. If $C_0'(\delta_0)\neq 0$ and $\delta\in\mathbb{R}$ in addition, then there exists a unique function $\delta^*(\varepsilon)=\delta_0+O(\varepsilon)$ such that for $|\varepsilon|+|\delta-\delta_0|$ small* (2.1) *has a double homoclinic loop or a large homoclinic loop (a large limit cycle, resp.) near L if and only if $\delta=\delta^*(\varepsilon)$($\mu(\delta-\delta^*(\varepsilon))<0$,resp.), where*

$$\mu=\operatorname{sgn}(C_0'(\delta_0)M_h(h,\delta_0)).$$

In particular, if

$$P(-x,-y,\varepsilon,\delta)=-P(x,y,\varepsilon,\delta),\ Q(-x,-y,\varepsilon,\delta)=-Q(x,y,\varepsilon,\delta), \tag{2.7}$$

then a double homoclinic loop exists when $\delta=\delta^*(\varepsilon)$, and two symmetric small limit cycles exist when $\mu(\delta-\delta^*(\varepsilon))>0$.

Han [8] proved that

$$M_h'(h,\delta)=\oint_{L_h}(f_x+g_y)\Big|_{\varepsilon=0}dt.$$

Hence

$$D_1(\delta_0)M_h'(h,\delta_0)>0,$$

which implies

$$\mu=\operatorname{sgn}(C_0'(\delta_0)D_1(\delta_0)).$$

Jiang and Han [24] proved that

$$M(h,\delta)=\sum_{k\geq 0}[C_{2k}(\delta)h^k+C_{2k+1}(\delta)h^{k+1}\ln h],\ 0<h\ll 1.$$

The authors [15, 20] obtained further results on the number of limit cycles bifurcating from a double homoclinic loop.

Then we consider limit cycle bifurcation near a heteroclinic loop. Suppose that $(2.1)_{\varepsilon=0}$ has a heteroclinic loop $\Gamma = \Gamma_1 \bigcup \Gamma_2$ passing through two hyperbolic saddle points S_1 and S_2, where $\Gamma_1 = \widehat{S_1S_2}$, $\Gamma_2 = \widehat{S_2S_1}$ are two heteroclinic orbits with endpoints S_1 and S_2. Suppose $H(S_1) = H(S_2) = 0$. Then near one side of Γ $(2.1)_{\varepsilon=0}$ has a family of closed orbits L_h defined by $H(x,y) = h$ for h near one side of $h = 0$. Let

$$\begin{aligned} &\bar{C}_0(\delta) = \bar{M}_1(\delta) + \bar{M}_2(\delta), \\ &\Delta_j(\varepsilon,\delta) = (f_x + g_y)_{(x,y)=\bar{S}_j}, \ j = 1,2, \end{aligned} \tag{2.8}$$

where $\bar{S}_j$ is the saddle point of (2.1) near S_j for $|\varepsilon|$ small, and

$$\bar{M}_j(\delta) = \int_{\Gamma_j} gdx - fdy\Big|_{\varepsilon=0}, \ j = 1,2.$$

Han et al [19] proved the following theorem.

Theorem 2.4. *Let* (2.8) *hold. Then*

(1) If $\bar{C}_0(\delta_0) \neq 0$ for some $\delta_0 \in \mathbb{R}^m$, then (2.1) *has no limit cycles near Γ for $|\varepsilon| + |\delta - \delta_0|$ small.*

(2) If $\bar{C}_0(\delta_0) = 0$, $\Delta_1(\varepsilon,\delta)\Delta_2(\varepsilon,\delta) \geq 0$ and $\Delta_1(0,\delta_0) + \Delta_2(0,\delta_0) \neq 0$, then (2.1) *has at most one limit cycle near Γ for $|\varepsilon| + |\delta - \delta_0|$ small. If $\bar{C}_0'(\delta_0) \neq 0$ and $\delta \in \mathbb{R}$ in addition, then there exists a unique function $\delta^*(\varepsilon) = \delta_0 + O(\varepsilon)$ such that for $|\varepsilon| + |\delta - \delta_0|$ small* (2.1) *has a homoclinic loop or a heteroclinic loop (a limit cycle, resp.) near Γ if and only if $\delta = \delta^*(\varepsilon)$($\mu(\delta - \delta^*(\varepsilon)) > 0$,resp.), where*

$$\mu = \operatorname{sgn}(\bar{C}_0'(\delta_0)M_h(h,\delta_0)) = \operatorname{sgn}(\bar{C}_0'(\delta_0)(\Delta_1(0,\delta_0) + \Delta_2(0,\delta_0)).$$

In particular, if (2.7) *holds, then* (2.1) *has a heteroclinic loop when $\delta = \delta^*(\varepsilon)$.*

We remark that the above theorem can be generalized to the case of heteroclinic loops passing through three or more saddle points.

For further study on heteroclinic bifurcation one can consult [22].

Finally, we consider a near-integrable system of the form

$$\begin{aligned} &\dot{x} = f_0(x,y) + \varepsilon f(x,y,\varepsilon,\delta), \\ &\dot{y} = g_0(x,y) + \varepsilon g(x,y,\varepsilon,\delta), \end{aligned} \tag{2.9}$$

where f_0, g_0, f and g are all C^∞ functions on a suitable region. Suppose that $(2.9)_{\varepsilon=0}$ has a C^1 first integral $I(x,y)$ whose level curve $I(x,y) = 0$ defines a heteroclinic loop $\Gamma = \bigcup_{j=1}^n \Gamma_j$ passing through n hyperbolic saddle points $S_1, \cdots, S_n$. Denote by $\bar{S}_j(\varepsilon,\delta)$ the saddle point of (2.9) near S_j for $|\varepsilon|$ small. Let $\lambda_{j1}(\varepsilon,\delta) < 0, \lambda_{j2}(\varepsilon,\delta) > 0$ be the eigenvalues of the linearized matrix of (2.9) at $\bar{S}_j$. Denote

$$\begin{aligned} &M(\delta) = \oint_\Gamma (I_x f + I_y g)\Big|_{\varepsilon=0} dt, \\ &r(\varepsilon,\delta) = r_1(\varepsilon,\delta) \cdots r_n(\varepsilon,\delta), \end{aligned}$$

where

$$r_j(\varepsilon,\delta) = -\frac{\lambda_{j1}(\varepsilon,\delta)}{\lambda_{j2}(\varepsilon,\delta)}, \ j = 1, \cdots, n.$$

Then the following theorem was obtained in [16].

Theorem 2.5. *Let $M(\delta) \in \mathbb{R}$. Then*

(1) If $M(\delta_0) \neq 0$ for some $\delta_0 \in \mathbb{R}^m$, then (2.9) has no limit cycles near Γ for $|\varepsilon| + |\delta - \delta_0|$ small.

(2) If (i) $M(\delta_0) = 0$, $\frac{\partial r}{\partial \varepsilon}(0, \delta_0) \neq 0$, and (ii) (2.9) has always $n-1$ heteroclinic orbits passing through the n saddle points $\bar{S}_j$, $j = 1, \cdots, n$ near Γ for $|\varepsilon| + |\delta - \delta_0|$ small, then (2.9) has at most one limit cycle near Γ for $|\varepsilon| + |\delta - \delta_0|$ small.

3. Limit cycles in codimension two bifurcations

First consider (1.3). From [33] when studying the existence of limit cycles of (1.3), rescaling variables and making some coordinate changes lead to a system of the form

$$\dot{x} = y, \ \dot{y} = x(1-x) + \varepsilon(\delta + x)y + O(\varepsilon^2), \tag{3.1}$$

where $(x, y) \in \mathbb{R}^2$, and $\varepsilon > 0$ is a small parameter.

Instead of (3.1), [5] and [28] deduced

$$\dot{x} = y, \ \dot{y} = -1 + x^2 + \varepsilon(\delta + x)y + O(\varepsilon^2),$$

and

$$\dot{x} = y, \ \dot{y} = -x(1-x) - \varepsilon(\delta + x)y + O(\varepsilon^2)$$

respectively. Clearly, these three systems are equivalent. We use (3.1) here.

For $\varepsilon = 0$ (3.1) has a family of closed orbits L_h defined by

$$\frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} = h, \ h \in (-\frac{1}{6}, 0).$$

The first order Melnikov function of it has the form

$$M(h, \delta) = I_0(h)\delta + I_1(h) = I_0(h)[\delta - P(h)], \ h \in (-\frac{1}{6}, 0),$$

where

$$P(h) = -\frac{I_1(h)}{I_0(h)}, \ I_j(h) = \oint_{L_h} x^j y dx, \ j = 0, 1.$$

By [33] or [20] we have the following theorem on the property of the function P.

Theorem 3.1. *The function P is C^∞ on $(-\frac{1}{6}, 0)$, and satisfies the following:*

(1) $P'(h) > 0$, $h \in (-\frac{1}{6}, 0)$;
(2) $P(-\frac{1}{6}+) = -1$, $P'(-\frac{1}{6}+) = \frac{1}{2}$;
(3) $P(0-) = -\frac{6}{7}$, $P'(0-) = +\infty$.

The conclusion (1) above in Theorem 3.1 is obtained by using Bogdanov's technique in [1] and due to Bogdanov. This conclusion can be used to obtain the uniqueness of limit cycles in Poincaré bifurcation, bifurcating from the family $\{L_h\}$. However, it can not be used to study the uniqueness of limit cycles in Hopf or homoclinic bifurcation. For more details, see [17].

Note that

$$I_0(0-) = \frac{6}{5}, \ I_0(h) = 2\pi(h + \frac{1}{6}) + O(|h + \frac{1}{6}|^2).$$

By conclusions (2) and (3) in Theorem 3.1 we can apply Theorems 2.1 and 2.2 respectively to obtain the uniqueness of limit cycles in Hopf and homoclinic bifurcations. Then based on these facts, we can further prove by contradiction that there exist two functions

$$\delta_1(\varepsilon) = -1 + O(\varepsilon),\ \delta_2(\varepsilon) = -\frac{6}{7} + O(\varepsilon),$$

such that for $\varepsilon > 0$ small (3.1) has a unique limit cycle if and only if $\delta_1(\varepsilon) < \delta < \delta_2(\varepsilon)$, and has a homolinic loop if and only if $\delta = \delta_2(\varepsilon)$. See [17, 33] for a very complete proof. The problem of the uniqueness of limit cycles in homoclinic bifurcation for (3.1) was first solved in [27] in 1992, and was not treated truly in many references.

Now, consider (1.4). By suitable rescaling one can get the following three systems from (1.4)

$$\dot{x} = y,\ \dot{y} = -x(1+x^2) + \varepsilon(\delta - x^2)y + \varepsilon^2 Q(x, y, \varepsilon), \tag{3.2}$$

$$\dot{x} = y,\ \dot{y} = x(1-x^2) + \varepsilon(\delta - x^2)y + \varepsilon^2 Q(x, y, \varepsilon), \tag{3.3}$$

and

$$\dot{x} = y,\ \dot{y} = -x(1-x^2) + \varepsilon(\delta - x^2)y + \varepsilon^2 Q(x, y, \varepsilon), \tag{3.4}$$

where $\varepsilon > 0$ is a small parameter, and Q is a smooth function and odd in (x, y).

For (3.2), we have

$$H(x, y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{x^4}{4},$$

which defines a family of closed curves L_h for $h > 0$. Then, the first order Melnikov function of (3.2) has the form

$$\begin{aligned} M(h, \delta) &= I_0(h)\delta - I_2(h) = I_0(h)[\delta - P(h)], \\ P(h) &= \tfrac{I_2(h)}{I_0(h)},\ I_j(h) = \oint_{L_h} x^j y dx,\ j = 0, 2 \end{aligned} \tag{3.5}$$

for $h > 0$. One can prove that $I_0(h) = 2\pi h + O(h^2)$ for $h > 0$ small, $P(0+) = 0$, $P'(0+) = \frac{1}{2}$, and $P'(h) > 0$ for $h > 0$. Thus by Theorem 2.1 we know that for any given $N > 0$ there exist a constant $\varepsilon_0 = \varepsilon_0(N) > 0$ and a smooth function $\delta^*(\varepsilon) = O(\varepsilon)$ such that for $0 < \varepsilon < \varepsilon_0$, $|\delta| < N$ (3.2) has a (unique) limit cycle if and only if $\delta > \delta^*(\varepsilon)$.

For (3.3), we have

$$H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4},$$

which defines a family of large closed curves L_h for $h > 0$ and two families of small closed curves for $h \in (-\frac{1}{4}, 0)$. Then, the first order Melnikov function M of (3.3) has the form (3.5) for $h > 0$. For $h \in (-\frac{1}{4}, 0)$, we can similarly define the functions M and P in (3.5). For the property of P we have from [3, 5]

Theorem 3.2. *For* (3.5) *the function P has the following properties:*

(1) $P'(-\frac{1}{4}+) = -\frac{1}{2}$, $P'(0-) = -\infty$ *and* $P'(h) < 0$ *for* $h \in (-\frac{1}{4}, 0)$*;*

(2) $P'(0+) = -\infty$, *and there exists* $h^* > 0$ *such that* $P'(h^*) = 0$, $P''(h^*) > 0$, *and* $(h - h^*)P'(h) > 0$ *for* $h > 0$, $h \neq h^*$*;*

(3) $P(-\frac{1}{4}+) = 1$, $P(0\pm) = \frac{4}{5}$, $P(+\infty) = +\infty$.

Note that

$$I_0(-\frac{1}{4}+) = 0,\ I_0'(-\frac{1}{4}+) > 0,\ I_0(0\pm) > 0.$$

Then based on Theorem 3.2 and applying Theorems 2.1, 2.2 and 2.3, one can obtain a complete bifurcation diagram as given in [3,5]. The bifurcation diagram near the double homoclinic loop L_0 was first strictly given in [18] in 1992.

For (3.4), we have

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4},$$

which defines a family of closed curves L_h for $h \in (0, \frac{1}{4})$. Then, the first order Melnikov function M of (3.4) has the form (3.5) for $h \in (0, \frac{1}{4})$. In this case, one can prove that

$$P(0+) = 0,\ P'(0+) = \frac{1}{2},\ I_0(h) = 2\pi h + O(h^2) \text{ for } 0 < h \ll 1,$$

and

$$P(\frac{1}{4}-) = \frac{1}{5},\ P'(\frac{1}{4}-) = +\infty,\ P'(h) > 0 \text{ for } h \in (0, \frac{1}{4}).$$

Then by Theorems 2.1 and 2.4 we know that there exist two smooth functions $\delta_1(\varepsilon) = O(\varepsilon)$ and $\delta_2(\varepsilon) = \frac{1}{5} + O(\varepsilon)$ such that for $\varepsilon > 0$ small (3.4) has a (unique) limit cycle if and only if $\delta_1(\varepsilon) < \delta < \delta_2(\varepsilon)$, and has a heteroclinic loop if and only if $\delta = \delta_2(\varepsilon)$.

Next, we consider (1.6). Under $abc \neq 0$ and by suitable scalings of variables (1.6) becomes

$$\begin{aligned} \dot{x} &= \varepsilon_1 x + Bxy + d_1 x^3 + d_2 xy^2, \\ \dot{y} &= \varepsilon_2 + \eta x^2 - y^2 + d_3 x^2 y + d_4 y^3, \end{aligned} \tag{3.6}$$

where $B \neq 0$, $\eta = \pm 1$ and $\eta B < 0$. The uniqueness of limit cycles of (3.6) was investigated in [4,6,7,31,32,34]. The main idea includes three steps as follows.

(1) Rescaling the variables in (3.6) leads to

$$\begin{aligned} \dot{x} &= Bxy + \varepsilon(\delta x + d_1 x^3 + d_2 xy^2), \\ \dot{y} &= -\eta(1 - x^2) - y^2 + \varepsilon(d_3 x^2 y + d_4 y^3), \end{aligned} \tag{3.7}$$

where $\varepsilon > 0$, $\delta \in \mathbb{R}$ with ε small. For $\varepsilon = 0$, (3.7) has an integrating factor x^q with $q + 1 = 2/B$ and is equivalent to the near-Hamiltonian system

$$\begin{aligned} \dot{x} &= x^q[Bxy + \varepsilon(\delta x + d_1 x^3 + d_2 xy^2)], \\ \dot{y} &= x^q[-\eta(1 - x^2) - y^2 + \varepsilon(d_3 x^2 y + d_4 y^3)] \end{aligned} \tag{3.8}$$

on $x > 0$. For $\varepsilon = 0$, (3.8) has a Hamiltonian function H which defines a family of closed orbits L_h for $h \in J$, where J is an open interval depending on B.

(2) It can be proved that the first order Melnikov function of (3.8) has the form

$$M(h, \delta) = (K_1 - (q+1)a)I_1(h) + K_3 I_3(h),$$

where

$$I_j(h) = \oint_{L_h} x^q y^j dx,\ j = 1, 3,$$

and

$$K_1 = -(q+3)d_1 - d_3,\ K_3 = \eta K_1 - (q+1)d_2 - 3d_4.$$

(3) (main step) Prove that the function $P(h) = I_3(h)/I_1(h)$ satisfies $P'(h) \neq 0$ for $h \in J$ if $K_3 \neq 0$.

Notice that for the case $B > 0$ with $\eta = -1$ $(3.8)_{\varepsilon=0}$ has a heteroclinic loop, denoted by L_0, as the outer boundary of the family $\{L_h\}$. Then clearly the discussion on Hopf and heteroclinic bifurcations was absent. For $\eta = -1$, letting $x + u^{B/2}$ (3.7) becomes a near-Hamiltonian system of the form

$$\begin{aligned}\dot{u} &= 2uy + 2\varepsilon u(\delta x + d_1 u^B + d_2 y^2)/B,\\ \dot{y} &= 1 - u^B - y^2 + \varepsilon y(d_3 u^B + d_4 y^2).\end{aligned}$$

The author [9] proved the following theorem using the above form.

Theorem 3.3. *Suppose $B > 0$ and $\eta = -1$ in (3.7). Let $K_3 \neq 0$. Then there exist a neighborhood U of L_0 and a C^1 function $\delta^*(\varepsilon) = \delta_0 + O(\varepsilon)$ for some $\delta_0 \in \mathbb{R}$ such that for $\varepsilon > 0$ and $|\delta - \delta_0|$ both small (3.7) has a unique limit cycle (a heteroclinic loop, resp.) in U if and only if $K_3(\delta - \delta^*(\varepsilon)) < 0$ ($\delta = \delta^*(\varepsilon)$, resp.). Moreover, the limit cycle, when exists, is stable (unstable) as $K_3 > 0$ ($K_3 < 0$).*

The above theorem can also be proved by using Theorem 2.5.

The uniqueness of limit cycles of (3.7) in Hopf bifurcation can be obtained by applying Theorem 2.1. Hence, we obtain that (3.6) has at most one limit cycle near the orgin for all $|\varepsilon_1|$ and $\varepsilon_2|$ small if $K_3 \neq 0$.

Finally, we consider (1.7). From [6,23,28,35], we know that making a suitable coordinate change to (1.7) and then rescaling the variables yield

$$\begin{aligned}\dot{x} &= x(\mu + x + by + \varepsilon(ex^2 + fxy + gy^2)),\\ \dot{y} &= y(-B\mu + cx - y + \varepsilon(\delta + hx^2 + jxy + ky^2)),\end{aligned} \tag{3.9}$$

where $\varepsilon > 0$ is small, $\delta \in \mathbb{R}$, $\mu = \pm 1$, $B = (1-c)/(1+b)$. If

$$\mu = -1,\ A \equiv -1 - bc > 0,\ 1 + b > 0,\ 1 - c > 0, \tag{3.10}$$

then $(3.7)_{\varepsilon=0}$ has a family of closed orbits $\{L_h\}$ having a 3-polycycle L_0 as its outer boundary. From [23] one can see that under (3.10) system (3.9) has a unique Hopf bifurcation curve $\delta = \delta_0^*(\varepsilon) = \delta_0 + O(\varepsilon)$ and a unique heteroclinic bifurcation curve $\delta = \delta_1^*(\varepsilon) = \delta_1 + O(\varepsilon)$, where $\delta_0, \delta_0 \in \mathbb{R}$ whose formulas were given in [11,23]. Let

$$\Delta = \delta_1 - \delta_0.$$

Zoladek [35] proved that the first order Melnikov function has at most one zero if $\Delta \neq 0$. Then using Theorems 2.1 and 2.5 the author [11] proved that there exists at most one limit cycle in Hopf and heteroclinic bifurcations respectively when $\Delta \neq 0$, and hence obtained the following conclusion.

Theorem 3.4. *Let (3.10) hold and $\Delta \neq 0$. Then for $\varepsilon > 0$ small (3.9) has a unique limit cycle (a heteroclinic loop, resp.) if and only if $\Delta\delta_0^*(\varepsilon)) < \Delta\delta < \Delta\delta_1^*(\varepsilon)$ ($\delta = \delta_1^*(\varepsilon)$, resp.).*

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