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Generalized Ito Formula and Some Stochastic Inclusions*

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Abstract This work is on a general integration by parts formula generalizing the Ito formula. An application is given to stochastic inclusions which have a different form than usual.

Keywords Ito formula, stochastic evolution inclusions

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1. Introduction

The Ito formula [3,4,7,13] is really about integration by parts in the setting where there is a stochastic integral. It is of fundamental importance in SPDE and related fields [3,4,7,9,17]. We establish an implicit Ito formula in this work and apply it to study a specific stochastic evolution inclusion.

1.1. The situation

Let $V \subseteq W, W' \subseteq V'$ be separable Banach spaces, such that V is dense in W and $B \in \mathcal{L}(W, W')$ satisfies

$$\langle Bw, w\rangle \geq 0,\ \langle Bu, v\rangle = \langle Bv, u\rangle . \tag{1.1}$$

Note that B does not need to be one to one. Also allowed is the case where B is the Riesz map. It could also happen that $V = W$. Let X have values in V and satisfy the following

$$BX(t) = BX_0 + \int_0^t Y(s)\,ds + B\int_0^t Z(s)\,dW(s), \tag{1.2}$$

$X_0 \in L^2(\Omega; W)$ and is $\mathcal{F}_0$ measurable, where Z is $\mathcal{L}_2\left(Q^{1/2}U, W\right)$ progressively measurable and

$$\|Z\|_{L^2\left([0,T]\times\Omega,\mathcal{L}_2\left(Q^{1/2}U,W\right)\right)} < \infty.$$

Here Q is a nonnegative self adjoint operator defined on U. See [21] for stochastic integrals.

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Assume that X, Y satisfy

$$BX, Y \in K' \triangleq L^{p'}([0,T] \times \Omega; V'),$$

the σ algebra of measurable sets defining K' will be the progressively measurable sets. Here $1/p' + 1/p = 1,\ p > 1$. In the following sense (1.2) holds: For a.e. ω, the equation holds in V' for all $t \in [0,T]$. Thus we are considering a particular representative X of K for which this happens. Also it is only assumed that $BX(t) = B(X(t))$ for a.e. t. Thus BX is the name of a function having values in V' for which $BX(t) = B(X(t))$ for a.e. t. Assume that X is progressively measurable and

$$X \in L^p([0,T] \times \Omega, V) \triangleq K.$$

Also $W(t)$ is a JJ^* Wiener process on U_1 in the above diagram. U_1 can be assumed to be U.

The **goal** is to prove the following Ito formula valid for a.e. t for each ω off a set of measure zero.

$$\langle BX(t), X(t)\rangle = \langle BX_0, X_0\rangle + \int_0^t \left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2}\right) ds + \int_0^t \left(Z \circ J^{-1}\right)^* BX \circ J dW. \tag{1.3}$$

The most significant feature of the last term is that it is a local martingale.

To understand the **goal**, the following fundamental deterministic result will be very helpful. It says essentially that if $(Bu)' \in L^{p'}(0,T;V')$ and $u \in L^p(0,T;V)$ then the map $u \to Bu(t)$ is continuous as a map from

$$X_1 \triangleq \left\{u \in L^p([0,T];V) : (Bu)' \in L^{p'}([0,T];V')\right\}$$

with

$$\|u\|_{X_1} \triangleq \|u\|_{L^p(0,T,V)} + \left\|(Bu)'\right\|_{L^{p'}(0,T;V')}$$

to W'.

Proposition 1.1. *Let $Y \in L^{p'}(0,T;V')$ and*

$$Bu(t) = Bu_0 + \int_0^t Y(s)\, ds \ \text{in } V',\ u_0 \in W, Bu(t) = B(u(t)) \ \text{for a.e. } t. \tag{1.4}$$

Thus $Y = (Bu)'$ as a weak derivative in the sense of V' valued distributions. It is known that $u \in L^p(0,T,V)$ for $p > 1$. Then $t \to Bu(t)$ is continuous into W' for t off a set of measure zero N and there exists a continuous function $t \to \langle Bu, u\rangle(t)$ such that for all $t \notin N, \langle Bu, u\rangle(t) = \langle B(u(t)), u(t)\rangle, Bu(t) = B(u(t))$, and for all t,

$$\frac{1}{2}\langle Bu, u\rangle(t) = \frac{1}{2}\langle Bu_0, u_0\rangle + \int_0^t \langle Y(s), u(s)\rangle\, ds.$$

Note that the formula (1.4) shows that $Bu_0 = Bu(0)$. Also it shows that $t \to \langle Bu, u\rangle(t)$ is continuous. To emphasize this a little more, Bu is the name of a function. $Bu(t) = B(u(t))$ for a.e. t and $t \to Bu(t)$ is continuous into V' on $[0,T]$ because of the integral equation.

The Ito formula to be developed in this paper is a probabilistic version of the above. Specifically, our main results about the Ito formula is as follows.

Theorem 1.1. *For ω off a set of measure zero, for every $t \notin N_\omega$, a set of measure zero dependent on ω,*

$$\langle BX(t), X(t)\rangle = \langle BX_0, X_0\rangle + \int_0^t \left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2}\right) ds$$

$$+2\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ J dW. \tag{1.5}$$

Also, there exists a unique continuous, progressively measurable function denoted as $\langle BX, X\rangle$ such that it equals $\langle BX(t), X(t)\rangle$ for a.e. t and $\langle BX, X\rangle(t)$ equals the right side of the above for all t. In addition to this,

$$E(\langle BX, X\rangle(t)) = E(\langle BX_0, X_0\rangle) + E\left(\int_0^t \left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2}\right) ds\right). \tag{1.6}$$

The quadratic variation of the stochastic integral in (1.5) *is dominated by*

$$C\int_0^t \|Z\|^2_{\mathcal{L}_2} \|BX\|^2_{W'} ds \tag{1.7}$$

for a suitable constant C. Also $t \to BX(t)$ is continuous with values in W' for $t \in N_\omega^C$.

It is a technical generalization of known results. It allows for the possibility that B is not one to one and this introduces many details which must be considered carefully. The case that B is the identity is discussed in [21]. It was originally due to Krylov [13].

We prove this theorem in Section 2 and provide in Section 3 an application to evolution inclusions in which there is a stochastic integral.

2. The Ito formula

In this section, we shall prove Theorem 1.1. We first give some background involving theorems from probability. Then we establish the Ito formula.

2.1. Background

Notation We shall list existing important theorems as propositions, and leave the word theorem only for our main results.

Concerning a general Ito formula involving Ito integrals in infinite dimensions, the level of generality might not be familiar and we begin with a brief summary of background theorems. The proofs can mostly be found in [21], [23], [11], [12]

A normal filtration, denoted as $\{\mathcal{F}_t\}$, is defined as follows.

1. Each $\mathcal{F}_t$ is a σ algebra and $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$;
2. $\mathcal{F}_s = \cap\{\mathcal{F}_t : t > s\}$;
3. Each $\mathcal{F}_t$ contains all sets of measure zero in $\mathcal{F}_T$.

It is assumed that all filtrations are normal in this paper.

The following lemma is fundamental to the presentation. A proof is found in [12].

Lemma 2.1. *Let* $\Phi : [0,T] \times \Omega \to V$, *be* $\mathcal{B}([0,T]) \times \mathcal{F}$ *measurable and suppose*

$$\Phi \in K \triangleq L^p([0,T] \times \Omega; E), \; p \geq 1.$$

Then there exists a sequence of nested partitions, $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$,

$$\mathcal{P}_k \triangleq \left\{t_0^k, \cdots, t_{m_k}^k\right\}$$

such that the step functions given by

$$\Phi_k^r(t) \triangleq \sum_{j=1}^{m_k} \Phi\left(t_j^k\right) \mathcal{X}_{(t_{j-1}^k, t_j^k]}(t),$$

$$\Phi_k^l(t) \triangleq \sum_{j=1}^{m_k} \Phi\left(t_{j-1}^k\right) \mathcal{X}_{[t_{j-1}^k, t_j^k)}(t)$$

both converge to Φ *in* K *as* $k \to \infty$ *and*

$$\lim_{k\to\infty} \max\left\{\left|t_j^k - t_{j+1}^k\right| : j \in \{0, \cdots, m_k\}\right\} = 0.$$

Also, each $\Phi\left(t_j^k\right), \Phi\left(t_{j-1}^k\right)$ *is in* $L^p(\Omega; E)$. *One can also assume that* $\Phi(0) = 0$. *The mesh points* $\left\{t_j^k\right\}_{j=0}^{m_k}$ *can be chosen to miss a given set of measure zero. In addition to this, we can assume that*

$$\left|t_j^k - t_{j-1}^k\right| = 2^{-n_k}$$

except for the case where $j = 1$ *or* $j = m_{n_k}$ *when this might not be so. In the case of the last subinterval defined by the partition, we can assume*

$$\left|t_m^k - t_{m-1}^k\right| = \left|T - t_{m-1}^k\right| \geq 2^{-(n_k+1)}.$$

The following lemma is convenient. The proof is a simple application of the Borel Cantelli theorem.

Lemma 2.2. *Let* $f_n \to f$ *in* $L^p([0,T] \times \Omega, E)$. *Then there exists a subsequence* n_k *and a set of measure zero* N *such that if* $\omega \notin N$, *then*

$$f_{n_k}(\cdot, \omega) \to f(\cdot, \omega)$$

in $L^p([0,T], E)$ *and for a.e.* t.

Because of this lemma, it can also be assumed that for a.e. ω, pointwise convergence is obtained on $[0,T]$ as well as convergence in $L^p([0,T])$. This kind of assumption will be tacitly made whenever convenient.

To begin with, here is a useful lemma about how to recognize a martingale.

Lemma 2.3. *Let* $\{X(t)\}$ *be a stochastic process adapted to the filtration* $\{\mathcal{F}_t\}$ *for* $t \geq 0, E(|X(t)|) < \infty$. *Then it is a martingale for the given filtration if for every stopping time* σ *it follows*

$$E(X(t)) = E(X(\sigma)).$$

In fact, it suffices to check this on stopping times which have two values.

The main result on the quadratic variation $[M]$ for a continuous local martingale is the following.

Proposition 2.1. *Let $M(t)$ be a continuous local martingale for $t \in [0,T]$ having values in H a separable Hilbert space adapted to the normal filtration $\{\mathcal{F}_t\}$ such that $M(0)=0$. Then there exists a unique continuous, increasing, nonnegative, local submartingale $[M](t)$ called the quadratic variation such that*

$$\|M(t)\|^2 - [M](t)$$

is a real local martingale and $[M](0)=0$. Here $t \in [0,T]$. If δ is any stopping time:

$$\left[M^\delta\right] = [M]^\delta .$$

Recall that $M^\delta(t) \triangleq M(\delta \wedge t)$.

The quadratic variation for the stochastic integral is obtained according to the following corollary.

Corollary 2.1. *Suppose that Φ is $\mathcal{L}_2\left(Q^{1/2}U, H\right)$ progressively measurable and has the localizing sequence. Then the quadratic variation, $[\Phi \cdot W]$ is given by the formula*

$$[\Phi \cdot W](t) = \int_0^t ||\Phi(s)||^2_{\mathcal{L}_2\left(Q^{1/2}U,H\right)} ds.$$

The main tool for dealing with stochastic equations is the Burkholder-Davis-Gundy (BDG) inequality which is a relationship between the maximal function

$$M^* = \sup_{t\in[0,T]} |M(t)|$$

of a martingale and the quadratic variation.

Proposition 2.2. *Let $\{M(t)\}$ be a continuous H valued martingale which is uniformly bounded, $M(0)=0$, where H is a separable Hilbert space and $t \in [0,T]$. Then if $F(r) = r^p$ for some $p \geq 1$, there are constants, C and c independent of such martingales M such that*

$$c\int_\Omega F\left(([M](T))^{1/2}\right) dP \leq \int_\Omega F(M^*)\, dP \leq C\int_\Omega F\left(([M](T))^{1/2}\right) dP,$$

where

$$M^*(\omega) \triangleq \sup\{||M(t)(\omega)|| : t \in [0,T]\}.$$

Proposition 2.3. *Let H be a Hilbert space and suppose $(M, \mathcal{F}_t), t \in [0,T]$ is a uniformly bounded continuous martingale with values in H. Also let $\{t_k^n\}_{k=1}^{m_n}$ be a sequence of partitions satisfying*

$$\lim_{n\to\infty} \max\left\{\left|t_i^n - t_{i+1}^n\right|, i = 0, \cdots, m_n\right\} = 0, \ \{t_k^n\}_{k=1}^{m_n} \subseteq \left\{t_k^{n+1}\right\}_{k=1}^{m_{n+1}}.$$

Then

$$[M](t) = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} \left|M\left(t\wedge t_{k+1}^n\right) - M\left(t\wedge t_k^n\right)\right|_H^2,$$

the limit taking place in $L^2(\Omega)$. In case that M is just a continuous local martingale, the above limit happens in probability.

Proposition 2.4. *Let $X(t)$ be a real valued stochastic process which is $\mathcal{F}_t$ adapted for a normal filtration $\mathcal{F}_t$, with the property that off a set of measure zero in Ω, $t \to X(t)$ is lower semicontinuous. Then*

$$\tau \triangleq \inf\{t : X(t) > \alpha\}$$

is a stopping time.

The following generalization will also be useful. It generalizes the usual notions of Fourier expansion.

Proposition 2.5. *There exists a countable set $\{e_i\}_{i=1}^{\infty}$ of vectors in V such that*

$$\langle Be_i, e_j\rangle = \delta_{ij},$$

and for each $x \in W$,

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i\rangle Be_i, \quad \text{and} \quad \langle Bx, x\rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i\rangle|^2 .$$

The series converges in W'.

2.2. Preliminaries

From the integral equation (1.2), if $\phi \in L^q(\Omega; V)$ and $\psi \in C_c^{\infty}(0,T)$ for $q = \max(p,2)$,

$$\begin{aligned}&\int_{\Omega}\int_0^T \left((BX)(t) - B\int_0^t Z(s)\, dW(s) - BX_0\right)\psi'\phi dt dP\\ =& \int_{\Omega}\int_0^T \int_0^t Y(s)\psi'(t)\, ds\phi dt dP,\end{aligned}$$

then the term on the right equals

$$\int_{\Omega}\int_0^T \int_s^T Y(s)\psi'(t)\, dt ds \phi(\omega)\, dP = \int_{\Omega}\left(-\int_0^T Y(s)\psi(s)\, ds\right)\phi(\omega)\, dP.$$

It follows that, since ϕ is arbitrary,

$$\int_0^T \left((BX)(t) - B\int_0^t Z(s)\, dW(s) - BX_0\right)\psi'(t)\, dt = -\int_0^T Y(s)\psi(s)\, ds$$

in $L^{q'}(\Omega; V')$ and so the weak time derivative of

$$t \to (BX)(t) - B\int_0^t Z(s)\, dW(s) - BX_0$$

equals Y in $L^{q'}\left([0,T]; L^{q'}(\Omega, V')\right)$. Thus, for a.e. t, say $t \notin \hat{N} \subseteq [0,T]$, $m\left(\hat{N}\right) = 0$,

$$B\left(X(t) - \int_0^t Z(s)\, dW(s)\right) = BX_0 + \int_0^t Y(s)\, ds \text{ in } L^{q'}(\Omega, V').$$

That is,

$$(BX)(t) = BX_0 + \int_0^t Y(s)\,ds + B\int_0^t Z(s)\,dW(s)$$

holds in $L^{q'}(\Omega, V')$ where $(BX)(t) = B(X(t))$ a.e. t, in addition to holding for all t for each ω. Now let $\{t_k^n\}_{k=1}^{m_n}{}_{n=1}^{\infty}$ be partitions for which, from Lemma 2.1, there are left and right step functions X_k^l and X_k^r, which converge in $L^p([0,T]\times\Omega;V)$ to X and such that each $\{t_k^n\}_{k=1}^{m_n}$ has empty intersection with the set of measure zero $\hat{N}$ where, in $L^{p'}(\Omega;V')$, $(BX)(t) \neq B(X(t))$ in $L^{q'}(\Omega;V')$. Thus for t_k a generic partition point,

$$BX(t_k) = B(X(t_k)) \text{ in } L^{q'}(\Omega;V').$$

Hence there is an exceptional set of measure zero, $N(t_k) \subseteq \Omega$, such that for $\omega \notin N(t_k)$,

$$BX(t_k)(\omega) = B(X(t_k,\omega)).$$

We define an exceptional set $N \subseteq \Omega$ to be the union of all these $N(t_k)$. There are countably many and so N is also a set of measure zero. Then for $\omega \notin N$ and t_k any mesh point at all, $BX(t_k)(\omega) = B(X(t_k,\omega))$. This will be important in what follows. In addition to this, from the integral equation, for each of these $\omega \notin N$, $BX(t)(\omega) = B(X(t,\omega))$ for all $t \notin N_\omega \subseteq [0,T]$ where N_ω is a set of Lebesgue measure zero. Thus the t_k from the various partitions are always in N_ω^C. By Proposition 2.5, there exists a countable set $\{e_i\}$ of vectors in V such that

$$\langle Be_i, e_j\rangle = \delta_{ij}$$

and for each $x \in W$,

$$\langle Bx, x\rangle = \sum_{i=0}^{\infty} |\langle Bx, e_i\rangle|^2, \ Bx = \sum_{i=1}^{\infty} \langle Bx, e_i\rangle Be_i.$$

Thus the conclusion of the above discussion is that at the mesh points, it is valid to write

$$\begin{aligned}\langle (BX)(t_k), X(t_k)\rangle &= \langle B(X(t_k)), X(t_k)\rangle \\ &= \sum_i \langle (BX)(t_k), e_i\rangle^2 = \sum_i \langle B(X(t_k)), e_i\rangle^2\end{aligned}$$

just as would be the case if $(BX)(t) = B(X(t))$ for every t.

In all which follows, the mesh points will be like this and an appropriate set of measure zero which may be replaced with a larger set of measure zero finitely many times is being neglected. Obviously, one can take a subsequence of the sequence of partitions described above without disturbing the above observations. We will denote these partitions as $\mathcal{P}_k$. As a case of this, we obtain the following

Lemma 2.4. *There exists a set of measure zero $N \subseteq \Omega$ and a dense subset of $[0,T]$, D such that for $\omega \notin N$, $BX(t,\omega) = B(X(t,\omega))$ for all $t \in D$.*

Proposition 2.6. *Let Z be progressively measurable and in*

$$L^2\left([0,T]\times\Omega, \mathcal{L}_2\left(Q^{1/2}U, W\right)\right).$$

Also suppose X is progressively measurable and in $L^2([0,T]\times\Omega,W)$. Let $\{t_j^n\}_{j=0}^{m_n}$ be a sequence of partitions of the sort in Lemma 2.1 such that if

$$X_n(t)\triangleq\sum_{j=0}^{m_n-1}X(t_j^n)\mathcal{X}_{[t_j^n,t_{j+1}^n)}(t)\triangleq X_n^l(t),$$

then $X_n\to X$ in $L^p([0,T]\times\Omega,W)$. Also, it can be assumed that none of these mesh points are in the exceptional set off which $BX(t)=B(X(t))$. (Thus it will make no difference whether we write $BX(t)$ or $B(X(t))$ in what follows for all t one of these mesh points.) Then the function of t given by

$$\sum_{j=0}^{m_n-1}\left\langle B\int_{t_j^n\wedge t}^{t_{j+1}^n\wedge t}ZdW,X(t_j^n)\right\rangle=\sum_{j=0}^{m_n-1}\left\langle BX(t_j^n),\int_{t_j^n\wedge t}^{t_{j+1}^n\wedge t}ZdW\right\rangle \tag{2.1}$$

is a local martingale which can be written in the form

$$\int_0^t\left(Z\circ J^{-1}\right)^*BX_n^l\circ JdW$$

where

$$X_n^l(t)=\sum_{k=0}^{m_n-1}X(t_k^n)\mathcal{X}_{[t_k^n,t_{k+1}^n)}(t).$$

Proof: The proof of this theorem amounts to a careful use of the definitions. One first assumes that $\langle BX(t_k^n),X(t_k^n)\rangle\in L^\infty(\Omega)$. The next step is to verify that (2.1) is a martingale using Lemma 2.3. Verification of the claimed formula follows from the definition of the stochastic integral applied to each of the sub-intervals and is mainly technical. The proof is completed using a stopping time argument.

The question of convergence as $n\to\infty$ is considered later.

2.3. The main estimate

The argument will be based on a formula which follows in the next lemma which is a technical generalization of one in [21].

Lemma 2.5. *Letting $M(t)\triangleq\int_0^tZ(u)dW(u)$, which has values in W, the following formula holds for a.e. ω for $0<s<t$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V,V'.*

$$\begin{aligned}\langle BX(t),X(t)\rangle&=\langle BX(s),X(s)\rangle+\\ &+2\int_s^t\langle Y(u),X(t)\rangle du+\langle B(M(t)-M(s)),M(t)-M(s)\rangle\\ &-\langle BX(t)-BX(s)-(M(t)-M(s)),X(t)-X(s)-(M(t)-M(s))\rangle\\ &+2\langle BX(s),M(t)-M(s)\rangle.\end{aligned}\tag{2.2}$$

Also for $t>0$,

$$\begin{aligned}\langle BX(t),X(t)\rangle&=\langle BX_0,X_0\rangle+2\int_0^t\langle Y(u),X(t)\rangle du+2\langle BX_0,M(t)\rangle+\\ &\langle BM(t),M(t)\rangle-\langle BX(t)-BX_0-BM(t),X(t)-X_0-M(t)\rangle.\end{aligned}\tag{2.3}$$

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The following estimate is important in proving the Ito formula.

Lemma 2.6. *For a.e. t,*

$$
\begin{aligned}
&E\left(\langle BX(t), X(t)\rangle\right) \\
&< C\left(||Y||_{K'}, ||X||_K, ||Z||_J, \|\langle BX_0, X_0\rangle\|_{L^1(\Omega)}\right) < \infty,
\end{aligned} \tag{2.4}
$$

where $J = L^2\left([0,T]\times\Omega; \mathcal{L}_2\left(Q^{1/2}U; W\right)\right)$. Or equivalently

$$
E\left(\sup_{t\in[0,T]}\sum_i \langle BX(t), e_i\rangle^2\right) \leq C\left(||Y||_{K'}, ||X||_K, ||Z||_J, \|\langle BX_0, X_0\rangle\|_{L^1(\Omega)}\right),
$$

where C is a continuous function of its arguments, increasing in each one, and $C(0,0,0,0) = 0$. Thus for a.e. ω,

$$
\sup_{t\notin N_\omega^C}\langle BX(t,\omega), X(t,\omega)\rangle \leq C(\omega) < \infty.
$$

Also for ω off a set of measure zero described earlier, $t \to BX(t)(\omega)$ is weakly continuous with values in W' on $[0,T]$. Also, $t\to\langle BX(t), X(t)\rangle$ is lower semi-continuous on N_ω^C.

Proof: Consider the formula in Lemma 2.5.

$$
\begin{aligned}
&\langle BX(t), X(t)\rangle = \langle BX(s), X(s)\rangle \\
&+2\int_s^t \langle Y(u), X(t)\rangle\, du + \langle B(M(t)-M(s)), M(t)-M(s)\rangle \\
&-\langle B(X(t)-X(s)-(M(t)-M(s))), X(t)-X(s)-(M(t)-M(s))\rangle \\
&+2\langle BX(s), M(t)-M(s)\rangle.
\end{aligned} \tag{2.5}
$$

Now let t_j denote a point of $\mathcal{P}_k$ from Lemma 2.1. Then for $t_j > 0, X(t_j)$ is just the value of X at t_j but when $t=0$, the definition of $X(0)$ in this step function is $X(0) \triangleq 0$. Thus

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\langle BX(t_{j+1}), X(t_{j+1})\rangle - \langle BX(t_j), X(t_j)\rangle \\
&+\langle BX(t_1), X(t_1)\rangle - \langle BX_0, X_0\rangle \\
&= \langle BX(t_m), X(t_m)\rangle - \langle BX_0, X_0\rangle.
\end{aligned}
$$

Using the formula in Lemma 2.5, for $t = t_m$ this yields

$$\begin{aligned}
&\langle BX(t_m), X(t_m)\rangle - \langle BX_0, X_0\rangle = 2\sum_{j=1}^{m-1}\int_{t_j}^{t_{j+1}} \langle Y(u), X_k^r(u)\rangle\, du \\
&+2\sum_{j=1}^{m-1}\left\langle B\int_{t_j}^{t_{j+1}} Z(u)\, dW, X(t_j)\right\rangle \\
&+\sum_{j=1}^{m-1}\langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j)\rangle \\
&-\sum_{j=1}^{m-1}\langle B(X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))), \\
&\quad X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))\rangle \\
&+2\int_0^{t_1}\langle Y(u), X(t_1)\rangle\, du + 2\left\langle BX_0, \int_0^{t_1} Z(u)\, dW\right\rangle + \langle BM(t_1), M(t_1)\rangle \\
&-\langle B(X(t_1) - X_0 - M(t_1)), X(t_1) - X_0 - M(t_1)\rangle .
\end{aligned} \tag{2.6}$$

First consider $2\int_0^{t_1}\langle Y(u), X(t_1)\rangle\, du + 2\left\langle BX_0, \int_0^{t_1} Z(u)\, dW\right\rangle + \langle BM(t_1), M(t_1)\rangle$. Each term converges to 0 for a.e. ω as $k \to \infty$ and in $L^1(\Omega)$. This follows right away for the second two terms from the Ito isometry and continuity properties of the stochastic integral. Consider the first term. This term is dominated by

$$\begin{aligned}
&\left(\int_0^{t_1} \|Y(u)\|^{p'} du\right)^{1/p'} \left(\int_0^T \|X_k^r(u)\|^p du\right)^{1/p} \\
&\leq C(\omega)\left(\int_0^{t_1} \|Y(u)\|^{p'} du\right)^{1/p'}, \qquad \left(\int_\Omega C(\omega)^p\, dP\right)^{1/p} < \infty.
\end{aligned}$$

Hence this converges to 0 for a.e. ω and also converges to 0 in $L^1(\Omega)$.

Not much is known about the last term in (2.6), but it is negative and is about to be neglected.

The term involving the stochastic integral equals

$$2\sum_{j=1}^{m-1}\left\langle B\int_{t_j}^{t_{j+1}} Z(u)\, dW, X(t_j)\right\rangle .$$

By Proposition 2.6 this equals

$$2\int_{t_1}^{t_m} \left(Z \circ J^{-1}\right)^* BX_k^l \circ J dW.$$

Also note that since $\langle BM(t_1), M(t_1)\rangle$ converges to 0 in $L^1(\Omega)$ and for a.e. ω, the sum involving

$$\langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j)\rangle$$

can be started at 0 rather than 1 at the expense of adding in a term which converges to 0 a.e. and in $L^1(\Omega)$. Thus (2.6) is of the form

$$\langle BX(t_m), X(t_m)\rangle - \langle BX_0, X_0\rangle = e(k) + 2\int_0^{t_m}\langle Y(u), X_k^r(u)\rangle\, du +$$

$$+2\int_0^{t_m}\left(Z\circ J^{-1}\right)^* BX_k^l\circ JdW$$
$$+\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_j\right)\right),M\left(t_{j+1}\right)-M\left(t_j\right)\right\rangle$$

$$-\sum_{j=1}^{m-1}\left\langle B\left(X\left(t_{j+1}\right)-X\left(t_j\right)-\left(M\left(t_{j+1}\right)-M\left(t_j\right)\right)\right),\right.$$
$$\left. X\left(t_{j+1}\right)-X\left(t_j\right)-\left(M\left(t_{j+1}\right)-M\left(t_j\right)\right)\right\rangle$$

$$-\left\langle B\left(X\left(t_1\right)-X_0-M\left(t_1\right)\right),X\left(t_1\right)-X_0-M\left(t_1\right)\right\rangle, \tag{2.7}$$

where $e\left(k\right)\to 0$ for a.e. ω and also in $L^1\left(\Omega\right)$.

By definition, $M\left(t_{j+1}\right)-M\left(t_j\right)=\int_{t_j}^{t_{j+1}}ZdW$. Now it follows, on discarding the negative terms,

$$\left\langle BX\left(t_m\right),X\left(t_m\right)\right\rangle-\left\langle BX_0,X_0\right\rangle\le e\left(k\right)+2\int_0^{t_m}\left\langle Y\left(u\right),X_k^r\left(u\right)\right\rangle du+$$

$$+2\int_0^{t_m}\left(Z\circ J^{-1}\right)^* BX_k^l\circ JdW+\sum_{j=0}^{m-1}\left\langle B\int_{t_j}^{t_{j+1}}ZdW,\int_{t_j}^{t_{j+1}}ZdW\right\rangle.$$

Therefore,

$$\sup_{t_m\in\mathcal{P}_k}\left\langle BX\left(t_m\right),X\left(t_m\right)\right\rangle\le\left\langle BX_0,X_0\right\rangle+e\left(k\right)+2\int_0^T\left|\left\langle Y\left(u\right),X_k^r\left(u\right)\right\rangle\right|du+$$

$$+2\sup_{t_m\in\mathcal{P}_k}\left|\int_0^{t_m}\left(Z\circ J^{-1}\right)^* BX_k^l\circ JdW\right|$$
$$+\sum_{j=0}^{m_k-1}\left\langle B\left(\int_{t_j}^{t_{j+1}}Z\left(u\right)dW\right),\int_{t_j}^{t_{j+1}}Z\left(u\right)dW\right\rangle,$$

where there are m_k+1 points in $\mathcal{P}_k$.

In order to take the expectation of both sides, let

$$\tau_p=\inf\left\{t:\left\langle BX_k^l\left(t\right),X_k^l\left(t\right)\right\rangle>p\right\}.$$

By right continuity this is a well defined stopping time. Then one obtain the above inequality for $\left(X_k^l\right)^{\tau_p}$ in place of X_k^l. Take the expectation and use the Ito isometry to obtain

$$\begin{aligned}
&\int_\Omega \left(\sup_{t_m \in \mathcal{P}_k} \left\langle B \left(X_k^l \right)^{\tau_p} (t_m), \left(X_k^l \right)^{\tau_p} (t_m) \right\rangle \right) dP \\
\leq & E\left(\langle BX_0, X_0 \rangle \right) + 2 \left|\left| Y \right|\right|_{K'} \left|\left| X_k^r \right|\right|_K \\
& + \|B\| \sum_{j=0}^{m_k - 1} \int_{t_j}^{t_{j+1}} \int_\Omega \left|\left| Z(u) \right|\right|^2 dP du \\
& + 2 \int_\Omega \left(\sup_{t \in [0,T]} \left| \int_0^t \mathcal{X}_{[0,\tau_p]} \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J dW \right| \right) dP + E\left(|e(k)| \right) \\
\leq & C + \|B\| \int_0^T \int_\Omega \|Z(u)\|^2 dP du + E\left(|e(k)| \right) \\
& + 2 \int_\Omega \left(\sup_{t \in [0,T]} \left| \int_0^t \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J dW \right| \right) dP \\
\leq & C + E\left(|e(k)| \right) + 2 \int_\Omega \left(\sup_{t \in [0,T]} \left| \int_0^t \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J dW \right| \right) dP,
\end{aligned} \tag{2.8}$$

where the convergence of X_k^r to X in K shows that the term $2 \left|\left| Y \right|\right|_{K'} \left|\left| X_k^r \right|\right|_K$ is bounded. Thus the constant C can be assumed to be a continuous function of

$$\left|\left| Y \right|\right|_{K'}, \left|\left| X \right|\right|_K, \left|\left| Z \right|\right|_J, \left\| \langle BX_0, X_0 \rangle \right\|_{L^1(\Omega)},$$

which equals zero when all are equal to zero and is increasing in each.

Consider the term involving the stochastic integral.

Let $\mathcal{M}(t) = \int_0^t \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J dW$. Then by Corollary 2.1

$$d[\mathcal{M}] = \left\| \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J \right\|^2 ds.$$

Applying the BDG inequality, Proposition 2.2 for $F(r) = r$ in that stochastic integral,

$$\begin{aligned}
& 2 \int_\Omega \left(\sup_{t \in [0,T]} \left| \int_0^t \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J dW \right| \right) dP \\
\leq & C \int_\Omega \left(\int_0^T \left\| \left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \circ J \right\|^2_{\mathcal{L}_2\left(Q^{1/2} U, \mathbb{R} \right)} ds \right)^{1/2} dP.
\end{aligned} \tag{2.9}$$

So let $\{g_i\}$ be an orthonormal basis for $Q^{1/2}U$ and consider the integrand in the above. It equals

$$\begin{aligned}
\sum_{i=1}^\infty \left(\left(\left(Z \circ J^{-1} \right)^* B \left(X_k^l \right)^{\tau_p} \right) \left(J(g_i) \right) \right)^2 &= \sum_{i=1}^\infty \left\langle B \left(X_k^l \right)^{\tau_p}, Z(g_i) \right\rangle^2 \\
&\leq \sum_{i=1}^\infty \left\langle B \left(X_k^l \right)^{\tau_p}, \left(X_k^l \right)^{\tau_p} \right\rangle \langle BZ(g_i), Z(g_i) \rangle \\
&\leq \left(\sup_{t_m \in \mathcal{P}_k} \left\langle B \left(X_k^l \right)^{\tau_p} (t_m), \left(X_k^l \right)^{\tau_p} (t_m) \right\rangle \right) \|B\| \|Z\|^2_{\mathcal{L}_2}.
\end{aligned}$$

It follows that the integral in (2.9) is dominated by

$$C\int_{\Omega}\sup_{t_m\in\mathcal{P}_k}\left\langle B\left(X_k^l\right)^{\tau_p}(t_m),\left(X_k^l\right)^{\tau_p}(t_m)\right\rangle^{1/2}\|B\|^{1/2}\left(\int_0^T\|Z\|_{\mathcal{L}_2}^2\,ds\right)^{1/2}dP.$$

Now return to (2.8). From what was just shown,

$$\begin{aligned}
&E\left(\sup_{t_m\in\mathcal{P}_k}\left\langle B\left(X_k^l\right)^{\tau_p}(t_m),\left(X_k^l\right)^{\tau_p}(t_m)\right\rangle\right)\\
\leq& C+E\left(|e(k)|\right)+2\int_{\Omega}\left(\sup_{t\in[0,T]}\left|\int_0^t\left(Z\circ J^{-1}\right)^*B\left(X_k^l\right)^{\tau_p}\circ J dW\right|\right)dP\\
\leq& C+C\int_{\Omega}\sup_{t_m\in\mathcal{P}_k}\left\langle B\left(X_k^l\right)^{\tau_p}(t_m),\left(X_k^l\right)^{\tau_p}(t_m)\right\rangle^{1/2}\cdot\\
&\|B\|^{1/2}\left(\int_0^T\|Z\|_{\mathcal{L}_2}^2\,ds\right)^{1/2}dP+E\left(|e(k)|\right)\\
\leq& C+\frac{1}{2}E\left(\sup_{t_m\in\mathcal{P}_k}\left\langle B\left(X_k^l\right)^{\tau_p}(t_m),\left(X_k^l\right)^{\tau_p}(t_m)\right\rangle\right)\\
&+C\|Z\|^2_{\mathcal{L}^2([0,T]\times\Omega,\mathcal{L}_2)}+E\left(|e(k)|\right).
\end{aligned}$$

It follows that

$$\frac{1}{2}E\left(\sup_{t_m\in\mathcal{P}_k}\left\langle B\left(X_k^l\right)^{\tau_p}(t_m),\left(X_k^l\right)^{\tau_p}(t_m)\right\rangle\right)\leq C+E\left(|e(k)|\right).$$

Now let $p\to\infty$ and use the monotone convergence theorem to obtain

$$E\left(\sup_{t_m\in\mathcal{P}_k}\left\langle BX_k^l(t_m),X_k^l(t_m)\right\rangle\right)=E\left(\sup_{t_m\in\mathcal{P}_k}\left\langle BX(t_m),X(t_m)\right\rangle\right)\leq C+E\left(|e(k)|\right). \tag{2.10}$$

As mentioned above, this constant C is a continuous function of

$$||Y||_{K'},||X||_K,||Z||_J,\|\langle BX_0,X_0\rangle\|_{L^1(\Omega,H)}$$

and equals zero when all of these quantities equal 0 and is increasing with respect to each of the above quantities. Also, for each $\varepsilon>0$,

$$E\left(\sup_{t_m\in\mathcal{P}_k}\left\langle BX(t_m),X(t_m)\right\rangle\right)\leq C+\varepsilon$$

whenever k is large enough.

Let D denote the union of all the $\mathcal{P}_k$. Thus, D is a dense subset of $[0,T]$ and it has just been shown, since the $\mathcal{P}_k$ are nested, that for a constant C, dependent only on the above quantities which is independent of $\mathcal{P}_k$,

$$E\left(\sup_{t\in D}\left\langle BX(t),X(t)\right\rangle\right)\leq C+\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$E\left(\sup_{t\in D}\langle BX(t), X(t)\rangle\right) \leq C. \tag{2.11}$$

Thus, enlarging N, for $\omega \notin N$,

$$\sup_{t\in D}\langle BX(t), X(t)\rangle = C(\omega) < \infty, \tag{2.12}$$

where $\int_\Omega C(\omega)\,dP < \infty$. By Proposition 2.5, there exists a countable set $\{e_i\}$ of vectors in V such that

$$\langle Be_i, e_j\rangle = \delta_{ij}$$

and for each $x \in W$,

$$\langle Bx, x\rangle = \sum_{i=0}^{\infty}\langle Bx, e_i\rangle^2,\ Bx = \sum_{i=1}^{\infty}\langle Bx, e_i\rangle Be_i.$$

Thus, for t not in a set of measure zero off which $BX(t) = B(X(t))$,

$$\langle BX(t), X(t)\rangle = \sum_{i=0}^{\infty}\langle BX(t), e_i\rangle^2 = \sup_m \sum_{k=1}^{m}\langle BX(t), e_i\rangle^2.$$

Now from the formula for $BX(t)$, it follows that BX is continuous into V'. For any $t \notin \hat{N}$, a set of measure zero, so that $(BX)(t) = B(X(t))$ in $L^{q'}(\Omega; V')$ and letting $t_k \to t$ where $t_k \in D$, Fatou's lemma implies

$$E(\langle BX(t), X(t)\rangle) = \sum_i E\left(\langle BX(t), e_i\rangle^2\right) = \sum_i \liminf_{k\to\infty} E\left(\langle BX(t_k), e_i\rangle^2\right)$$

$$\begin{aligned}
&\leq \liminf_{k\to\infty}\sum_i E\left(\langle BX(t_k), e_i\rangle^2\right) = \liminf_{k\to\infty} E(\langle BX(t_k), X(t_k)\rangle) \\
&\leq C\left(||Y||_{K'}, ||X||_K, ||Z||_J, \|\langle BX_0, X_0\rangle\|_{L^1(\Omega)}\right).
\end{aligned}$$

In addition to this, for arbitrary $t \in [0,T]$, and $t_k \to t$ from D,

$$\sum_i \langle BX(t), e_i\rangle^2 \leq \liminf_{k\to\infty}\sum_i\langle BX(t_k), e_i\rangle^2 \leq \sup_{s\in D}\langle BX(s), X(s)\rangle.$$

Hence

$$\begin{aligned}
\sup_{t\in[0,T]}\sum_i\langle BX(t), e_i\rangle^2 &\leq \sup_{s\in D}\langle BX(s), X(s)\rangle \\
&= \sup_{s\in D}\sum_i\langle BX(s), e_i\rangle^2 \leq \sup_{t\in[0,T]}\sum_i\langle BX(t), e_i\rangle^2.
\end{aligned}$$

It follows that

$$\sup_{t\in[0,T]}\sum_i\langle BX(t), e_i\rangle^2$$

is measurable and for ω off a set of measure zero, $\sup_{t\in[0,T]}\sum_i \langle BX(t), e_i\rangle^2$ is bounded above.

Also for $t \notin N_\omega$ and a given $\omega \notin N$, letting $t_k \to t$ for $t_k \in D$,

$$\begin{aligned}\langle BX(t), X(t)\rangle &= \sum_i \langle BX(t), e_i\rangle^2 \leq \liminf_{k\to\infty} \sum_i \langle BX(t_k), e_i\rangle^2 \\ &= \liminf_{k\to\infty} \langle BX(t_k), X(t_k)\rangle \leq \sup_{t\in D} \langle BX(t), X(t)\rangle\end{aligned}$$

and so

$$\sup_{t\notin N_\omega} \langle BX(t), X(t)\rangle \leq \sup_{t\in D} \langle BX(t), X(t)\rangle \leq \sup_{t\notin N_\omega} \langle BX(t), X(t)\rangle .$$

From (2.12),

$$\sup_{t\notin N_\omega} \langle BX(t), X(t)\rangle = C(\omega) \text{ a.e. } \omega,$$

where $\int_\Omega C(\omega)\, dP < \infty$. In particular, $\sup_{t\notin N_\omega} \langle BX(t), X(t)\rangle$ is bounded for a.e. ω say for $\omega \notin N$ where N includes the earlier sets of measure zero. This shows that $BX(t)$ is bounded in W' for $t \in N_\omega^C$.

If $v \in V$, then for $\omega \notin N$,

$$\lim_{t\to s} \langle BX(t), v\rangle = \langle BX(s), v\rangle .$$

Therefore, since for such ω, $\|BX(t)\|_{W'}$ is bounded for $t \notin N_\omega$, the above holds for all $v \in W$ also. Therefore, for a.e. ω, $t \to BX(t,\omega)$ is weakly continuous with values in W' for $t \notin N_\omega$.

Note also that

$$\begin{aligned}\|BX\|_{W'}^2 &\triangleq \left(\sup_{\|y\|_W\leq 1} \langle BX, y\rangle\right)^2 \leq \sup_{\|y\|\leq 1} \left(\langle BX, X\rangle^{1/2} \langle By, y\rangle^{1/2}\right)^2 \\ &\leq \langle BX, X\rangle \|B\|\end{aligned}$$

$$\begin{aligned}\int_0^T \int_\Omega \|BX(t)\|^2 \, dPdt &\leq \int_\Omega \int_0^T \|B\| \langle BX(t), X(t)\rangle \, dtdP \\ &\leq C\left(||Y||_{K'}, ||X||_K, ||Z||_J, \|\langle BX_0, X_0\rangle\|_{L^1(\Omega)}\right) \|B\| T. \end{aligned} \tag{2.13}$$

Eventually, it is shown that in fact, the function $t \to BX(t,\omega)$ is continuous with values in W'. The above shows that $BX \in L^2([0,T]\times\Omega, W')$.

With the last lemma, we can simplify one of the formulas derived earlier **in the case that** $X_0 \in L^p(\Omega, V)$ so that $X - X_0 \in L^p([0,T]\times\Omega, V)$. Refer to (2.7). One term there is

$$\langle B(X(t_1) - X_0 - M(t_1)), X(t_1) - X_0 - M(t_1)\rangle$$

$$\leq 2\langle B(X(t_1) - X_0), X(t_1) - X_0\rangle + 2\langle BM(t_1), M(t_1)\rangle .$$

It was observed above that $2\langle BM(t_1), M(t_1)\rangle \to 0$ a.e. and also in $L^1(\Omega)$ as $k \to \infty$. Apply the above lemma to $\langle B(X(t_1) - X_0), X(t_1) - X_0\rangle$ using $[0, t_1]$

instead of $[0,T]$. The new X_0 equals 0. Then from the estimate (2.4), it follows that

$$E\left(\left\langle B\left(X\left(t_1\right)-X_0\right), X\left(t_1\right)-X_0\right\rangle\right) \rightarrow 0$$

as $k \rightarrow \infty$. Taking a subsequence, we could also assume that

$$\left\langle B\left(X\left(t_1\right)-X_0\right), X\left(t_1\right)-X_0\right\rangle \rightarrow 0$$

a.e. ω as $k \rightarrow \infty$. Then, using this subsequence, it would follow from (2.7),

$$\begin{aligned}
\left\langle BX\left(t_m\right), X\left(t_m\right)\right\rangle-\left\langle BX_0, X_0\right\rangle = e(k)+2 \int_0^{t_m}\left\langle Y(u), X_k^r(u)\right\rangle du+ \\
+2 \int_0^{t_m}\left(Z \circ J^{-1}\right)^* BX_k^l \circ J dW \\
+\sum_{j=0}^{m-1}\left\langle B\left(M\left(t_{j+1}\right)-M\left(t_j\right)\right), M\left(t_{j+1}\right)-M\left(t_j\right)\right\rangle \\
-\sum_{j=1}^{m-1}\left\langle B\left(\Delta X\left(t_j\right)-\Delta M\left(t_j\right)\right), \Delta X\left(t_j\right)-\Delta M\left(t_j\right)\right\rangle \qquad (2.14)
\end{aligned}$$

where $e(k) \rightarrow 0$ in $L^1(\Omega)$ and a.e. ω and

$$\Delta X\left(t_j\right) \triangleq X\left(t_{j+1}\right)-X\left(t_j\right),$$

$\Delta M\left(t_j\right)$ being defined similarly. Note how this eliminated the need to consider the term

$$\left\langle B\left(X\left(t_1\right)-X_0-M\left(t_1\right)\right), X\left(t_1\right)-X_0-M\left(t_1\right)\right\rangle$$

in passing to a limit.

In the case that X_0 is not assumed to be in $L^p(\Omega, V)$, let $Z_{0k} \in L^p(\Omega, V) \cap L^2(\Omega, W), Z_{0k} \rightarrow X_0$ in $L^2(\Omega, W)$. Then from the usual arguments involving the Cauchy Schwarz inequality,

$$\begin{aligned}
\left\langle B\left(X\left(t_1\right)-X_0\right), X\left(t_1\right)-X_0\right\rangle^{1/2} \leq \left\langle B\left(X\left(t_1\right)-Z_{0k}\right), X\left(t_1\right)-Z_{0k}\right\rangle^{1/2} \\
+\left\langle B\left(Z_{0k}-X_0\right), Z_{0k}-X_0\right\rangle^{1/2}.
\end{aligned}$$

Also, restoring the superscript to identify the parition,

$$B\left(X\left(t_1^k\right)-Z_{0k}\right)=B\left(X_0-Z_{0k}\right)+\int_0^{t_1^k} Y(s)\, ds+B \int_0^{t_1^k} Z(s)\, dW.$$

Of course $\left\|X-Z_{0k}\right\|_K$ is not bounded, but for each k it is finite. There is a sequence of partitions $\mathcal{P}_k, \left\|\mathcal{P}_k\right\| \rightarrow 0$ such that all the above holds. In the definitions of K, K', J replace $[0,T]$ with $[0,t]$ and let the resulting spaces be denoted by K_t, K_t', J_t. Let n_k denote a subsequence of $\{k\}$ such that

$$\left\|X-Z_{0k}\right\|_{K_{t_1^{n_k}}}<1/k.$$

Then from the above lemma,

$$E\left(\left\langle B\left(X\left(t_1^{n_k}\right)-Z_{0k}\right), X\left(t_1^{n_k}\right)-Z_{0k}\right\rangle\right)$$

$$\leq C\left(\|Y\|_{K'_{t_1^{n_k}}}, \|X-Z_{0k}\|_{K_{t_1^{n_k}}}, \|Z\|_{J_{t_1^{n_k}}}, \left\langle B\left(X_0-Z_{0k}\right), X_0-Z_{0k}\right\rangle_{L^1(\Omega)}\right)$$

$$\leq C\left(\|Y\|_{K'_{t_1^{n_k}}}, \frac{1}{k}, \|Z\|_{J_{t_1^{n_k}}}, \left\langle B\left(X_0-Z_{0k}\right), X_0-Z_{0k}\right\rangle_{L^1(\Omega)}\right).$$

Hence

$$E\left(\left\langle B\left(X\left(t_1^{n_k}\right)-X_0\right), X\left(t_1^{n_k}\right)-X_0\right\rangle\right)$$

$$\leq 2E\left(\left\langle B\left(X\left(t_1^{n_k}\right)-Z_{0k}\right), X\left(t_1^{n_k}\right)-Z_{0k}\right\rangle\right)+2E\left(\left\langle B\left(Z_{0k}-X_0\right), Z_{0k}-X_0\right\rangle\right)$$

$$\leq 2C\left(\|Y\|_{K'_{t_1^{n_k}}}, \frac{1}{k}, \|Z\|_{J_{t_1^{n_k}}}, \left\langle B\left(X_0-Z_{0k}\right), X_0-Z_{0k}\right\rangle_{L^1(\Omega)}\right)$$
$$+2\|B\|\left\|Z_{0k}-X_0\right\|_{L^2(\Omega, W)}^2,$$

which converges to 0 as $k\to\infty$. It follows that there exists a suitable subsequence such that (2.14) holds even in the case that X_0 is only known to be in $L^2(\Omega, W)$. From now on, assume this subsequence for the partitions $\mathcal{P}_k$. Thus k will really be n_k and it suffices to consider the limit as $k\to\infty$ of the equation of (2.14).

Remark 2.1. The reason for the above observations is to argue that, even when X_0 is only in $L^2(\Omega, W)$, one can neglect

$$\left\langle B\left(X\left(t_1\right)-X_0-M\left(t_1\right)\right), X\left(t_1\right)-X_0-M\left(t_1\right)\right\rangle$$

in passing to the limit as $k\to\infty$ provided a suitable subsequence is used.

2.4. Convergence

We prove next that the stochastic integral $\int_0^t\left(Z\circ J^{-1}\right)^* BX_n^l\circ JdW$ converges as $n\to\infty$ in some sense to

$$\int_0^t\left(Z\circ J^{-1}\right)^* BX\circ JdW, \tag{2.15}$$

which is also a local martingale.

Notice that $Z\circ J^{-1}$ maps $JQ^{1/2}U$ to W and so $\left(Z\circ J^{-1}\right)^*$ maps W' to $\left(JQ^{1/2}U\right)'$. Thus

$$\left(Z\circ J^{-1}\right)^* BX\in\left(JQ^{1/2}U\right)', \text{ so } \left(Z\circ J^{-1}\right)^* BX\circ J\in Q^{1/2}(U)'=\mathcal{L}_2\left(Q^{1/2}U,\mathbb{R}\right).$$

Thus it has values in the right space. The problem is that the integrand is not in $L^2\left([0,T]\times\Omega;\mathcal{L}_2\left(Q^{1/2}U,\mathbb{R}\right)\right)$.

By assumption, $t\to BX(t)$ is continuous into V' thanks to the integral equation solved, and also $BX(t)=B(X(t))$ for $t\notin N_\omega$ a set of measure zero. For such t, it follows from Proposition 2.5,

$$\langle BX(t), X(t)\rangle=\sum_i\langle BX(t), e_i\rangle_{V',V}^2 \text{ a.e. } \omega$$

and so $t \to \sum_i \langle BX(t), e_i\rangle^2$ is lower semicontinuous and as just explained, it equals $\langle BX(t), X(t)\rangle$ for a.e. t, this for each $\omega \notin N$, a single set of measure zero. Also, $t \to \sum_i \langle BX(t), e_i\rangle_{V',V}^2$ is progressively measurable and lower semicontinuous in t so by Proposition 2.4, one can define a stopping time

$$\tau_p \triangleq \inf\left\{t : \sum_i \langle BX(t), e_i\rangle_{V',V}^2 > p\right\}, \tau_0 \triangleq 0. \tag{2.16}$$

Instead of referring to this Proposition, consider

$$\tau_p^m \triangleq \inf\left\{t : \sum_{i=1}^m \langle BX(t), e_i\rangle_{V',V}^2 > p\right\}$$

which is clearly a stopping time because $t \to \sum_{i=1}^m \langle BX(t), e_i\rangle_{V',V}^2$ is a continuous process. Then observe that $\tau_p = \sup_m \tau_p^m$. Then

$$[\tau_p \le t] = \cup_m \left[\tau_p^m \le t\right] \in \mathcal{F}_t.$$

That $\tau_p = \infty$ for all p large enough follows from Lemma 2.6.

Lemma 2.7. *$\tau_p = \infty$ for all p large enough off a set of measure zero for τ_p defined above. Moreover,*

$$P\left(\int_0^T \left|\left(Z \circ J^{-1}\right)^* BX \circ J\right|^2 dt < \infty\right) = 1,$$

so that $\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW$ can be defined as a local martingale.

Proof: Let

$$A \triangleq \left\{\omega : \int_0^T \left|\left(Z \circ J^{-1}\right)^* BX \circ J\right|^2 dt = \infty\right\}.$$

Then from the assumption that $\tau_p = \infty$ for all p large enough, it follows that

$$A = \cup_{m=1}^\infty A \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty]).$$

Now

$$P(A \cap [\tau_m = \infty]) \le P\left(\omega : \int_0^T \mathcal{X}_{[0,\tau_m]} \left|\left(Z \circ J^{-1}\right)^* BX \circ J\right|^2 dt = \infty\right). \tag{2.17}$$

Consider the integrand. Letting $\{g_i\}$ be an orthonormal basis for $Q^{1/2}U$, the

$$\left|\left(Z \circ J^{-1}\right)^* BX \circ J\right|^2$$

is defined as

$$\sum_i \left[\left(\left(Z \circ J^{-1}\right)^* BX \circ J\right)(g_i)\right]^2 \triangleq \sum_i \left[\left(Z \circ J^{-1}\right)^* BX(Jg_i)\right]^2$$

$$\triangleq \sum_i \left[BX\left(Z\circ J^{-1}\left(Jg_i\right)\right)\right]^2$$
$$= \sum_i \left[(BX)\left(Zg_i\right)\right]^2$$
$$\le \sum_i \|BX\|^2 \|Zg_i\|_W^2 .$$

Incorporating the stopping time, for a.e. t,

$$\langle BX, X\rangle (t) = \langle BX(t), X(t)\rangle \le m$$

and

$$\begin{aligned}|\langle B(X(t)), w\rangle| &\le \langle B(X(t)), X(t)\rangle^{1/2} \|B\|^{1/2} \|w\|_W \\ &= \left(\sum_i \langle BX(t), e_i\rangle_{V',V}^2\right)^{1/2} \|B\|^{1/2} \|w\|_W \\ &\le \sqrt{m}\,\|B\|^{1/2}\|w\|_W, \text{ so } \|BX(t)\| \le m\|B\|^{1/2}.\end{aligned}$$

Thus the integrand satisfies for a.e. t

$$\mathcal{X}_{[0,\tau_m]}\left|\left(Z\circ J^{-1}\right)^* BX\circ J\right|^2 \le m\|B\|\,\|Z\|_{\mathcal{L}_2}^2 .$$

Hence, from (2.17),

$$P\left(A\cap[\tau_m=\infty]\right) \le P\left(\omega : \int_0^T \|Z\|_{\mathcal{L}_2}^2 m\|B\|\,dt = \infty\right).$$

However,

$$\int_\Omega\int_0^T \|Z\|_{\mathcal{L}_2}^2 m\|B\|\,dtdP < \infty$$

by the assumptions on Z. Therefore, $P\left(A\cap[\tau_m=\infty]\right)=0$. It follows that

$$P(A) = \sum_m P\left(A\cap\left([\tau_m=\infty]\setminus[\tau_{m-1}<\infty]\right)\right) = \sum_m 0 = 0.$$

It follows that $P\left(\int_0^T \left|\left(Z\circ J^{-1}\right)^* BX\circ J\right|^2 dt<\infty\right)=1$ and $\int_0^t \left(Z\circ J^{-1}\right)^* BX\circ JdW$ a local martingale.

As part of Lemma 2.6, see (2.13), it was shown that $BX\in L^2\left([0,T]\times\Omega, W'\right)$. Therefore, there exist partitions of $[0,T]$ like the above such that

$$BX_k^r, BX_k^l \to BX \text{ in } L^2\left([0,T]\times\Omega, W'\right)$$

in addition to the convergence of X_k^l, X_k^r to X in K. From now on, the argument will involve a subsequence of these.

Lemma 2.8. *There exists a subsequence still denoted with the subscript k and an enlarged set of measure zero N including the earlier one such that $BX_k^l(t), BX_k^r(t)$ also converges pointwise a.e. t to $BX(t)$ in W' and $X_k^l(t), X_k^r(t)$ converge pointwise a.e. in V to $X(t)$ for $\omega\notin N$ as well as having convergence of $X_k^l(\cdot,\omega)$ to $X(\cdot,\omega)$ in $L^p([0,T];V)$ and $BX_k^l(\cdot,\omega)$ to $BX(\cdot,\omega)$ in $L^2([0,T];W)$.*

Proof: Consider a subsequence such that

$$\int_\Omega \int_0^T \left\| BX_{n_k}^r(t) - BX(t) \right\|_{W'}^2 dtdP + \int_\Omega \int_0^T \left\| X_{n_k}^r(t) - X(t) \right\|_V^p dtdP +$$

$$\int_\Omega \int_0^T \left\| BX_{n_k}^l(t) - BX(t) \right\|_{W'}^2 dtdP + \int_\Omega \int_0^T \left\| X_{n_k}^l(t) - X(t) \right\|_V^p dtdP < 4^{-k}.$$

Then use a Borel Cantelli argument to obtain the desired pointwise convergence.

We denote these subsequences as $\{X_k^r\}_{k=1}^\infty, \{X_k^l\}_{k=1}^\infty$, and prove

Lemma 2.9. *In the above context, let $X(s) - X_k^l(s) \triangleq \Delta_k(s)$. Then the integral*

$$\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW$$

exists as a local martingale and the following limit is valid for the subsequence of Lemma 2.8

$$\lim_{k\to\infty} P\left(\left[\sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k \circ JdW\right| \geq \varepsilon\right]\right) = 0.$$

That is,

$$\sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k \circ JdW\right|$$

converges to 0 in probability.

Proof: Let τ_m be as in (2.16) and

$$A_k \triangleq \left\{\omega : \sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k \circ JdW\right| \geq \varepsilon\right\}.$$

Then

$$A_k \cap \{\omega : \tau_m = \infty\} \subseteq \left\{\omega : \sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k^{\tau_m} \circ JdW\right| \geq \varepsilon\right\}.$$

By BDG inequality,

$$\begin{aligned} P\left(A_k \cap \{\omega : \tau_m = \infty\}\right) &\leq \frac{C}{\varepsilon} \int_\Omega \sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k^{\tau_m} \circ JdW\right| dP \\ &\leq \frac{C}{\varepsilon} \int_\Omega \left(\int_0^T \|Z\|_{\mathcal{L}_2}^2 \|B\Delta_k^{\tau_m}\|^2 dt\right)^{1/2} dP \\ &\leq \frac{C}{\varepsilon} \left(\int_\Omega \int_0^T \|Z\|_{\mathcal{L}_2}^2 \|B\Delta_k^{\tau_m}\|^2 dtdP\right)^{1/2}. \end{aligned}$$

Recall that if $\langle Bx, x\rangle \leq m$, then $\|Bx\|_{W'} \leq m^{1/2} \|B\|^{1/2}$. Then the integrand is bounded for a.e. t by $\|Z\|_{\mathcal{L}_2}^2 4m \|B\|$. Use the result of Lemma 2.8 and the dominated convergence theorem to conclude that the above converges to 0 as $k \to \infty$. Then from the fact that $\tau_m = \infty$ for all m large enough,

$$P(A_k) = \sum_{m=1}^{\infty} P(A_k \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty])).$$

Now $\sum_m P([\tau_m = \infty] \setminus [\tau_{m-1} < \infty]) = 1$ and so, one can apply the dominated convergence theorem to conclude that

$$\lim_{k\to\infty} P(A_k) = \sum_{m=1}^{\infty} \lim_{k\to\infty} P(A_k \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty])) = 0.$$

Lemma 2.10. *Let X_k^l be as in Lemma 2.1 corresponding to X above. Let X_k^l and X_k^r both converge to X in K and also*

$$BX_k^l, BX_k^r \to BX \text{ in } L^2([0,T] \times \Omega, W').$$

Say

$$X_k^l(t) = \sum_{j=0}^{m_k} X(t_j) \mathcal{X}_{[t_j,t_{j+1})}(t), \tag{2.18}$$

$$BX_k^l(t) = \sum_{j=0}^{m_k} BX(t_j) \mathcal{X}_{[t_j,t_{j+1})}(t). \tag{2.19}$$

Then the sum in (2.19) is progressively measurable into W'. As mentioned earlier, we can take $X(0) \triangleq 0$ in the definition of the "left step function".

Proof: This follows right away from the definition of progressively measurable.

We shall take a further subsequence.

Lemma 2.11. *Let $X(s) - X_k^l(s) \triangleq \Delta_k(s)$. Then the following limit occurs.*

$$\lim_{k\to\infty} P\left(\left[\sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k \circ JdW\right| \geq \varepsilon\right]\right) = 0.$$

The stochastic integral

$$\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW$$

makes sense because BX is W' progressively measurable and is in $L^2([0,T] \times \Omega; W')$. Also, there exists a further subsequence, still denoted as k such that

$$\int_0^t \left(Z \circ J^{-1}\right)^* BX_k^l \circ JdW \to \int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW$$

uniformly on $[0,T]$ for a.e. ω.

Proof: This follows from Lemma 2.9. The last conclusion follows from the usual use of the Borel Cantelli lemma. We obtain a further subsequence, still denoted with k such that

$$P\left(\left[\sup_{t\in[0,T]} \left|\int_0^t \left(Z \circ J^{-1}\right)^* B\Delta_k \circ JdW\right| \geq \frac{1}{k}\right]\right) < 2^{-k}.$$

From now on, the sequence will either be this subsequence or a further subsequence.

2.5. The implicit Ito formula

First, the Ito formula is valid on the partition points:

Lemma 2.12. *Let D be the union of all the positive mesh points for all the $\mathcal{P}_k$. Also assume $X_0 \in L^2(\Omega; W)$. Then for $\omega \notin N$ the exceptional set of measure zero in Ω and every $t \in D$,*

$$\langle BX(t), X(t)\rangle = \langle BX_0, X_0\rangle + \int_0^t \left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2}\right) ds$$

$$+2\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW, \tag{2.20}$$

where $\langle BZ, Z\rangle_{\mathcal{L}_2} \triangleq \left(R^{-1}BZ, Z\right)_{\mathcal{L}_2(Q^{1/2}U,W)}$ for R the Riesz map from W to W'.

Proof: Note first that for $\{g_i\}$ an orthonormal basis for $Q^{1/2}(U)$,

$$\left(R^{-1}BZ, Z\right)_{\mathcal{L}_2} \triangleq \sum_i \left(R^{-1}BZ(g_i), Z(g_i)\right)_W = \sum_i \langle BZ(g_i), Z(g_i)\rangle_{W'W} \geq 0.$$

Let $t \in D$. Then $t \in \mathcal{P}_k$ for all k large enough. Consider (2.14),

$$\langle BX(t), X(t)\rangle - \langle BX_0, X_0\rangle = e(k) + 2\int_0^t \langle Y(u), X_k^r(u)\rangle du$$

$$+2\int_0^t \left(Z \circ J^{-1}\right)^* BX_k^l \circ JdW + \sum_{j=0}^{q_k-1} \langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j)\rangle$$

$$-\sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), \Delta X(t_j) - \Delta M(t_j)\rangle \tag{2.21}$$

where $t_{q_k} = t,\ \Delta X(t_j) = X(t_{j+1}) - X(t_j)$ and $e(k) \to 0$ in probability. By Lemma 2.11 the stochastic integral on the right converges uniformly for $t \in [0, T]$ to

$$2\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW$$

for ω off a set of measure zero. The deterministic integral on the right converges uniformly for $t \in [0, T]$ to

$$2\int_0^t \langle Y(u), X(u)\rangle du$$

thanks to Lemma 2.8.

$$\left|\int_0^t \langle Y(u), X(u)\rangle du - \int_0^t \langle Y(u), X_k^r(u)\rangle du\right| \leq \int_0^T \|Y(u)\|_{V'} \|X(u) - X_k^r(u)\|_V$$
$$\leq \|Y\|_{L^{p'}([0,T])} 2^{-k}$$

for all k large enough. Consider the fourth term. It equals

$$\sum_{j=0}^{q_k-1} \left(R^{-1}B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j)\right)_W, \tag{2.22}$$

where R^{-1} is the Riesz map from W to W'. This equals

$$\frac{1}{4}\left(\sum_{j=0}^{q_k-1}\left\|R^{-1}BM\left(t_{j+1}\right)+M\left(t_{j+1}\right)-\left(R^{-1}BM\left(t_j\right)+M\left(t_j\right)\right)\right\|^2\right.$$
$$\left.-\sum_{j=0}^{q_k-1}\left\|R^{-1}BM\left(t_{j+1}\right)-M\left(t_{j+1}\right)-\left(R^{-1}BM\left(t_j\right)-M\left(t_j\right)\right)\right\|^2\right).$$

From Proposition 2.3, as $k\to\infty$, the above converges in probability to ($t_{q_k}=t$)

$$\frac{1}{4}\left(\left[R^{-1}BM+M\right](t)-\left[R^{-1}BM-M\right](t)\right).$$

However, from the description of the quadratic variation of M, the above equals

$$\frac{1}{4}\left(\int_0^t\left\|R^{-1}BZ+Z\right\|_{\mathcal{L}_2}^2 ds-\int_0^t\left\|R^{-1}BZ-Z\right\|_{\mathcal{L}_2}^2 ds\right),$$

which equals

$$\int_0^t\left(R^{-1}BZ,Z\right)_{\mathcal{L}_2}ds\triangleq\int_0^t\langle BZ,Z\rangle_{\mathcal{L}_2}ds.$$

This is what was desired.

Note that in the case of a Gelfand triple, when $W=H=H'$, the term $\langle BZ,Z\rangle_{\mathcal{L}_2}$ will end up reducing to nothing more than $\|Z\|_{\mathcal{L}_2}^2$.

Thus all the terms in (2.21) converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

$$\langle BX(t),X(t)\rangle-\langle BX_0,X_0\rangle=2\int_0^t\langle Y(u),X(u)\rangle du$$
$$+2\int_0^t\left(Z\circ J^{-1}\right)^*BX\circ JdW+\int_0^t\langle BZ,Z\rangle_{\mathcal{L}_2}ds-a$$

where a is the limit in probability of the term

$$\sum_{j=1}^{q_k-1}\left\langle B\left(\Delta X\left(t_j\right)-\Delta M\left(t_j\right)\right),\Delta X\left(t_j\right)-\Delta M\left(t_j\right)\right\rangle. \tag{2.23}$$

Let P_n be the projection onto $\operatorname{span}(e_1,\cdots,e_n)$ where $\{e_k\}$ is an orthonormal basis for W with each $e_k\in V$. Then using

$$BX\left(t_{j+1}\right)-BX\left(t_j\right)-\left(BM\left(t_{j+1}\right)-BM\left(t_j\right)\right)=\int_{t_j}^{t_{j+1}}Y(s)\,ds,$$

the troublesome term of (2.23) above is of the form

$$\sum_{j=1}^{q_k-1}\int_{t_j}^{t_{j+1}}\left\langle Y(s),\Delta X\left(t_j\right)-\Delta M\left(t_j\right)\right\rangle ds$$

$$= \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), \Delta X(t_j) - P_n \Delta M(t_j) \rangle \, ds$$

$$+ \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), -(I - P_n) \Delta M(t_j) \rangle \, ds,$$

which equals

$$\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), X(t_{j+1}) - X(t_j) - P_n (M(t_{j+1}) - M(t_j)) \rangle \, ds \tag{2.24}$$

$$+ \sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), -(I - P_n)(M(t_{j+1}) - M(t_j)) \rangle. \tag{2.25}$$

The reason for the P_n is to get $P_n(M(t_{j+1}) - M(t_j))$ in V. The sum in (2.25) is dominated by

$$\left(\sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), (\Delta X(t_j) - \Delta M(t_j)) \rangle \right)^{1/2}$$

$$\cdot \left(\sum_{j=1}^{q_k-1} \left| \langle B(I - P_n) \Delta M(t_j), (I - P_n) \Delta M(t_j) \rangle \right|^2 \right)^{1/2}. \tag{2.26}$$

Now it is known from the above that

$$\sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), (\Delta X(t_j) - \Delta M(t_j)) \rangle$$

converges in probability to $a \geq 0$. Taking expectation of the square of the other factor, it is no larger than

$$\|B\| E \left(\sum_{j=1}^{q_k-1} \|(I - P_n) \Delta M(t_j)\|_W^2 \right)$$

$$= \|B\| E \left(\sum_{j=1}^{q_k-1} \left\| (I - P_n) \int_{t_j}^{t_{j+1}} Z(s) \, dW(s) \right\|_W^2 \right)$$

$$= \|B\| \sum_{j=1}^{q_k-1} E \left(\left\| \int_{t_j}^{t_{j+1}} (I - P_n) Z(s) \, dW(s) \right\|^2 \right)$$

$$= \|B\| \sum_{j=1}^{q_k-1} E \left(\int_{t_j}^{t_{j+1}} \|(I - P_n) Z(s)\|_{\mathcal{L}_2(Q^{1/2}U, W)}^2 \, ds \right)$$

$$\leq \|B\| E \left(\int_0^T \|(I - P_n) Z(s)\|_{\mathcal{L}_2(Q^{1/2}U, H)}^2 \, ds \right)$$

letting $\{g_i\}$ be an orthonormal basis for $Q^{1/2}U$,

$$= \|B\| \int_\Omega \int_0^T \sum_{i=1}^\infty \|(I - P_n) Z(s)(g_i)\|_W^2 \, dsdP. \tag{2.27}$$

The integrand $\sum_{i=1}^\infty \|(I - P_n) Z(s)(g_i)\|_W^2$ converges to 0 as $n \to \infty$. Also, it is dominated by

$$\sum_{i=1}^\infty \|Z(s)(g_i)\|_W^2 \triangleq \|Z\|^2_{\mathcal{L}_2(Q^{1/2}U,W)},$$

which is given to be in $L^1([0,T] \times \Omega)$. Therefore, from the dominated convergence theorem, the expression in (2.27) converges to 0 as $n \to \infty$.

Thus the expression in (2.26) is of the form $f_k g_{nk}$ where f_k converges in probability to $a^{1/2}$ as $k \to \infty$ and g_{nk} converges in probability to 0 as $n \to \infty$ independently of k. Now this implies that $f_k g_{nk}$ converges in probability to 0.

Now consider the other term (2.24) using the n just determined. This term is of the form

$$\sum_{j=1}^{q_k - 1} \int_{t_j}^{t_{j+1}} \langle Y(s), X(t_{j+1}) - X(t_j) - P_n(M(t_{j+1}) - M(t_j))\rangle \, ds$$

$$\begin{aligned} &= \sum_{j=1}^{q_k - 1} \int_{t_j}^{t_{j+1}} \left\langle Y(s), X_k^r(s) - X_k^l(s) - P_n\left(M_k^r(s) - M_k^l(s)\right)\right\rangle ds \\ &= \int_{t_1}^{t} \left\langle Y(s), X_k^r(s) - X_k^l(s) - P_n\left(M_k^r(s) - M_k^l(s)\right)\right\rangle ds, \end{aligned}$$

where M_k^r denotes the step function $M_k^r(t) = \sum_{i=0}^{m_k - 1} M(t_{i+1}) \mathcal{X}_{(t_i, t_{i+1}]}(t)$ and M_k^l is defined similarly. The term $\int_{t_1}^t \left\langle Y(s), P_n\left(M_k^r(s) - M_k^l(s)\right)\right\rangle ds$ converges to 0 for $a.e.$ ω as $k \to \infty$ thanks to continuity of $t \to M(t)$. However, more is needed than this. Define the stopping time

$$\tau_p = \inf\{t > 0 : \|M(t)\|_W > p\}.$$

Then $\tau_p = \infty$ for all p large enough, this for $a.e.$ ω. Let

$$A_k = \left[\left|\int_{t_1}^t \left\langle Y(s), P_n\left(M_k^r(s) - M_k^l(s)\right)\right\rangle ds\right| > \varepsilon\right],$$

$$P(A_k) = \sum_{p=0}^\infty P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])). \tag{2.28}$$

Now

$$\begin{aligned} &P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) \le P(A_k \cap ([\tau_p = \infty])) \\ &\le P\left(\left[\left|\int_{t_1}^t \left\langle Y(s), P_n\left((M^{\tau_p})_k^r(s) - (M^{\tau_p})_k^l(s)\right)\right\rangle ds\right| > \varepsilon\right]\right). \end{aligned}$$

This is so because if $\tau_p = \infty$, then it has no effect but also it could happen that the defining inequality may hold even if $\tau_p < \infty$ hence the inequality. This is no larger than an expression of the form

$$\frac{C_n}{\varepsilon} \int_\Omega \int_0^T \|Y(s)\|_{V'} \left\| (M^{\tau_p})_k^r(s) - (M^{\tau_p})_k^l(s) \right\|_W dsdP. \tag{2.29}$$

The inside integral converges to 0 by continuity of M. Also, thanks to the stopping time, the inside integral is dominated by an expression of the form

$$\int_0^T \|Y(s)\|_{V'} 2pds$$

and this is a function in $L^1(\Omega)$ by assumption on Y. It follows that the integral in (2.29) converges to 0 as $k \to \infty$ by the dominated convergence theorem. Hence

$$\lim_{k\to\infty} P(A_k \cap ([\tau_p = \infty])) = 0.$$

Since the sets $[\tau_p = \infty] \setminus [\tau_{p-1} < \infty]$ are disjoint, the sum of their probabilities is finite. Hence there is a dominating function in (2.28) and so, by the dominated convergence theorem applied to the sum,

$$\lim_{k\to\infty} P(A_k) = \sum_{p=0}^{\infty} \lim_{k\to\infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) = 0.$$

Thus $\int_{t_1}^t \left\langle Y(s), P_n\left(M_k^r(s) - M_k^l(s)\right)\right\rangle ds$ converges to 0 in probability as $k \to \infty$.

Now consider

$$\begin{aligned}\left|\int_{t_1}^t \left\langle Y(s), X_k^r(s) - X_k^l(s)\right\rangle ds\right| &\le \int_0^T |\langle Y(s), X_k^r(s) - X(s)\rangle| ds \\ &\quad + \int_0^T \left|\left\langle Y(s), X_k^l(s) - X(s)\right\rangle\right| ds \\ &\le 2\|Y(\cdot,\omega)\|_{L^{p'}(0,T)} 2^{-k}\end{aligned}$$

for all k large enough, this by Lemma 2.8. Therefore,

$$\sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), \Delta X(t_j) - \Delta M(t_j)\rangle$$

converges to 0 in probability. This establishes the desired formula for $t \in D$.

Finally, we prove that the formula (2.20) is valid for all $t \in N_\omega^C$ and complete

Proof. [Proof of Theorem 1.1] Let $t \in N_\omega^C \setminus D$. For $t > 0$, let $t(k)$ denote the largest point of $\mathcal{P}_k$ which is less than t. Suppose $t(m) < t(k)$. Hence $m \le k$. Then

$$BX(t(m)) = BX_0 + \int_0^{t(m)} Y(s)\,ds + B\int_0^{t(m)} Z(s)\,dW(s),$$

a similar formula holding for $X(t(k))$. Thus for $t > t(m), t \notin N_\omega$,

$$B(X(t) - X(t(m))) = \int_{t(m)}^{t} Y(s)\,ds + B\int_{t(m)}^{t} Z(s)\,dW(s)$$

which is the same sort of thing studied so far except that it starts at $t(m)$ rather than at 0 and $BX_0 = 0$. Therefore, from Lemma 2.12 it follows

$$\begin{aligned}
&\langle B(X(t(k)) - X(t(m))), X(t(k)) - X(t(m))\rangle \\
&= \int_{t(m)}^{t(k)} \left(2\langle Y(s), X(s) - X(t(m))\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2}\right) ds \\
&+2\int_{t(m)}^{t(k)} \left(Z\circ J^{-1}\right)^* B(X(s) - X(t(m)))\circ JdW.
\end{aligned} \tag{2.30}$$

Consider that last term. It equals

$$2\int_{t(m)}^{t(k)} \left(Z\circ J^{-1}\right)^* B\left(X(s) - X_m^l(s)\right)\circ JdW. \tag{2.31}$$

This is dominated by

$$\begin{aligned}
&2\left|\int_0^{t(k)} \left(Z\circ J^{-1}\right)^* B\left(X(s) - X_m^l(s)\right)\circ JdW\right. \\
&\left.-\int_0^{t(m)} \left(Z\circ J^{-1}\right)^* B\left(X(s) - X_m^l(s)\right)\circ JdW\right| \\
&\leq 4\sup_{t\in[0,T]}\left|\int_0^{t} \left(Z\circ J^{-1}\right)^* B\left(X(s) - X_m^l(s)\right)\circ JdW\right|.
\end{aligned}$$

In Lemma 2.11 the above expression was shown to converge to 0 in probability. Therefore, by the usual appeal to the Borel Canteli lemma, there is a subsequence still referred to as $\{m\}$, such that it converges to 0 pointwise in ω for all ω off some set of measure 0 as $m \to \infty$. It follows that there is a set of measure 0 including the earlier one such that for ω not in that set, (2.31) converges to 0 in $\mathbb{R}$. Similar reasoning shows that the first term on the right in the non stochastic integral of (2.30) is dominated by an expression of the form

$$4\int_0^T \left|\left\langle Y(s), X(s) - X_m^l(s)\right\rangle\right| ds,$$

which clearly converges to 0 thanks to Lemma 2.8. Finally, it is obvious that

$$\lim_{m\to\infty}\int_{t(m)}^{t(k)} \langle BZ, Z\rangle_{\mathcal{L}_2}\,ds = 0 \text{ for a.e. } \omega$$

due to the assumptions on Z. For $\{g_i\}$ an orthonormal basis of $Q^{1/2}(U)$,

$$\langle BZ, Z\rangle_{\mathcal{L}_2} \triangleq \sum_i \left(R^{-1}BZ(g_i), Z(g_i)\right) = \sum_i \langle BZ(g_i), Z(g_i)\rangle$$

$$\leq \|B\| \sum_i \|Z(g_i)\|_W^2 \in L^1(0,T) \text{ a.e.}$$

This shows that for ω off a set of measure 0

$$\lim_{m,k\to\infty} \langle B(X(t(k)) - X(t(m))), X(t(k)) - X(t(m))\rangle = 0.$$

Then for $x \in W$,

$$\begin{aligned} &|\langle B(X(t(k)) - X(t(m))), x\rangle| \\ &\leq \langle B(X(t(k)) - X(t(m))), X(t(k)) - X(t(m))\rangle^{1/2} \langle Bx, x\rangle^{1/2} \\ &\leq \langle B(X(t(k)) - X(t(m))), X(t(k)) - X(t(m))\rangle^{1/2} \|B\|^{1/2} \|x\|_W \end{aligned}$$

and so

$$\lim_{m,k\to\infty} \|BX(t(k)) - BX(t(m))\|_{W'} = 0.$$

Recall t is arbitrary in N_ω^C and $\{t(k)\}$ is a sequence converging to t. Then the above has shown that $\{BX(t(k))\}_{k=1}^\infty$ is a convergent sequence in W'. Does it converge to $BX(t)$? Let $\xi(t) \in W'$ be what it converges to. Letting $v \in V$ then, since the integral equation shows that $t \to BX(t)$ is continuous into V',

$$\langle \xi(t), v\rangle = \lim_{k\to\infty} \langle BX(t(k)), v\rangle = \langle BX(t), v\rangle,$$

and now, since V is dense in W, this implies $\xi(t) = BX(t) = B(X(t))$. Recall also that it was shown earlier that BX is weakly continuous into W'. Hence, the strong convergence of $\{BX(t(k))\}_{k=1}^\infty$ in W' implies that it converges to $BX(t)$, this for any $t \in N_\omega^C$.

For every $t \in D$ and for ω off the exceptional set of measure zero described earlier,

$$\langle B(X(t)), X(t)\rangle = \langle BX_0, X_0\rangle + \int_0^t \left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2} ds\right) ds$$

$$+2\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ JdW. \tag{2.32}$$

Does this formula hold for all $t \in [0,T]$? Maybe not. However, it will hold for $t \notin N_\omega$. Indeed, let $t \notin N_\omega$.

$$|\langle BX(t(k)), X(t(k))\rangle - \langle BX(t), X(t)\rangle|$$

$$\begin{aligned} &\leq |\langle BX(t(k)), X(t(k))\rangle - \langle BX(t), X(t(k))\rangle| \\ &\quad + |\langle BX(t), X(t(k))\rangle - \langle BX(t), X(t)\rangle| \end{aligned}$$

$$= |\langle B(X(t(k)) - X(t)), X(t(k))\rangle| + |\langle B(X(t(k)) - X(t)), X(t)\rangle|$$

using the Cauchy Schwarz inequality on each term,

$$\begin{aligned} &\leq \langle B(X(t(k)) - X(t)), X(t(k)) - X(t)\rangle^{1/2} \\ &\quad \cdot \left(\langle BX(t(k)), X(t(k))\rangle^{1/2} + \langle BX(t), X(t)\rangle^{1/2}\right). \end{aligned}$$

As before, one can use the lower semicontinuity of

$$t \to \langle B(X(t(k)) - X(t)), X(t(k)) - X(t)\rangle$$

on N_ω^C along with the boundedness of $\langle BX(t), X(t)\rangle$ also shown earlier **off** N_ω to conclude

$$\begin{aligned}&|\langle BX(t(k)), X(t(k))\rangle - \langle BX(t), X(t)\rangle|\\ &\leq C\langle B(X(t(k)) - X(t)), X(t(k)) - X(t)\rangle^{1/2}\end{aligned}$$

$$\leq C \lim\inf_{m\to\infty} \langle B(X(t(k)) - X(t(m))), X(t(k)) - X(t(m))\rangle^{1/2} < \varepsilon$$

provided k is sufficiently large. Since ε is arbitrary,

$$\lim_{k\to\infty} \langle BX(t(k)), X(t(k))\rangle = \langle BX(t), X(t)\rangle.$$

It follows that the formula (2.32) is valid for all $t \notin N_\omega$. Now define the function $\langle BX, X\rangle(t)$ as

$$\langle BX, X\rangle(t) \triangleq \begin{cases} \langle B(X(t)), X(t)\rangle, t \notin N_\omega, \\ \text{The right side of (2.32) if } t \in N_\omega. \end{cases}$$

Then in short, $\langle BX, X\rangle(t)$ equals the right side of (2.32) for all $t \in [0, T]$ and is consequently progressively measurable and continuous. Furthermore, for a.e. t, this function equals $\langle B(X(t)), X(t)\rangle$. Since it is known on a dense subset, it must be unique.

This implies that $t \to BX(t)$ is continuous with values in W' for $t \notin N_\omega$, which we justify next.

The fact that the formula (2.32) holds for all $t \notin N_\omega$ implies that $t \to \langle BX(t), X(t)\rangle$ is continuous on N_ω^C. Then for $x \in W$,

$$|\langle BX(t) - BX(s), x\rangle| \leq \langle B(X(t) - X(s)), X(t) - X(s)\rangle^{1/2} \|B\|^{1/2} \|x\|_W. \tag{2.33}$$

Also

$$\begin{aligned}&\langle B(X(t) - X(s)), X(t) - X(s)\rangle\\ &= \langle BX(t), X(t)\rangle + \langle BX(s), X(s)\rangle - 2\langle BX(t), X(s)\rangle.\end{aligned}$$

By weak continuity of $t \to BX(t)$ shown earlier,

$$\lim_{t\to s} \langle BX(t), X(s)\rangle = \langle BX(s), X(s)\rangle.$$

Therefore,

$$\lim_{t\to s} \langle B(X(t) - X(s)), X(t) - X(s)\rangle = 0$$

and so the inequality (2.33) implies the continuity of $t \to BX(t)$ into W' for $t \notin N_\omega$. Note that by assumption, this function is continuous into V' for all t. It was also shown that it is weakly continuous into W' on $[0, T]$ and hence it is bounded in W'.

Now consider the claim about the expectation. Since the stochastic integral

$$2\int_0^t \left(Z \circ J^{-1}\right)^* BX \circ J dW$$

is only a local martingale, it is necessary to employ a stopping time. We use the function $\langle BX, X\rangle$ to define this stopping time as

$$\tau_p \triangleq \inf\{t > 0 : \langle BX, X\rangle (t) > p\}.$$

This is the first hitting time of a continuous process and so it is a valid stopping time. Using this, leads to

$$\langle BX, X\rangle^{\tau_p}(t) = \langle BX_0, X_0\rangle + \int_0^t \mathcal{X}_{[0,\tau_p]}(s)\left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2} ds\right) ds$$

$$+2\int_0^t \mathcal{X}_{[0,\tau_p]}(s)\left(Z\circ J^{-1}\right)^* BX^{\tau_p}\circ JdW. \tag{2.34}$$

By continuity of $\langle BX, X\rangle, \tau_p = \infty$ for all p large enough. Take expectation of both sides of the above. In the integrand of the last term, BX refers to the function $BX(t,\omega) \triangleq B(X(t,\omega))$ and so it is progressively measurable because X is assumed to be so. Hence BX^{τ_p} is also progressively measurable and for a.e. Also, for a.e. $s, \|BX(s\wedge\tau_p)\|_{W'} \leq \sqrt{p}\sqrt{\|B\|}$. Therefore, one can take expectations and get

$$E\left(\langle BX, X\rangle^{\tau_p}(t)\right) = E\left(\langle BX_0, X_0\rangle\right)$$

$$+E\left(\int_0^t \mathcal{X}_{[0,\tau_p]}(s)\left(2\langle Y(s), X(s)\rangle + \langle BZ, Z\rangle_{\mathcal{L}_2} ds\right) ds\right).$$

Now let $p\to\infty$ and use the monotone convergence theorem on the left and the dominated convergence theorem on the right to obtain the desired result (1.6). The claim about the quadratic variation follows from Corollary 2.1. □

3. Application to a stochastic evolution inclusions

We apply our Ito formula to study a specific stochastic inclusion. We arrange this section as follows. In the first subsection 3.1, we explain the set up and our results. In subsection 3.2, we provide and prove some preliminaries. In the final subsection 3.3, we prove our main theorems.

3.1. Setup and background

Let U be dense in V with the embedding compact, U being a separable Hilbert space. It is always possible to get such a space. We will let $r > \max(2, \hat{p})$ where $\hat{p} \geq p$ and $\hat{p}$ will be involved in a growth estimate below, and $\mathcal{U}_r = L^r([0,T]; U)$. Also, for $I = \left[0, \hat{T}\right], \hat{T} < T$, we will denote as $\mathcal{V}_I$ the space $L^p(I; V)$ with a similar usage of this notation in other situations. If $u \in \mathcal{V}$ the symbol for $L^p([0,T]; V)$, then we will always consider $u \in \mathcal{V}_I$ also by simply considering its restriction to I. With this convention, it is clear that if u is measurable into $\mathcal{V}$ then it is also measurable into $\mathcal{V}_I$.

Next are conditions on the evolutionary set-valued operator $A : \mathcal{V}_I \to \mathcal{P}(\mathcal{V}_I')$ for $A(u,\omega)$ a convex closed set in $\mathcal{V}_I'$.

1. **growth estimate**
Assume the specific estimate for $u \in \mathcal{V}_I$.

$$\sup \left\{ \|u^*\|_{\mathcal{V}_I'} : u^* \in A(u,\omega) \right\} \leq a(\omega) + b(\omega) \|u\|_{\mathcal{V}_I}^{\hat{p}-1} \tag{3.1}$$

where $a(\omega), b(\omega)$ are nonnegative, $\hat{p} \geq p$.

2. **coercivity estimate**
Also assume the coercivity condition: valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

$$\inf \left(\int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle Bu, u \rangle \, ds : u^* \in A(u,\omega) \right)$$
$$\geq \delta(\omega) \int_0^t \|u\|_V^p \, ds - m(\omega), \tag{3.2}$$

where $m(\omega)$ is some nonnegative constant for fixed ω, and $\delta(\omega) > 0$. No uniformity in ω is necessary.

3. **limit conditions**
Let U be as before a Banach space dense and compact in V and that if $u_i \rightharpoonup u$ in $\mathcal{V}_I$ and $u_i^* \in A(u_i, \omega)$ with $(Bu_n)' \to (Bu)'$ weakly in $\mathcal{U}_{rI}'$, $(r > \max(p,2))$, then if

$$\limsup_{i\to\infty} \langle u_i^*, u_i - u \rangle_{\mathcal{V}_I', \mathcal{V}_I} \leq 0, \tag{3.3}$$

it follows that for all $v \in \mathcal{V}_I$, there exists $u^*(v) \in Au$ such that

$$\liminf_{i\to\infty} \langle u_i^*, u_i - v \rangle_{\mathcal{V}_I', \mathcal{V}_I} \geq \langle u^*(v), u - v \rangle_{\mathcal{V}_I', \mathcal{V}_I} \ . \tag{3.4}$$

Typically one obtains this kind of thing from Proposition 3.5 applied to lower order terms along with some sort of compactness of the embedding of V into W.

4. **measurability condition**
For $\omega \to u(\cdot,\omega)$ measurable into $\mathcal{V}$,

$$\omega \to A(\mathcal{X}_I u(\cdot,\omega), \omega) \text{ has a measurable selection into } \mathcal{V}_I'. \tag{3.5}$$

In some of the results to be stated, the following condition on measurability is also assumed.

Condition 3.1. For each $t \leq T$, if $\omega \to u(\cdot,\omega)$ is $\mathcal{F}_t$ measurable into $\mathcal{V}_{[0,t]}$, then there exists a $\mathcal{F}_t$ measurable selection of $A\left(\mathcal{X}_{[0,t]} u(\cdot,\omega), \omega\right)$ into $\mathcal{V}_{[0,t]}'$.

First of all, one has the following existence of measurable solutions.

Proposition 3.1. *Suppose $p > 1$ and the conditions on A,1 - 4. Also let u_0 be measurable into W and f measurable into $\mathcal{V}'$. Let $B \in \mathcal{L}(W, W')$ be nonnegative and self adjoint as described above. Let $\sigma > 0$ be small. Then there exist functions u, u^* measurable into $\mathcal{V}_{[0,T-\sigma]} \times \mathcal{V}_{[0,T-\sigma]}'$ such that $u^*(\omega) \in A\left(\mathcal{X}_{[0,T-\sigma]} u(\omega), \omega\right)$ for each ω and for $t \leq T - \sigma$, for each ω,*

$$Bu(t) - Bu_0 + \int_0^t u^*(s) \, ds = \int_0^t f(s) \, ds.$$

In case $p \geq 2$, the proof of this theorem can be obtained from the same arguments given in [15] applied to the special sequence of Lemma 3.1.

For the details of $p < 2$, see [2].

Proposition 3.1 gives an existence theorem for an evolution inclusion for each ω such that the resulting solution (u, u^*) is measurable into $\mathcal{V} \times \mathcal{V}'$. From Lemma 3.2, this means that these functions have representatives which are product measurable. The next theorem says that in fact, the unique solution is progressively measurable in the case of the progressive measurability condition 3.1.

Theorem 3.1. *Assume the above conditions, 1 - 4, and Condition 3.1. Let u_0 be $\mathcal{F}_0$ measurable and $(t,\omega) \to \mathcal{X}_{[0,t]}(t) f(t,\omega)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ product measurable into V' for each t. Also assume that for each ω, there is at most one solution (u, u^*) to the evolution equation*

$$Bu(\omega)(t) - Bu_0(\omega) + \int_0^t u^*(\cdot,\omega)\, ds = \int_0^t f(s,\omega)\, ds, \qquad u^*(\cdot,\omega) \in A(u(\cdot,\omega),\omega), \tag{3.6}$$

for $t \in [0,T]$. Then there exists a unique solution $(u(\cdot,\omega), u^*(\cdot,\omega))$ in $\mathcal{V}_{[0,T]} \times \mathcal{V}'_{[0,T]}$ to the above integral equation for each ω with $t \in (0,T)$. This solution satisfies that $(t,\omega) \to (u(t,\omega), u^*(t,\omega))$ is progressively measurable into $V \times V'$.

We prove this theorem in the last subsection. Using a routine stopping time argument, we can give an extension of this to the following proposition, the proof of which we omit. obtain the following

Proposition 3.2. *Assume the above conditions, 1 - 4, and Condition 3.1. Let u_0 be $\mathcal{F}_0$ measurable and $(t,\omega) \to \mathcal{X}_{[0,t]}(t) f(t,\omega)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$ product measurable into V' for each t. Also let $t \to q(t,\omega)$ be continuous and q is progressively measurable into $V, q(0,\omega) = 0$. Suppose that there is at most one solution to*

$$Bu(t,\omega) + \int_0^t z(s,\omega)\, ds = \int_0^t f(s,\omega)\, ds + Bu_0(\omega) + Bq(t,\omega), \tag{3.7}$$

for each ω. Then there exists a unique solution u to the above integral equation and it is progressively measurable and so is z. Moreover, for each ω, both $Bu(t,\omega) = B(u(t,\omega))$ a.e. t and $z(\cdot,\omega) \in A(u(\cdot,\omega),\omega)$. Also, for each $a \in [0,T]$,

$$Bu(t,\omega) + \int_a^t z(s,\omega)\, ds = \int_a^t f(s,\omega)\, ds + Bu(a,\omega) + Bq(t,\omega) - Bq(a,\omega)$$

Remark 3.1. In this proposition, an additive noise term involves, which, however, has values in V. The proof proceeds by using the last theorem on a modified operator A which is not possible unless q has values in V. Eventually, we want to consider the case where we have a stochastic integral with values in W. This is where it is important to have the Ito formula.

Taking $q(t,\omega) = \int_0^t \Phi_n dW$, we obtain the following

Proposition 3.3. *Assume 1 - 4, and the progressively measurable condition 3.1. Also assume there is at most one solution to the integral equation* (3.8). *Then there exists a $\mathcal{P}$ measurable u_n such that also z_n is progressively measurable*

$$Bu_n(t,\omega) - Bu_0(\omega) + \int_0^t z_n(s,\omega)\, ds = \int_0^t f(s,\omega)\, ds + B\int_0^t \Phi_n dW, \tag{3.8}$$

where for each ω, $z_n(\cdot,\omega) \in A(u_n(\cdot,\omega),\omega)$. *The function* $Bu_n(t,\omega) = B(u_n(t,\omega))$ *for a.e.* t.

To pass to a limit as $n \to \infty$, we make an assumption of monotonicity. This will also ensure uniqueness assumed above. For $z_i(\cdot,\omega) \in A(u_i,\omega)$ and for all λ large enough,

$$\langle \lambda Bu_1(t) + !z_1(t) - (\lambda Bu_2(t) + z_2(t)), u_1(t) - u_2(t)\rangle \geq \delta \|u_1(t) - u_2(t)\|_{\hat{V}}^{\alpha}, \ \alpha \geq 1 \tag{3.9}$$

where $\hat{V}$ will be a Banach space such that V is dense in $\hat{V}$ and the embedding is continuous. As mentioned, this is not surprising in the case of most interest where there is a Gelfand triple, $V \subseteq H = H' \subseteq V'$, $B = I$, and A does not involve memory terms. One simply takes $\hat{V} = H$ and assumes that $\lambda I + A(\cdot,\omega)$ is monotone. Then if this extra monotonicity holds, we can pass to a limit and obtain the following theorem in which we specialize the growth condition to

$$\sup\left\{\|u^*\|_{\mathcal{V}_I'} : u^* \in A(u,\omega)\right\} \leq a(\omega) + b(\omega)\|u\|_{\mathcal{V}_I}^{p-1}. \tag{3.10}$$

Our main result on stochastic inclusion is the following

Theorem 3.2. *Assume 1 - 4, in which 1 is replaced with* (3.10). *Also assume Condition 3.1, and monotonicity condition* (3.9). *Then there exists a* $\mathcal{P}$ *measurable* u *such that also* z *is progressively measurable and*

$$Bu(t,\omega) - Bu_0(\omega) + \int_0^t z(s,\omega)\,ds = \int_0^t f(s,\omega)\,ds + B\int_0^t \Phi dW \tag{3.11}$$

where for each ω, $z(\cdot,\omega) \in A(u(\cdot,\omega),\omega)$. *The function* $Bu(t,\omega) = B(u(t,\omega))$ *for a.e.* t. *Here*

$$\Phi \in L^{\alpha}\left(\Omega; L^{\infty}\left([0,T], \mathcal{L}_2\left(Q^{1/2}U, W\right)\right)\right) \cap L^2\left([0,T]\times\Omega, \mathcal{L}_2\left(Q^{1/2}U, W\right)\right).$$

Remark 3.2. We could include the more general one used earlier by introducing a regularizing term εF where F is a duality map from U to U' for a suitable Hilbert space U which imbedds into V. Take a limit as $\varepsilon \to 0$ and use some of the same arguments.

A stopping time argument and uniqueness for fixed ω yield

Corollary 3.1. *Instead of letting*

$$\Phi \in L^{\alpha}\left(\Omega; L^{\infty}\left([0,T], \mathcal{L}_2\left(Q^{1/2}U, W\right)\right)\right) \cap L^2\left([0,T]\times\Omega, \mathcal{L}_2\left(Q^{1/2}U, W\right)\right), \alpha > 2,$$

assume that $\Phi \in L^2\left([0,T]\times\Omega; \mathcal{L}_2\left(Q^{1/2}U, W\right)\right)$ *and that* $t \to \Phi(t,\omega)$ *is continuous into* $\mathcal{L}_2\left(Q^{1/2}U, W\right)$. *Then there exists a unique solution to the integral equation* (3.11).

Remark 3.3. One can replace Φ with $\sigma(u)$ provided B maps W one to one onto W'. This includes the most common case of a Gelfand triple in which $B = I$ and $V \subseteq H = H' \subseteq V'$. Taking into consideration of the length of the current paper, we choose not to include this generality.

3.2. Preliminaries

The main difficulty in dealing with stochastic inclusions is retaining progressive measurability of limits. The following two lemmas on measurable selection was proved in [14].

Lemma 3.1. *Let V be a reflexive separable Banach space with dual V', and let p, p' be such that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let the functions $t \to u_n(t,\omega)$, for $n \in \mathbb{N}$, be in $L^p([0,T];V) \triangleq \mathcal{V}$ and $(t,\omega) \to u_n(t,\omega)$ be $\mathcal{B}([0,T]) \times \mathcal{F} \triangleq \mathcal{P}$ measurable into V. Suppose*

$$\|u_n(\cdot,\omega)\|_{\mathcal{V}} \le C(\omega),$$

for all n. Then, there exists a product measurable function u such that $t \to u(t,\omega)$ is in $\mathcal{V}$ and for each ω a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot,\omega) \to u(\cdot,\omega)$ weakly in $\mathcal{V}$.

Lemma 3.2. *Let $f(\cdot,\omega) \in \mathcal{V}'$ and assume that $\omega \to f(\cdot,\omega)$ is measurable into $\mathcal{V}'$. Then, for each ω, there exists a representative $\hat{f}(\cdot,\omega) \in \mathcal{V}'$, $\hat{f}(\cdot,\omega) = f(\cdot,\omega)$ in $\mathcal{V}'$, such that $(t,\omega) \to \hat{f}(t,\omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}$ product measurable. If $f(\cdot,\omega) \in \mathcal{V}'$ and $(t,\omega) \to f(t,\omega)$ is product measurable, then $\omega \to f(\cdot,\omega)$ is measurable into $\mathcal{V}'$. The same statement holds true when $\mathcal{V}'$ is replaced with $\mathcal{V}$.*

The following two compact embedding theorems will be used. The first one is due to Simon [22] and the second is in Lions [16].

Proposition 3.4. *Let $q > 1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Let S be defined by*

$$\Big\{u \text{ such that } ||u(t)||_E \le R \text{ for all } t \in [a,b], \text{ and } \|u(s) - u(t)\|_X \le R|t-s|^{1/q}\Big\}.$$

Thus S is bounded in $L^\infty(a,b,E)$ and in addition, the functions are uniformly Holder continuous into X. Then $S \subseteq C([a,b];W)$ and if $\{u_n\} \subseteq S$, there exists a subsequence $\{u_{n_k}\}$ which converges to a function $u \in C([a,b];W)$ in the following way.

$$\lim_{k\to\infty} ||u_{n_k} - u||_{\infty,W} = 0.$$

Proposition 3.5. *Let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Let $p \ge 1$, let $q > 1$, and define*

$$S \triangleq \{u \in L^p([a,b];E) : \text{for some } C,\ \|u(t) - u(s)\|_X \le C|t-s|^{1/q}$$
$$\text{and } ||u||_{L^p([a,b];E)} \le R\}.$$

Thus S is bounded in $L^p([a,b];E)$ and Holder continuous into X. Then S is precompact in $L^p([a,b];W)$. This means that if $\{u_n\}_{n=1}^\infty \subseteq S$, it has a subsequence $\{u_{n_k}\}$ which converges in $L^p([a,b];W)$.

These results are usually stated for a condition on the weak derivative, but here we use appropriate bounds in a Holder space. This is because the stochastic integrals typically are nowhere differentiable although a Holder condition will be available under suitable assumptions. The proofs work the same way.

Adding in stochastic integrals with values in W, now let

$$\Phi \in L^2\left([0,T] \times \Omega; \mathcal{L}_2\left(Q^{1/2}U, W\right)\right),$$

where U is some Hilbert space. Let an orthonormal basis for $Q^{1/2}U$ be $\{g_i\}$ and an orthonormal basis for W be $\{f_i\}$. Then $\{f_i \otimes g_i\}$ is an orthonormal basis for $\mathcal{L}_2\left(Q^{1/2}U, W\right)$. Hence,

$$\Phi = \sum_i \sum_j \Phi_{ij} f_i \otimes g_j,$$

where $f_i \otimes g_j (y) \triangleq (g_j, y)_{Q^{1/2}U} f_i$. Let E be a separable real Hilbert space which is dense in V. Then without loss of generality, one can assume that the orthonormal basis for W are all vectors in E. Thus for the orthogonal projection of Φ onto the closed subspace $\operatorname{span}(\{f_i \otimes g_i\}, i, j \leq n)$, given by

$$\Phi_n \triangleq \sum_{i=1}^n \sum_{j=1}^n \Phi_{ij} f_i \otimes g_j,$$

$\Phi_n \in L^2\left([0,T] \times \Omega; \mathcal{L}_2\left(Q^{1/2}U, E\right)\right)$ and also

$$\lim_{n\to\infty} \|\Phi_n - \Phi\|_{L^2\left([0,T]\times\Omega;\mathcal{L}_2\left(Q^{1/2}U,W\right)\right)} = 0$$

and $\int_0^t \Phi_n dW$ is continuous and progressively measurable into E hence into V. We can take a subsequence such that $\|\Phi_n - \Phi\|_{L^2\left([0,T]\times\Omega;\mathcal{L}_2\left(Q^{1/2}U,W\right)\right)} < 2^{-n}$ and this will be assumed.

Note that if P_n is the orthogonal projection onto $\operatorname{span}(f_1, \cdots, f_n)$, then

$$\begin{aligned}
|P_n \Phi(y)|_W &= \left| P_n \sum_i \sum_j \Phi_{ij} f_i \otimes g_j (y) \right|_W \\
&= \left| P_n \sum_i \sum_j \Phi_{ij} f_i (y, g_j) \right|_W \\
&= \left| \sum_{i=1}^n \sum_j \Phi_{ij} f_i (y, g_j) \right|_W \\
&\geq \left| \sum_{i=1}^n \sum_{j=1}^n \Phi_{ij} f_i (y, g_j) \right|_W = |\Phi_n(y)|_W .
\end{aligned}$$

Thus

$$\left| \int_s^t \Phi_n dW \right|_W \leq \left| \int_s^t P_n \Phi dW \right|_W = \left| P_n \int_s^t \Phi dW \right|_W \leq \left| \int_s^t \Phi dW \right|_W .$$

The following corollary will be useful.

Corollary 3.2. *Let Φ_n be as described above. Then*

$$\|\Phi_n(t,\omega)\|_{\mathcal{L}_2\left(Q^{1/2}U,W\right)} \leq \|\Phi(t,\omega)\|_{\mathcal{L}_2\left(Q^{1/2}U,W\right)},$$

where $\|\Phi_n(t,\omega)\|_{\mathcal{L}_2\left(Q^{1/2}U,W\right)} \uparrow \|\Phi(t,\omega)\|_{\mathcal{L}_2\left(Q^{1/2}U,W\right)}$. *Let*

$$\Phi \in L^\alpha\left(\Omega; L^\infty\left([0,T], \mathcal{L}_2\left(Q^{1/2}U, W\right)\right)\right) \cap L^2\left([0,T] \times \Omega, \mathcal{L}_2\left(Q^{1/2}U, W\right)\right),$$

where $\alpha > 2$. Here U is some Hilbert space. Then off a set of measure zero, the stochastic integrals $\int_0^t \Phi_n dW$ satisfy

$$\sup_n \sup_{t\neq s} \frac{\left\|\int_s^t \Phi_n dW\right\|}{|t-s|^{\gamma}} < C(\omega), \gamma < 1/2, \gamma = \frac{(\alpha/2)-1}{\alpha}.$$

Proof: Let, $\alpha > 2$. As explained above, $\left|\int_s^r \Phi_n dW\right| \leq \left|\int_s^r \Phi dW\right|$. Thus by the BDG inequality:

$$\sup_n \left|\int_s^r \Phi_n dW\right| \leq \left|\int_s^r \Phi dW\right|,$$

$$\begin{aligned}\int_\Omega \left(\left|\int_s^t \Phi dW\right|\right)^\alpha dP &\leq C\int_\Omega \left(\int_s^t \|\Phi\|^2 d\tau\right)^{\alpha/2} dP \\ &\leq C\int_\Omega \|\Phi\|^\alpha_{L^\infty\left([0,T],\mathcal{L}_2\left(Q^{1/2}U,H\right)\right)} |t-s|^{\alpha/2} \\ &\leq C\|\Phi\|^\alpha_{L^\alpha\left(\Omega;L^\infty\left([0,T],\mathcal{L}_2\left(Q^{1/2}U,W\right)\right)\right)} |t-s|^{\alpha/2} \\ &\triangleq C|t-s|^{\alpha/2}.\end{aligned}$$

Then by the Kolmogorov Čentsov theorem, for γ as given,

$$E\left(\sup_{0\leq s<t\leq T}\sup_n \frac{\left|\int_s^t \Phi_n dW\right|}{(t-s)^\gamma}\right) \leq E\left(\sup_{0\leq s<t\leq T} \frac{\left|\int_s^t \Phi dW\right|}{(t-s)^\gamma}\right) \leq C,$$

where $\gamma < \beta/\alpha$, $\beta + 1 = \alpha/2$. Thus for $\gamma < \frac{(\alpha/2)-1}{\alpha}$,

$$\sup_n \sup_{0\leq s<t\leq T} \frac{\left|\int_s^t \Phi_n dW\right|}{(t-s)^\gamma} \leq C(\omega)$$

for all ω off a set of measure zero.

We will also need the following lemma about measurability.

Lemma 3.3. *Suppose that f_n is progressively measurable and converges weakly to $\bar{f}$ in*

$$L^\alpha([0,T]\times\Omega, X, \mathcal{B}([0,T])\times\mathcal{F}_T), \ \alpha > 1,$$

where X is a reflexive separable Banach space. Also suppose that for each $\omega \notin N$ a set of measure zero,

$$f_n(\cdot,\omega) \to f(\cdot,\omega) \ \textit{weakly in } L^\alpha(0,T,X).$$

Then there is an enlarged set of measure zero, still denoted as N such that for $\omega \notin N$,

$$\bar{f}(\cdot,\omega) = f(\cdot,\omega) \ \textit{in } L^\alpha(0,T,X).$$

Also $\bar{f}$ is progressively measurable.

Proof: By the Pettis theorem, $\bar{f}$ is progressively measurable. Letting

$$\phi \in L^{\alpha'}\left([0,T]\times\Omega, X', \mathcal{B}\left([0,T]\right)\times\mathcal{F}_T\right),$$

it is known that for $a.e.$ ω,

$$\int_0^T \left\langle \phi\left(t,\omega\right), f_n\left(t,\omega\right)\right\rangle dt \to \int_0^T \left\langle \phi\left(t,\omega\right), f\left(t,\omega\right)\right\rangle dt.$$

Therefore, the function of ω on the right is at least $\mathcal{F}_T$ measurable. Now let $g \in L^\infty\left(\Omega, X', \mathcal{F}_T\right)$ and let $\psi \in C\left([0,T]\right)$. Then for $1 < p \leq \alpha$,

$$\begin{aligned}
&\int_\Omega \left| \int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), f_n\left(t,\omega\right)\right\rangle dt \right|^p dP \\
\leq\ & C\left(T\right)\int_\Omega \|g\|^p_{L^\infty(\Omega,X')}\int_0^T \left|\psi\left(t\right)\right|^p \left\|f_n\left(t,\omega\right)\right\|_X^p dtdP \\
\leq\ & C\left(T,g,\psi\right)\int_\Omega\int_0^T \left\|f_n\left(t,\omega\right)\right\|_X^p dtdP \leq C < \infty
\end{aligned}$$

for some C. Since $\int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), f_n\left(t,\omega\right)\right\rangle dt$ is bounded in $L^p\left(\Omega\right)$ independent of n because $\int_\Omega\int_0^T \left\|f_n\left(t,\omega\right)\right\|_X^p dtdP$ is given to be bounded, it follows that the functions

$$\omega \to \int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), f_n\left(t,\omega\right)\right\rangle dt$$

are uniformly integrable and so it follows from the Vitali convergence theorem that

$$\int_\Omega\int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), f_n\left(t,\omega\right)\right\rangle dtdP \to \int_\Omega\int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), f\left(t,\omega\right)\right\rangle dtdP.$$

But also from the assumed weak convergence to $\bar{f}$

$$\int_\Omega\int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), f_n\left(t,\omega\right)\right\rangle dtdP \to \int_\Omega\int_0^T \left\langle g\left(\omega\right)\psi\left(t\right), \bar{f}\left(t,\omega\right)\right\rangle dtdP.$$

It follows that

$$\int_\Omega \left\langle g\left(\omega\right), \int_0^T \left(f-\bar{f}\right)\psi\left(t\right)dt\right\rangle dP = 0.$$

This is true for every such $g \in L^\infty\left(\Omega, X'\right)$, and so for a fixed $\psi \in C\left([0,T]\right)$ and the Riesz representation theorem,

$$\int_\Omega \left\| \int_0^T \left(f-\bar{f}\right)\psi\left(t\right)dt\right\|_X dP = 0.$$

Therefore, there exists N_ψ such that if $\omega \notin N_\psi$, then

$$\int_0^T \left(f-\bar{f}\right)\psi\left(t\right)dt = 0.$$

Enlarge N, the exceptional set to also include $\cup_{\psi\in\mathcal{D}}N_\psi$ where $\mathcal{D}$ is a countable dense subset of $C\left([0,T]\right)$. Therefore, if $\omega \notin N$, then the above holds for all $\psi \in C\left([0,T]\right)$. It follows that for such ω, $f\left(t,\omega\right) = \bar{f}\left(t,\omega\right)$ for a.e. t thanks to density of $C\left([0,T]\right)$. Therefore, $f\left(\cdot,\omega\right) = \bar{f}\left(\cdot,\omega\right)$ in $L^\alpha\left(0,T,X\right)$ for all $\omega \notin N$.

3.3. Proof of theorems

Proof. [Proof of Theorem 3.1] First note that by Proposition 3.1, there exists a solution on $[0,T-\sigma]$ for each small $\sigma>0$. Then by uniqueness, there exists a solution on $(0,T)$. Let $\mathcal{T}$ denote subsets of $(0,T-\sigma]$ which contain $T-\sigma$ such that for $S\in\mathcal{T}$, there exists a solution u_S for each ω to the above integral equation on $[0,T-\sigma]$ such that $(t,\omega)\to\mathcal{X}_{[0,s]}(t)\,u_S(t,\omega)$ is $\mathcal{B}([0,s])\times\mathcal{F}_s$ measurable for each $s\in S$. Then $\{T-\sigma\}\in\mathcal{T}$. If S,S' are in $\mathcal{T}$, then $S\le S'$ will mean that $S\subseteq S'$ and also $u_S(t,\omega)=u_{S'}(t,\omega)$ in V for all $t\in S$, similar for u_S^* and $u_{S'}^*$. Note how we are considering a particular representative of a function in $\mathcal{V}_{[0,T-\sigma]}$ and $\mathcal{V}'_{[0,T-\sigma]}$ because of the pointwise condition. Now let $\mathcal{C}$ denote a maximal chain. Is $\cup\mathcal{C}\triangleq S_\infty$ all of $(0,T-\sigma]$? What is u_{S_∞}? Define $u_{S_\infty}(t,\omega)$ the common value of $u_S(t,\omega)$ for all S in $\mathcal{C}$, which contain $t\in S_\infty$. If $s\in S_\infty$, then it is in some $S\in\mathcal{C}$ and so the product measurability condition holds for this s. Thus S_∞ is a maximal element of the partially ordered set. Is S_∞ all of $(0,T-\sigma]$? Suppose $\hat{s}\notin S_\infty, T-\sigma>\hat{s}>0$.

From Theorem 3.1 there exists a solution to the integral equation 3.6 on $[0,\hat{s}]$ called u_1 such that $(t,\omega)\to u_1(t,\omega)$ is $\mathcal{B}([0,\hat{s}])\times\mathcal{F}_{\hat{s}}$ measurable, similar for u_1^*. By the same theorem, there is a solution on $[0,T-\sigma]$, u_2 which is $\mathcal{B}([0,T-\sigma])\times\mathcal{F}_{[0,T-\sigma]}$ measurable. Now by uniqueness, $u_2(\cdot,\omega)=u_1(\cdot,\omega)$ in $\mathcal{V}_{[0,\hat{s}]}$, similar for u_i^*. Therefore, no harm is done in re-defining u_2,u_2^* on $[0,\hat{s}]$ so that $u_2(t,\omega)=u_1(t,\omega)$, for all $t\in[0,\hat{s}]$, similar for u^*. Denote these functions as $\hat{u},\hat{u}^*$. By uniqueness, $u_{S_\infty}(\cdot,\omega)=\hat{u}(\cdot,\omega)$ in $L^p([0,\hat{s}],V)$. Thus no harm is done by re-defining $\hat{u}(s,\omega)$ to equal $u_{S_\infty}(s,\omega)$ for $s<\hat{s}$ and $u_1(\hat{s},\omega)$ at $\hat{s}$. As to $s>\hat{s}$ also redefine $\hat{u}(s,\omega)\triangleq u_{S_\infty}(s,\omega)$ for such s. By uniqueness, the two are equal in $\mathcal{V}_{[\hat{s},T-\sigma]}$ and so no change occurs in the solution of the integral equation. Now S_∞ was not maximal after all. $S_\infty\cup\{\hat{s}\}$ is larger. This contradiction shows that in fact, $S_\infty=(0,T-\sigma]$. Thus there exists a unique progressively measurable solution to 3.6 on $[0,T-\sigma]$ for each small σ. Thus we can simply use uniqueness to conclude the existence of a unique progressively measurable solution on $[0,T)$. □

Proof. [Proof of Theorem 3.2] Now apply this Ito formula to Theorem 3.3 in which we make the assumptions on $\|u_0\|\in L^2(\Omega)$ and that $f\in L^{p'}([0,T]\times\Omega;V')$ where the σ algebra is $\mathcal{P}$ the progressively measurable σ algebra, and

$$\Phi\in L^2\left(\Omega,L^2\left([0,T],\mathcal{L}_2\left(Q^{1/2}U,W\right)\right)\right),$$

which implies that the same is true of Φ_n. This yields, from the assumed estimates, an expression of the form where $\delta>0$ is a suitable constant.

$$\begin{aligned}&\frac{1}{2}\langle Bu_n,u_n\rangle(t)-\frac{1}{2}\langle Bu_0,u_0\rangle+\delta\int_0^t\|u_n(s)\|_V^p\,ds\\ \le{}&\lambda\int_0^t\langle Bu_n,u_n\rangle(s)\,ds+\int_0^t\langle f,u_n\rangle_{V',V}\,ds+\int_0^t c(s,\omega)\,ds\\ &+\int_0^t\langle B\Phi_n,\Phi_n\rangle_{\mathcal{L}_2}\,ds+M_n(t)\end{aligned}\tag{3.12}$$

where $c\in L^1([0,T]\times\Omega)$. Then taking expectations or using that part of the Ito formula,

$$\frac{1}{2}E(\langle Bu_n,u_n\rangle(t))+\delta E\left(\int_0^T\|u_n(s)\|_V^p\,ds\right)$$

$$\leq \lambda \int_0^t E\left(\langle Bu_n, u_n\rangle(s)\right) ds + \int_0^t E\left(\langle f, u_n\rangle_{V',V}\right) ds + C\left(\Phi, u_0\right).$$

Then by Gronwall's inequality and some simple manipulations,

$$E\left(\langle Bu_n, u_n\rangle(t)\right) + E\left(\int_0^T \|u_n(s)\|_V^p\, ds\right) \leq C\left(T, f, u_0, \Phi\right).$$

Then using obvious estimates and Gronwall's inequality in (3.12), this yields an inequality of the form

$$\langle Bu_n, u_n\rangle(t) - \langle Bu_0, u_0\rangle + \int_0^t \|u_n(s)\|_V^p\, ds \leq C(f,\lambda,c) + \|B\| \int_0^t \|\Phi_n\|_{\mathcal{L}_2}^2\, ds + M_n^*(t),$$

where the random variable $C(f,\lambda,c)$ is nonnegative and is integrable. Now $t \to M_n^*(t)$ is increasing as is the integral on the right. Hence it follows that, modifying the constants,

$$\begin{aligned}&\sup_{s\in[0,t]} \langle Bu_n, u_n\rangle(s) + \int_0^t \|u_n(s)\|_V^p\, ds \\ &\leq C(f,\lambda,c,u_0) + 2\|B\| \int_0^t \|\Phi_n\|_{\mathcal{L}_2}^2\, ds + 2M_n^*(t).\end{aligned} \tag{3.13}$$

Next take the expectation of both sides and use the BDG inequality along with the description of the quadratic variation of the martingale $M_n(t)$. This yields

$$\begin{aligned}&E\left(\sup_{s\in[0,t]} \langle Bu_n, u_n\rangle(s)\right) + E\left(\int_0^t \|u_n(s)\|_V^p\, ds\right) \\ &\leq C + 2\|B\| E\left(\int_0^t \|\Phi_n\|_{\mathcal{L}_2}^2\, ds\right) + C\int_\Omega \left(\int_0^t \|Bu_n\|_W^2 \|\Phi_n\|_{\mathcal{L}_2}^2\, ds\right)^{1/2} dP.\end{aligned}$$

Now $\|Bw\| = \sup_{\|v\|\leq 1} \langle Bw, v\rangle \leq \langle Bw, w\rangle^{1/2}$. Also $\int_0^t \|\Phi_n\|_{\mathcal{L}_2}^2\, ds \leq \int_0^T \|\Phi\|_{\mathcal{L}_2}^2\, ds$ and so the above inequality implies

$$\begin{aligned}&E\left(\sup_{s\in[0,t]} \langle Bu_n, u_n\rangle(s)\right) + E\left(\int_0^t \|u_n(s)\|_V^p\, ds\right) \\ &\leq C(f,\lambda,c,\Phi) + C\int_\Omega \sup_{s\in[0,t]} \langle Bu_n, u_n\rangle^{1/2}(s) \left(\int_0^t \|\Phi\|_{\mathcal{L}_2}^2\right)^{1/2} dP.\end{aligned}$$

Then adjusting the constants yields

$$\begin{aligned}&\frac{1}{2}E\left(\sup_{s\in[0,T]} \langle Bu_n, u_n\rangle(s)\right) + E\left(\int_0^T \|u_n(s)\|_V^p\, ds\right) \\ &\leq C + C\int_\Omega \int_0^T \|\Phi\|_{\mathcal{L}_2}^2\, dtdP = C.\end{aligned} \tag{3.14}$$

If needed, you could use a stopping time to be sure that

$$E\left(\sup_{s\in[0,T]} \langle Bu_n, u_n\rangle(s)\right) < \infty$$

and then let it converge to ∞.

From the integral equation,

$$Bu_n(t) - Bu_m(t) + \int_0^t z_n - z_m ds = B\int_0^t (\Phi_n - \Phi_m) dW.$$

Then using the monotonicity assumption and the Ito formula,

$$\frac{1}{2}\langle Bu_n - Bu_m, u_n - u_m\rangle(t) \leq \lambda \int_0^t \langle Bu_n - Bu_m, u_n - u_m\rangle dss$$

$$+ \int_0^t \langle B(\Phi_n - \Phi_m), \Phi_n - \Phi_m\rangle d + \int_0^t \left((\Phi_n - \Phi_m)\circ J^{-1}\right)^* B(u_n - u_m)\circ JdW$$

and so, from Gronwall's inequality, there is a constant C which is independent of m, n such that

$$\langle Bu_n - Bu_m, u_n - u_m\rangle(t) \leq CM_{nm}(t) \leq CM_{nm}^*(T) + C\int_0^t \|\Phi_n - \Phi_m\|_{\mathcal{L}_2}^2 ds,$$

where M_{nm} refers to that local martingale on the right. Thus also

$$\sup_{t\in[0,T]} \langle Bu_n - Bu_m, u_n - u_m\rangle(t) \leq CM_{nm}(t) \leq CM_{nm}^*(T) + C\int_0^T \|\Phi_n - \Phi_m\|_{\mathcal{L}_2}^2. \tag{3.15}$$

Taking the expectation and using the BDG inequality again in a similar manner to the above,

$$E\left(\sup_{t\in[0,T]} \langle Bu_n - Bu_m, u_n - u_m\rangle(t)\right) \leq C\int_\Omega\int_0^T \|\Phi_n - \Phi_m\|_{\mathcal{L}_2}^2 dtdP.$$

Now the right side converges to 0 as $m, n \to \infty$ and so there is a subsequence, denoted with the index k such that whenever $m > k$,

$$E\left(\sup_{t\in[0,T]} \langle Bu_k - Bu_m, u_k - u_m\rangle(t)\right) \leq \frac{1}{2^k}.$$

Note how this implies

$$\int_\Omega\int_0^T \langle Bu_k - Bu_m, u_k - u_m\rangle dtdP \leq \frac{T}{2^k}. \tag{3.16}$$

Then consider the martingales $M_k(t)$ considered earlier. One of these is of the form

$$M_k = \int_0^t \left(\Phi_k \circ J^{-1}\right)^* Bu_k \circ JdW.$$

Then by the Burkholder Davis Gundy inequality and modifying constants as appropriate,

$$E\left((M_k - M_{k+1})^*\right)$$

$$\leq C\int_\Omega \left(\int_0^T \left\|\left(\Phi_k\circ J^{-1}\right)^* Bu_k - \left(\Phi_{k+1}\circ J^{-1}\right)^* Bu_{k+1}\right\|^2 dt\right)^{1/2} dP$$

$$\leq C \int_\Omega \left(\begin{array}{c} \int_0^T \|\Phi_k - \Phi_{k+1}\|^2 \langle Bu_k, u_k \rangle \\ + \|\Phi_{k+1}\|^2 \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle dt \end{array} \right)^{1/2} dP$$

$$\leq C \int_\Omega \left(\int_0^T \|\Phi_k - \Phi_{k+1}\|^2 \langle Bu_k, u_k \rangle dt \right)^{1/2}$$

$$+ C \int_\Omega \left(\int_0^T \|\Phi_{k+1}\|^2 \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle dt \right)^{1/2} dP$$

$$\leq C \int_\Omega \sup_t \langle Bu_k, u_k \rangle^{1/2} \left(\int_0^T \|\Phi_k - \Phi_{k+1}\|^2 dt \right)^{1/2} dP$$

$$+ C \int_\Omega \sup_t \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle^{1/2} \left(\int_0^T \|\Phi_{k+1}\|^2 dt \right)^{1/2} dP$$

$$\leq C \left(\int_\Omega \sup_t \langle Bu_k, u_k \rangle dP \right)^{1/2} \left(\int_\Omega \int_0^T \|\Phi_k - \Phi_{k+1}\|^2 dt dP \right)^{1/2}$$

$$+ C \left(\int_\Omega \sup_t \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle dP \right)^{1/2} \left(\int_\Omega \int_0^T \|\Phi_{k+1}\|^2 dt dP \right)^{1/2}.$$

From the above inequality, (3.14) and after adjusting the constants, the above is no larger than an expression of the form $C \left(\frac{1}{2}\right)^{k/2}$ which is a summable sequence. Then

$$\sum_k \int_\Omega \sup_{t \in [0,T]} |M_k(t) - M_{k+1}(t)| \, dP < \infty.$$

Thus $\{M_k\}$ is a Cauchy sequence in M_T^1, a space of continuous martingales such that the norm is $\|M\| \triangleq E(M^*(T))$ and so there is a continuous martingale M such that

$$\lim_{k \to \infty} E\left(\sup_t |M_k(t) - M(t)| \right) = 0.$$

Taking a further subsequence if needed, one can also have

$$P\left(\sup_t |M_k(t) - M(t)| > \frac{1}{k} \right) \leq \frac{1}{2^k}$$

and so by the Borel Cantelli lemma, there is a set of measure zero such that off this set, $\sup_t |M_k(t) - M(t)|$ converges to 0. Hence for such $\omega, M_k^*(T)$ is bounded independent of k. Thus for ω off a set of measure zero, (3.13) implies that for such ω,

$$\sup_{s \in [0,T]} \langle Bu_k, u_k \rangle (s) + \int_0^T \|u_k(s)\|_V^p ds \leq C(\omega),$$

where $C(\omega)$ does not depend on the index k, this for the subsequence just described which will be the sequence of interest in what follows. Using the boundedness assumption for A, one also obtains an estimate of the form

$$\sup_{s \in [0,T]} \langle Bu_r, u_r \rangle (s) + \int_0^T \|u_r(s)\|_V^p ds + \int_0^T \|z_r\|_{V'}^{p'} \leq C(\omega). \tag{3.17}$$

The idea here is to take weak limits converging to a function u and then identify $z(\cdot,\omega)$ as being in $A(u,\omega)$ but this will involve a difficulty. It will require a use of the above Ito formula and this will need u to be progressively measurable. By uniqueness, it would seem that this could be concluded by arguing that one does not need to take a subsequence due to uniqueness but the problem is that we won't know the limit of the sequence is a solution unless we use the Ito formula. This is why we make the extra assumption that for $z_i(\cdot,\omega) \in A(u_i,\omega)$ and for all λ large enough,

$$\langle \lambda B u_1 + z_1 - (\lambda B u_2 + z_2), u_1 - u_2 \rangle \geq \delta \|u_1 - u_2\|_{\hat{V}}^{\alpha}, \ \alpha \geq 1 \tag{3.18}$$

where here $\hat{V}$ will be a Banach space such that V is dense in $\hat{V}$ and the embedding is continuous. As mentioned, this is automatic in the case of most interest where there is a Gelfand triple and $B = I$ but here, since B is not one to one, we assume it. Then using the integral equation with the conclusion of the Ito formula above,

$$\begin{aligned} &E(\langle B(u_n - u_m), u_n - u_m \rangle (t)) + E\left(\int_0^t \|u_n - u_m\|_{\hat{V}}^{\alpha} ds\right) \\ \leq\ &E\left(\int_0^t \|B\| \|\Phi_n - \Phi_m\|_{\mathcal{L}_2}^2 ds\right) \triangleq e(m,n). \end{aligned}$$

Hence, the right side converges to 0 as $m, n \to \infty$ from the dominated convergence theorem. In particular,

$$E\left(\int_0^T \|u_n - u_m\|_{\hat{V}}^{\alpha} ds\right) \leq E\left(\int_0^T \|B\| \|\Phi_n - \Phi_m\|_{\mathcal{L}_2}^2 ds\right) \triangleq e(m,n). \tag{3.19}$$

Then also

$$P\left(\int_0^T \|u_n - u_m\|_{\hat{V}}^{\alpha} ds > \lambda\right) \leq \frac{e(m,n)}{\lambda}$$

and so there exists a subsequence, denoted by r such that

$$P\left(\int_0^T \|u_r - u_{r+1}\|_{\hat{V}}^{\alpha} ds \leq 2^{-r}\right) < 2^{-r}.$$

Thus, by the Borel Cantelli lemma, there is a further enlarged set of measure zero, still denoted as N such that for $\omega \notin N$,

$$\int_0^T \|u_r - u_{r+1}\|_{\hat{V}}^{\alpha} ds \leq 2^{-r}$$

for all r large enough. Hence, by the usual proof of completeness, for these ω,

$$\{u_r(\cdot,\omega)\}$$

is Cauchy in $L^{\alpha}\left([0,T], \hat{V}\right)$ and also $u_r(t,\omega)$ converges to some $u(t,\omega)$ pointwise in $\hat{V}$ for a.e. t. In addition, from (3.19) these functions are a Cauchy sequence in $L^{\alpha}\left([0,T] \times \Omega; \hat{V}\right)$ with respect to the σ algebra of progressively measurable sets. Thus from Lemma 3.3, it can be assumed that for ω off the set of measure zero,

$(t,\omega) \to u(t,\omega)$ is progressively measurable. From now on, this will be the sequence or a further subsequence. For $\omega \notin N$, a set of measure zero and (3.17), there is a further subsequence for which the following convergences occur as $r \to \infty$.

$$u_r \to u \text{ weakly in } \mathcal{V}, \tag{3.20}$$

$$B(u_r) \to B(u) \text{ weakly in } \mathcal{V}', \tag{3.21}$$

$$z_r \to z \text{ weakly in } \mathcal{V}', \tag{3.22}$$

$$\left(B\left(u_r - \int_0^{(\cdot)} \Phi_r dW\right)\right)' \to \left(B\left(u - \int_0^{(\cdot)} \Phi dW\right)\right)' \text{ weakly in } \mathcal{V}', \tag{3.23}$$

$$\int_0^{(\cdot)} \Phi_r dW \to \int_0^{(\cdot)} \Phi dW \text{ uniformly in } C([0,T];W), \tag{3.24}$$

$$Bu_r(t) \to Bu(t) \text{ weakly in } V', \tag{3.25}$$

$$Bu(0) = Bu_0, \tag{3.26}$$

$$Bu(t) = B(u(t)) \text{ a.e. } t. \tag{3.27}$$

In addition to this, we can choose the subsequence such that

$$\sup_r \sup_{t \neq s} \frac{\left\| \int_s^t \Phi_r dW \right\|}{|t-s|^\gamma} < C(\omega) < \infty. \tag{3.28}$$

This is thanks to Corollary 3.2. The boundedness of the operator A, in particular the given estimates, imply that z_r is bounded in $L^{p'}([0,T] \times \Omega, V')$. Thus a subsequence can be obtained which yields weak convergence of z_r in $L^{p'}([0,T] \times \Omega, V')$ and then Lemma 3.3 may be applied to conclude that off a set of measure zero, z is progressively measurable.

The claim (3.25) and (3.26) follow from the continuity of the evaluation map defined on X. The claim in (3.27) follows from (3.21) and the convergence (3.25). To see this, let $\psi \in C_c^\infty(0,T)$.

$$\begin{aligned}\int_0^T Bu(t)\psi(t)\,dt &= \lim_{r\to\infty} \int_0^T Bu_r(t)\psi(t)\,dt \\ &= \lim_{r\to\infty} \int_0^T B(u_r(t))\psi(t)\,dt = \int_0^T B(u(t))\psi(t)\,dt.\end{aligned}$$

Since this is true for all such ψ, it follows that $Bu(t) = B(u(t))$ for a.e. t. Passing to a limit in the integral equation yields the following for ω off a set of measure zero,

$$Bu(t,\omega) - Bu_0(\omega) + \int_0^t z(s,\omega)\,ds = \int_0^t f(s,\omega)\,ds + B\int_0^t \Phi_n dW.$$

In the following claim, we use that $\Phi \in L^2\left(\Omega, L^\infty\left([0,T], \mathcal{L}_2\left(Q^{1/2}U, W\right)\right)\right)$.

Claim: $\lim_{r\to\infty} \int_0^T \left(\Phi_r \circ J^{-1}\right)^* Bu_r \circ J dW = \int_0^T \left(\Phi \circ J^{-1}\right)^* Bu \circ J dW$ off a set of measure zero.

Proof of claim:

$$E\left(\left|\int_0^T \left(\Phi_r\circ J^{-1}\right)^* Bu_r\circ JdW-\int_0^T \left(\Phi\circ J^{-1}\right)^* Bu\circ JdW\right|\right)$$

$$\leq E\left(\left|\int_0^T \left(\Phi_r\circ J^{-1}\right)^* Bu_r\circ JdW-\int_0^T \left(\Phi\circ J^{-1}\right)^* Bu_r\circ JdW\right|\right)$$
$$+E\left(\left|\int_0^T \left(\Phi\circ J^{-1}\right)^* Bu_r\circ JdW-\int_0^T \left(\Phi\circ J^{-1}\right)^* Bu\circ JdW\right|\right).$$

By the BDG inequality,

$$\leq \int_\Omega\left(\int_0^T\|\Phi_r-\Phi\|^2\langle Bu_r,u_r\rangle\right)^{1/2}dP+\int_\Omega\left(\int_0^T\|\Phi\|^2\langle Bu_r-Bu,u_r-u\rangle\right)^{1/2}$$

$$\leq \int_\Omega\sup_t\langle Bu_r(t),u_r(t)\rangle^{1/2}\left(\int_0^T\|\Phi_r-\Phi\|^2dt\right)^{1/2}dP$$
$$+\int_\Omega\|\Phi_n\|_{L^\infty([0,T],\mathcal{L}_2)}\left(\int_0^T\langle Bu_r-Bu,u_r-u\rangle\right)^{1/2}dP$$
$$\leq\left(\int_\Omega\sup_t\langle Bu_r(t),u_r(t)\rangle dP\right)^{1/2}\left(\int_\Omega\int_0^T\|\Phi_r-\Phi\|^2dt\right)^{1/2}$$
$$+\left(\int_\Omega\|\Phi_n\|^2_{L^\infty([0,T],\mathcal{L}_2)}\right)^{1/2}\left(\int_\Omega\int_0^T\langle Bu_r-Bu,u_r-u\rangle dtdP\right)^{1/2}. \tag{3.29}$$

Letting the e_i be the special vectors of Proposition 2.5,

$$\int_\Omega\int_0^T\langle Bu_r-Bu,u_r-u\rangle dtdP=\int_\Omega\int_0^T\sum_i\langle Bu_r-Bu,e_i\rangle^2dtdP$$
$$=\int_\Omega\int_0^T\sum_i\liminf_{p\to\infty}\langle Bu_r-Bu_p,e_i\rangle^2dtdP$$
$$\leq\liminf_{p\to\infty}\int_\Omega\int_0^T\sum_i\langle Bu_r-Bu_p,e_i\rangle^2dtdP$$
$$=\liminf_{p\to\infty}\int_\Omega\int_0^T\sum_i\langle Bu_r-Bu_p,e_i\rangle^2dtdP$$
$$=\liminf_{p\to\infty}\int_\Omega\int_0^T\langle Bu_r-Bu_p,u_r-u_p\rangle dtdP.$$

Now by (3.16), the last expression is no larger than $T/2^r$ and so

$$\int_\Omega\int_0^T\langle Bu_r-Bu,u_r-u\rangle dtdP\leq\frac{T}{2^r}.$$

Then, from (3.29),

$$E\left(\left|\int_0^T \left(\Phi_r\circ J^{-1}\right)^* Bu_r\circ JdW - \int_0^T \left(\Phi\circ J^{-1}\right)^* Bu\circ JdW\right|\right)$$

$$\leq \left(\int_\Omega \sup_t \left\langle Bu_r(t), u_r(t)\right\rangle dP\right)^{1/2}\left(\int_\Omega\int_0^T \left\|\Phi_r-\Phi\right\|^2 dt\right)^{1/2} + C\left(\frac{T}{2^r}\right)^{1/2}$$

$$\leq C\left(\int_\Omega\int_0^T \left\|\Phi_r-\Phi\right\|^2 dt\right)^{1/2} + C\left(\frac{T}{2^r}\right)^{1/2} < C2^{-r} + C\left(\frac{T}{2^r}\right)^{1/2},$$

which clearly converges to 0 as $r\to\infty$. Since the right side is summable, one obtains also pointwise convergence. This proves the claim.

From the above considerations using the space $\hat{V}$, it follows that this u is the same as the one just obtained in the sense that for ω off N, the two are equal for a.e. t. Thus we take u to be this common function. Hence there is a set of measure zero such that $(t,\omega)\to\mathcal{X}_{N^C}u(t,\omega)$ is progressively measurable in the above convergences. From the measurability of u_r, u, we can obtain a dense countable subset $\{t_k\}$ and an enlarged set of measure zero N such that for $\omega\notin N, Bu(t_k,\omega) = B(u(t_k,\omega))$ and $Bu_r(t_k,\omega) = B(u_r(t_k,\omega))$ for all t_k and r. This uses the same argument as in Lemma 2.4.

It remains to verify that $z(\cdot,\omega)\in A(u(\cdot,\omega),\omega)$. It follows from the above considerations that the Ito formula above can be used at will. Assume that for a given $\omega\notin N,\ Bu(T,\omega) = B(u(T,\omega))$, similar for Bu_r. If not, just do the following argument for all T' close to T, letting T' be in the dense subset just described. Then from the integral equation solved, and letting $\{e_i\}$ be the special set described in Proposition 2.5 and suppressing the dependence on ω,

$$\begin{aligned}&\sum_{i=1}^\infty \left\langle Bu_r(T), e_i\right\rangle^2 - \sum_{i=1}^\infty \left\langle Bu_0, e_i\right\rangle + 2\int_0^T \left\langle z_r, u_r\right\rangle ds\\ &= 2\int_0^T \left\langle f, u_r\right\rangle ds + 2\int_0^T \left(\Phi_r\circ J^{-1}\right)^* Bu_r\circ JdW.\end{aligned}$$

Thus also

$$\begin{aligned}&2\int_0^T \left\langle z_r, u_r\right\rangle ds = -\sum_{i=1}^\infty \left\langle Bu_r(T), e_i\right\rangle^2 + \sum_{i=1}^\infty \left\langle Bu_0, e_i\right\rangle\\ &+2\int_0^T \left\langle f, u_r\right\rangle ds + 2\int_0^T \left(\Phi_r\circ J^{-1}\right)^* Bu_r\circ JdW.\end{aligned} \tag{3.30}$$

A similar formula to (3.30) holds for u. Thus

$$\begin{aligned}&2\int_0^T \left\langle z, u\right\rangle ds = -\sum_{i=1}^\infty \left\langle Bu(T), e_i\right\rangle^2 + \sum_{i=1}^\infty \left\langle Bu_0, e_i\right\rangle\\ &+2\int_0^T \left\langle f, u\right\rangle ds + 2\int_0^T \left(\Phi\circ J^{-1}\right)^* Bu\circ JdW.\end{aligned}$$

It follows from (3.24) and the other convergences that

$$\limsup_{r\to\infty} \int_0^T \langle z_r, u_r \rangle \, ds \le \int_0^T \langle z, u \rangle \, ds.$$

Hence

$$\limsup_{r\to\infty} \langle z_r, u_r - u \rangle_{\mathcal{V}', \mathcal{V}} \le 0.$$

Now from the limit condition, for any $v \in \mathcal{V}$, there exists a $z(v) \in A(u(\cdot,\omega),\omega)$ such that

$$\begin{aligned}\langle z, u - v \rangle_{\mathcal{V}', \mathcal{V}} &\ge \liminf_{r\to\infty} \left(\langle z_r, u_r - u \rangle + \langle z_r, u - v \rangle \right) \\ &\ge \liminf_{r\to\infty} \langle z_r, u_r - v \rangle \ge \langle z(v), u - v \rangle .\end{aligned}$$

The reason the limit condition applies is the estimate (3.28) and the convergence (3.23) which shows that

$$B\left(u_r - \int_0^{(\cdot)} \Phi_r dW \right)$$

satisfy a Holder condition into V'. Then the estimate (3.28) implies that the $B\int_0^{(\cdot)} \Phi_r dW$ are bounded in a Holder norm and so the same is true of the Bu_r. Thus the situation of the lim inf limit condition is obtained. Then it follows from separation theorems and the fact that $A(u(\cdot,\omega),\omega)$ is closed and convex that $z(\cdot,\omega) \in A(u(\cdot,\omega),\omega)$. □

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