Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 3, September 2024, 612–622 DOI:10.12150/jnma.2024.612

Positive Solutions for a Stationary Prey-Predator Model with Density-Dependent Diffusion and Hunting Cooperation*

Zhuoru Han[1] and Shanbing Li[1,†]

Abstract This paper concerns a stationary prey-predator model with density-dependent diffusion and hunting cooperation under homogeneous Dirichlet boundary conditions. Based on the spectral analysis, the asymptotic stability of trivial and semi-trivial solutions is obtained. Moreover, the sufficient conditions for the existence of positive solutions are established by using degree theory in cones. Our analytical results suggest that density-dependent diffusion and hunting cooperation obviously influence on the positive solutions.

Keywords Prey-predator model, density-dependent diffusion, hunting cooperation, positive solutions

MSC(2010) 35J55, 92D40.

1. Introduction

The present paper is concerned with the following Dirichlet problem of quasilinear elliptic equations:

$$\begin{cases} -d_u \Delta u = ru - u^2 - (1+\alpha v)uv, & x \in \Omega, \\ -\Delta\left[\left(d_v + \frac{\beta}{1+\gamma u}\right)v\right] = mv - v^2 + c(1+\alpha v)uv, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, the parameters $d_u, d_v, \beta, \gamma, \alpha, r, c$ are positive constants and m may change sign. System (1.1) is the stationary problem of a prey-predator model in which unknown functions $u = u(x)$ and $v = v(x)$ denote the stationary population densities of the prey and the predator in the habitat Ω, respectively. In the reaction terms, r and m are the growth rates of respective species; α describes predator cooperation in hunting; c accounts for

†the corresponding author.
Email address: 20071212570@stu.xidian.edu.cn(Z. Han), lishanbing@xidian.edu.cn(B. Li)
[1]School of Mathematics and Statistics, Xidian University, Xi'an, 710071, PR China
*The work was partially supported by the National Science Foundation of China (11901446), the Postdoctoral Science Foundation of China (2021T140530), the Innovation Capability Support Program of Shaanxi (2024RS-CXTD-88), the National Science Foundation in Shaanxi Province of China (2024JC-YBMS-002), and the Fundamental Research Funds for the Central Universities (ZYTS24074).

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

the intrinsic predation rate. In the diffusion terms, $d_u\Delta u$ and $d_v\Delta v$ denote the linear diffusion driven by the dispersive force associated with random movement of each species, while the nonlinear diffusion $\Delta\left(\frac{\beta v}{1+\gamma u}\right)$ describes a situation in which the predator chases the prey: β is the cross-diffusion pressures, and γ represents the interference rate from the prey in the chase by the predator. For more details on the backgrounds of density-dependent diffusion and hunting cooperation, we refer to [1] and [11].

When $\alpha = 0$ and $\gamma = 0$, (1.1) is reduced to the classical Lotka-Volterra prey-predator model which has received extensive study in the last decade (see [2,8,9,16] and references therein). When $\alpha = 0$ and $\gamma > 0$, K. Kuto and his collaborators established the existence of positive solutions by the bifurcation theory in [4,6] and discussed the limiting behavior of positive solutions in [5, 7]. However, as far as we know, there are few works on the positive solutions in the case where $\alpha > 0$ and $\gamma > 0$. It is worth noting that, although the literature on hunting cooperation is limited till now, some recent works can be found which address the effect of cooperative hunting [3, 12, 14, 15, 17] and the references therein.

The purpose of this paper is to establish the asymptotic stability of trivial and semi-trivial solutions and provide the sufficient conditions for the existence of positive solutions. To present our main result, we introduce some notations. For any given $d > 0$ and $q(x) \in C(\overline{\Omega})$, the eigenvalue problem

$$-d\Delta\phi + q(x)\phi = \lambda\phi, \quad x \in \Omega, \qquad \phi = 0, \quad x \in \partial\Omega$$

has an infinite sequence of eigenvalues denoted by $\{\lambda_i(d, q(x))\}_{i=1}^{\infty}$. Additionally, for any given $d > 0$, the logistic equation

$$-d\Delta\phi = a\phi - \phi^2, \quad x \in \Omega, \qquad \phi = 0, \quad x \in \partial\Omega$$

admits a unique positive solution if and only if $\lambda_1(d, -a) < 0$, which is denoted by $\theta_{d,a}$.

Our first theorem gives the asymptotic stability of trivial and semi-trivial solutions.

Theorem 1.1. *The following statements hold true.*

(1) *Trivial solution $(0, 0)$ is asymptotically stable if $\lambda_1(d_u, -r) > 0$ and $\lambda_1(d_v + \beta, -m) > 0$, while it is unstable if $\lambda_1(d_u, -r) < 0$ or $\lambda_1(d_v + \beta, -m) < 0$.*

(2) *Assume that $\lambda_1(d_v + \beta, -m) < 0$. Then $(0, \theta_{d_v+\beta,m})$ is asymptotically stable if*
$$\lambda_1(d_u, (1 + \alpha\theta_{d_v+\beta,m})\theta_{d_v+\beta,m} - r) > 0;$$
while it is unstable if
$$\lambda_1(d_u, (1 + \alpha\theta_{d_v+\beta,m})\theta_{d_v+\beta,m} - r) < 0.$$

(3) *Assume that $\lambda_1(d_u, -r) < 0$. Then $(\theta_{d_u,r}, 0)$ is asymptotically stable if*
$$\lambda_1\left(1, -\frac{(m + c\theta_{d_u,r})(1 + \gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta}\right) > 0;$$
while it is unstable if
$$\lambda_1\left(1, -\frac{(m + c\theta_{d_u,r})(1 + \gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta}\right) < 0.$$

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

Our second theorem yields sufficient conditions for the existence of positive solutions.

Theorem 1.2. *Suppose that* $c\alpha r < 1$ *and* $\lambda_1(d_u, -r) < 0$. *Then the following statements hold true.*

(1) Assume that $\lambda_1(d_v + \beta, -m) < 0$. *Then (1.1) has a positive solution if*

$$\lambda_1(d_u, (1 + \alpha\theta_{d_v+\beta,m})\theta_{d_v+\beta,m} - r) < 0.$$

(2) Assume that $\lambda_1(d_v + \beta, -m) > 0$. *Then (1.1) has a positive solution if*

$$\lambda_1\left(1, -\frac{(m + c\theta_{d_u,r})(1 + \gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta}\right) < 0.$$

(3) Assume that $\lambda_1(d_v + \beta, -m) = 0$. *Then (1.1) always has a positive solution.*

Remark 1.1. Theorem 1.1 and Theorem 1.2 imply that the fixed point index of a trivial or semi-trivial solution is 1 if it is stable and 0 if it is unstable, where the stability means the linearized stability of any non-negative solution of (1.1).

2. Proof of main results

2.1. Proof of Theorem 1.1

In this subsection we will complete the proof of Theorem 1.1 by analysing the spectrum of the linearized operator around some non-negative solution of (1.1). Since these three cases in Theorem 1.1 can be studied in a similar manner, we only prove (3).

Proof of Theorem 1.1(3). The linearized system of (1.1) with respect to (u, v) at $(\theta_{d_u,r}, 0)$ is given by

$$\begin{cases} -d_u\Delta\phi - (r - 2\theta_{d_u,r})\phi + \theta_{d_u,r}\psi = \lambda\phi, & x \in \Omega, \\ -\Delta\left[\left(d_v + \frac{\beta}{1+\gamma\theta_{d_u,r}}\right)\psi\right] - (m + c\theta_{d_u,r})\psi = \lambda\psi, & x \in \Omega, \\ \phi = \psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Let

$$\Psi = \left(d_v + \frac{\beta}{1 + \gamma\theta_{d_u,r}}\right)\psi.$$

Then (2.1) can be expressed as

$$\begin{cases} -d_u\Delta\phi - (r - 2\theta_{d_u,r})\phi + \frac{\theta_{d_u,r}(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\Psi = \lambda\phi, & x \in \Omega, \\ -\Delta\Psi - \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\Psi = \lambda\frac{1+\gamma\theta_{d_u,r}}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\Psi, & x \in \Omega, \\ \phi = \Psi = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

If $\Psi = 0$, then

$$\lambda = \lambda_i(d_u, -r + 2\theta_{d_u,r}) \geq \lambda_1(d_u, -r + 2\theta_{d_u,r})$$

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

for some $i \geq 1$. Since $\theta_{d_u,r}$ is stable, it follows from the monotonicity of the first eigenvalue with respect to the potential that $\lambda > \lambda_1(d_u, -r+\theta_{d_u,r}) = 0$. On the other hand, if $\Psi \neq 0$, then $\lambda \geq \widetilde{\lambda}$, where $\widetilde{\lambda}$ is the first eigenvalue for the second equation of (2.2). Moreover, $\widetilde{\lambda}$ is given by the following variational characterization:

$$\widetilde{\lambda} = \inf_{\phi \in H_0^1(\Omega), \phi \neq 0} \left\{ \frac{\int_\Omega |\nabla \phi|^2 dx - \int_\Omega \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r}+\beta} \phi^2 dx}{\int_\Omega \frac{1+\gamma\theta_{d_u,r}}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\phi^2 dx} \right\}.$$

We now decide the sign of $\widetilde{\lambda}$. Let $\overline{\phi}$ be the positive eigenfunction associated to

$$\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right).$$

Then it follows from the variational characterization of the first eigenvalue that

$$\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right) = \frac{\int_\Omega |\nabla\overline{\phi}|^2 dx - \int_\Omega \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\overline{\phi}^2 dx}{\int_\Omega \overline{\phi}^2 dx}.$$

When $\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right) < 0$, we have

$$\int_\Omega |\nabla\overline{\phi}|^2 dx - \int_\Omega \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\overline{\phi}^2 dx < 0.$$

Hence, we deduce from the fact that $\theta_{d_u,r} < r$ that

$$\begin{aligned}
\widetilde{\lambda} &\leqslant \frac{\int_\Omega |\nabla\overline{\phi}|^2 dx - \int_\Omega \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\overline{\phi}^2 dx}{\int_\Omega \frac{1+\gamma\theta_{d_u,r}}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\overline{\phi}^2 dx} \\
&< \frac{d_v+d_v\gamma r+\beta}{1+\gamma r}\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right) \\
&< 0.
\end{aligned}$$

When $\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right) > 0$, we have

$$\int_\Omega |\nabla\phi|^2 dx - \int_\Omega \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\phi^2 dx > 0$$

for any $\phi \in H_0^1(\Omega)$ and $\phi \neq 0$. Let $\widetilde{\phi}$ be the positive eigenfunction associated to $\widetilde{\lambda}$, where $\widetilde{\phi} \in H_0^1$ and $\widetilde{\phi} > 0$. Then

$$\begin{aligned}
\widetilde{\lambda} &= \frac{\int_\Omega |\nabla\widetilde{\phi}|^2 dx - \int_\Omega \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\widetilde{\phi}^2 dx}{\int_\Omega \frac{1+\gamma\theta_{d_u,r}}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\widetilde{\phi}^2 dx} \\
&> \frac{d_v+d_v\gamma r+\beta}{1+\gamma r}\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right) \\
&> 0.
\end{aligned}$$

Therefore, we use the linearization principle developed by Potier-Ferry [13] to conclude that all eigenvalues of (2.1) are positive if $\lambda_1\left(1, -\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right) > 0$,

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

and hence, $(\theta_{d_u,r},0)$ is asymptotically stable; while (2.1) has at least one negative eigenvalue if $\lambda_1\left(1,-\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right)<0$, and hence, $(\theta_{d_u,r},0)$ is unstable. This completes the proof of Theorem 1.1(3).

2.2. Proof of Theorem 1.2

In this subsection we would complete the proof of Theorem 1.2 by using the theory of fixed point index in positive cones. Let

$$V=\left(d_v+\frac{\beta}{1+\gamma u}\right)v.$$

Then (1.1) can be expressed as

$$\begin{cases}-d_u\Delta u=ru-u^2-\left(1+\alpha\frac{(1+\gamma u)V}{d_v+d_v\gamma u+\beta}\right)\frac{(1+\gamma u)uV}{d_v+d_v\gamma u+\beta}, & x\in\Omega,\\ -\Delta V=\frac{(1+\gamma u)V}{d_v+d_v\gamma u+\beta}\left(m-\frac{(1+\gamma u)V}{d_v+d_v\gamma u+\beta}+c\left(1+\alpha\frac{(1+\gamma u)V}{d_v+d_v\gamma u+\beta}\right)u\right), & x\in\Omega,\\ u=V=0, & x\in\partial\Omega.\end{cases} \tag{2.3}$$

Particularly, if $u=0$, then V satisfies

$$-\Delta V=\frac{V}{d_v+\beta}\left(m-\frac{V}{d_v+\beta}\right),\quad x\in\Omega,\qquad V=0,\quad x\in\partial\Omega.$$

Under the homogeneous Dirichlet boundary condition, this logistic equation has a unique positive solution $(d_v+\beta)\theta_{d_v+\beta,m}$ if and only if $\lambda_1(d_v+\beta,-m)<0$. Thus, (2.3) has a semi-trivial solution $(0,(d_v+\beta)\theta_{d_v+\beta,m})$ if $\lambda_1(d_v+\beta,-m)<0$. Likewise, (2.3) has another semi-trivial solution $(\theta_{d_u,r},0)$ if $\lambda_1(d_u,-r)<0$.

We first give a priori estimates of any positive solution.

Lemma 2.1. *Assume that $c\alpha r<1$ and (u,V) is any positive solution of (2.3). Then*

$$0\leqslant u(x)\leqslant r\quad \textit{and}\quad 0\leqslant V(x)\leqslant (d_v+\beta)\frac{m+cr}{1-c\alpha r}$$

for all $x\in\Omega$.

Proof. Suppose that $\|u\|_\infty=\max_{x\in\overline{\Omega}}u(x)=u(x_0)>0$ for some $x_0\in\Omega$. It follows from the first equation of (2.3) that

$$0\leq -d_u\Delta u(x_0)\leq ru(x_0)-u^2(x_0),$$

and hence, $u(x_0)\leq r$. This means that $0\leqslant u(x)\leqslant r$ for all $x\in\Omega$.

Suppose that $\|V\|_\infty=\max_{x\in\overline{\Omega}}V(x)=V(x_1)>0$ for some $x_1\in\Omega$. It follows from the second equation of (2.3) that

$$\begin{aligned}0\leq -\Delta V(x_1)&=v(x_1)(m-v(x_1)+c(1+\alpha v(x_1))u(x_1))\\&\leq v(x_1)(m+cr-(1-c\alpha r)v(x_1)),\end{aligned}$$

and hence, $v(x_1)<\frac{m+cr}{1-c\alpha r}$. Thus,

$$V(x)\leqslant V(x_1)=\left(d_v+\frac{\beta}{1+\gamma u(x_1)}\right)v(x_1)\leqslant (d_v+\beta)\frac{m+cr}{1-c\alpha r}$$

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

for all $x \in \Omega$. $\square$

We next introduce some functional spaces and operators. Let $C_0(\overline{\Omega}) = \{\phi(x) \in C(\overline{\Omega}) : \phi(x) = 0 \text{ on } \partial\Omega\}$ and $K(\overline{\Omega}) = \{\phi(x) \in C_0(\overline{\Omega}) : \phi(x) \geq 0 \text{ in } \Omega\}$. Set $E = C_0(\overline{\Omega}) \oplus C_0(\overline{\Omega})$ and $W = K(\overline{\Omega}) \oplus K(\overline{\Omega})$. Clearly, $W \subset E$. For any $\phi = (\phi_1, \phi_2) \in W$ and some $\iota > 0$, we define $W_\phi = \{\varphi \in E : \phi + \iota\varphi \in W\}$ and $S_\phi = \overline{W}_\phi \cap (-\overline{W}_\phi)$. Suppose that $T : E \to E$ is a compact linear operator satisfying $T(\overline{W}_\phi) \subseteq \overline{W}_\phi$. Then for any $\psi \in S_\phi$, $T\psi \in S_\phi$, and so T maps S_ϕ into itself. Thus, T induces a compact linear mapping $\widetilde{T}$ of $\widetilde{S}_\phi$ into itself, where $\widetilde{S}_\phi$ is the quotient space $E \backslash S_\phi$. Let $\widetilde{W}_\phi$ be the image of $\overline{W}_\phi$ under the quotient mapping $E \to E \backslash S_\phi$. Then $\widetilde{T}(\widetilde{W}_\phi) \subseteq \widetilde{W}_\phi$, since $T(\overline{W}_\phi) \subseteq \overline{W}_\phi$. One can refer to [10] for the fixed point index with respect to the positive cone.

Define

$$O := \left\{ (u, V) \in W : u(x) \leq r + 1, V(x) \leq (d_v + \beta)\frac{m + cr}{1 - c\alpha r} + 1, x \in \Omega \right\}.$$

It follows from Lemma 2.1 that all non-negative solutions of (2.3) lie in the interior of O, denoted by int O, with respect to W. By selecting a sufficiently large constant $M > 0$, we have

$$ru - u^2 - (1 + \alpha v)uv + Mu > 0 \quad \text{and} \quad mv - v^2 + c(1 + \alpha v)uv + MV > 0,$$

where $v = \frac{(1+\gamma u)V}{d_v + d_v\gamma u + \beta}$. For any $s \in (0, \infty)$, we consider a parameter of operators by

$$T_s \begin{pmatrix} u \\ V \end{pmatrix} = (-\Delta + M)^{-1} \begin{pmatrix} T_{s,1} + Mu \\ T_{s,2} + MV \end{pmatrix},$$

where

$$T_{s,1} = \frac{s}{d_u}\left(ru - u^2 - \left(1 + \alpha\frac{(1 + \gamma u)V}{d_v + d_v\gamma u + \beta}\right)\frac{(1 + \gamma u)uV}{d_v + d_v\gamma u + \beta}\right),$$

and

$$T_{s,2} = \frac{s(1 + \gamma u)V}{d_v + d_v\gamma u + \beta}\left(m - \frac{(1 + \gamma u)V}{d_v + d_v\gamma u + \beta} + c\left(1 + \alpha\frac{(1 + \gamma u)V}{d_v + d_v\gamma u + \beta}\right)u\right).$$

By virtue of the regularity theory for elliptic operators, T_s is a completely continuous operator in E.

Let $r(T)$ be the spectral radius of some operator T and $\deg_W(I - T_s, \text{int } O)$ be the Leray-Schauder degree for $I - T_s$ in the interior of O with respect to W. We can prove the following lemmas.

Lemma 2.2. *Assume that* $c\alpha r < 1$. *Then* $\deg_W(I - T_1, \text{int } O) = 1$.

Proof. By using a procedure similar to that in the proof of Lemma 2.1, we can prove that $0 \leqslant u_s(x) \leqslant r$ and $0 \leqslant V_s(x) \leqslant (d_v + \beta)\frac{m+cr}{1-c\alpha r}$ for all $x \in \Omega$, where (u_s, V_s) is any fixed point of T_s. Thus, for any $s \in (0, \infty)$, T_s has no fixed point on ∂O with respect to W, where ∂O denotes the boundary of O. Hence, the homotopy invariance of the degree ensures that $\deg_W(I - T_s, \text{int } O)$ is independent of s. Moreover, a standard comparison argument implies that $(0, 0)$ is the only fixed point of

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

T_s in W, provided $0 < s < \min\{d_u\lambda_1(1,0)/r, (d_v+\beta)\lambda_1(1,0)/m\}$. Consequently, the excision property shows that $\deg_W(I - T_s, \operatorname{int} O) = \operatorname{index}_W(T_s, (0,0))$ for any $0 < s < \min\{d_u\lambda_1(1,0)/r, (d_v+\beta)\lambda_1(1,0)/m\}$.

We now calculate the index of $(0,0)$. The Fréchet derivative of T_s at $(0,0)$ is given by

$$DT_s(0,0) = (-\Delta + M)^{-1} \begin{pmatrix} \frac{sr}{d_u} + M & 0 \\ 0 & \frac{sm}{d_v+\beta} + M \end{pmatrix}.$$

When $0 < s < \min\{d_u\lambda_1(1,0)/r, (d_v+\beta)\lambda_1(1,0)/m\}$, it is easy to check that 1 is not an eigenvalue of $DT_s(0,0)$ in $\overline{W}_{(0,0)}\backslash\{(0,0)\}$, where $\overline{W}_{(0,0)} = K(\overline{\Omega}) \oplus K(\overline{\Omega})$. Because $S_{(0,0)} = \{(0,0)\}$, $\widetilde{DT}_s(0,0)$ is identified with $DT_s(0,0)$. When $\lambda_1(d_u, -sr) > 0$ and $\lambda_1(d_v+\beta, -sm) > 0$, Lemmas 2.3 and 2.4 in [9] ensure that $r[(-\Delta+M)^{-1}(sr/d_u + M)] < 1$ and $r[(-\Delta+M)^{-1}(sm/(d_v+\beta)+M)] < 1$. Thus, $r(DT_s(0,0)) < 1$, and so Proposition 2 in [10] yields that $\operatorname{index}_W(T_s, (0,0)) = 1$. The desired result is obtained. □

Lemma 2.3. *Assume that $c\alpha r < 1$ and $\lambda_1(d_u, -r) < 0$. If $\lambda_1(d_v+\beta, -m) \neq 0$, then $\operatorname{index}_W(T_1, (0,0)) = 0$.*

Proof. If $\lambda_1(d_u, -r) < 0$ and $\lambda_1(d_v+\beta, -m) \neq 0$, then it is easy to check that 1 is not an eigenvalue of $DT_1(0,0)$ in $\overline{W}_{(0,0)}\backslash\{(0,0)\}$. Moreover, when $\lambda_1(d_u, -r) < 0$, Lemmas 2.3 and 2.4 in [9] ensure that $r[(-\Delta+M)^{-1}(r/d_u + M)] > 1$. As mentioned above, we identify $\widetilde{DT}_s(0,0)$ with $DT_s(0,0)$, and hence, $r(\widetilde{DT}_1(0,0)) = r(DT_1(0,0)) > 1$. Therefore, Proposition 2 in [10] yields that $\operatorname{index}_W(T_1, (0,0)) = 0$. □

Lemma 2.4. *Assume that $c\alpha r < 1$ and $\lambda_1(d_u, -r) < 0$. If*

$$\lambda_1\left(1, -\frac{(m + c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta}\right) < 0 \ (resp. > 0),$$

then $\operatorname{index}_W(T_1, (\theta_{d_u,r}, 0)) = 0$ $(resp. = 1)$.

Proof. The Fréchet derivative of T_1 at $(\theta_{d_u,r}, 0)$ is given by

$$DT_1(\theta_{d_u,r}, 0) = (-\Delta+M)^{-1} \begin{pmatrix} \frac{1}{d_u}(r - 2\theta_{d_u,r}) + M & -\frac{1}{d_u}\frac{\theta_{d_u,r}(1+\gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r}+\beta} \\ 0 & \frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta} + M \end{pmatrix}.$$

Let

$$\mathbb{T}_1 = (-\Delta+M)^{-1}\left(\frac{1}{d_u}(r - 2\theta_{d_u,r}) + M\right)$$

and

$$\mathbb{T}_2 = (-\Delta+M)^{-1}\left(\frac{(m + c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta} + M\right).$$

In view of the definitions of W_ϕ and S_ϕ, it is clear that $\overline{W}_{(\theta_{d_u,r},0)} = C_0(\overline{\Omega}) \oplus K(\overline{\Omega})$ and $S_{(\theta_{d_u,r},0)} = C_0(\overline{\Omega}) \oplus \{0\}$. We identify $\widetilde{DT}_1(\theta_{d_u,r}, 0)$ with $\mathbb{T}_2$.

We claim that $(I - DT_1(\theta_{d_u,r}, 0))(h,k) \neq 0$ for any $(h,k) \in \overline{W}_{(\theta_{d_u,r},0)}\backslash\{(0,0)\}$ if

$$\lambda_1\left(1, -\frac{(m + c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta}\right) \neq 0.$$

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

Otherwise, we assume that there exists some $(h,k)\in\overline{W}_{(\theta_{d_u,r},0)}\backslash\{(0,0)\}$ such that $(I-DT_1(\theta_{d_u,r},0))(h,k)=0$. Thus, we have

$$\begin{cases}(-\Delta+M)^{-1}\left[\left(\frac{1}{d_u}(r-2\theta_{d_u,r})+M\right)h-\frac{1}{d_u}\frac{\theta_{d_u,r}(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}k\right]=h, & x\in\Omega,\\ (-\Delta+M)^{-1}\left(\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}+M\right)k=k, & x\in\Omega,\\ h=k=0, & x\in\partial\Omega.\end{cases}$$

If $k\equiv 0$ in Ω, then the first equation implies that $\mathbb{T}_1 h=h$ in Ω, where $h\not\equiv 0$ in Ω, and hence, 1 is an eigenvalue of $\mathbb{T}_1$. Thus, we have $r(\mathbb{T}_1)\geq 1$. On the other hand, because $\theta_{d_u,r}$ is the unique positive solution of $-d_u\Delta\phi=r\phi-\phi^2$ with the homogeneous Dirichlet boundary condition, the monotonicity of the first eigenvalue with respect to the potential shows that $\lambda_1(d_u,2\theta_{d_u,r}-r)>\lambda_1(d_u,\theta_{d_u,r}-r)=0$. As a result, Lemmas 2.3 and 2.4 in [9] yield that $r(\mathbb{T}_1)<1$. This is a contradiction, and hence, $k\not\equiv 0$ in Ω. Therefore, in view of the second equation, it follows from the Krein-Rutman theorem that $r(\mathbb{T}_2)=1$ since $k\in K(\overline{\Omega})\backslash\{0\}$. However, by virtue of Lemmas 2.3 and 2.4 in [9], $\lambda_1\left(1,-\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right)\neq 0$ implies that $r(\mathbb{T}_2)\neq 1$. This contradiction yields the desired result.

We now apply Proposition 2 in [10] to calculate the index of $(\theta_{d_u,r},0)$. Suppose

$$\lambda_1\left(1,-\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right)<0.$$

Then Lemmas 2.3 and 2.4 in [9] show that $r(\mathbb{T}_2)>1$, and hence, $r(\widetilde{DT}_1(\theta_{d_u,r},0))>1$. Consequently, it follows from Proposition 2 in [10] that $\mathrm{index}_W(T_1,(\theta_{d_u,r},0))=0$. Suppose

$$\lambda_1\left(1,-\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right)>0.$$

Then Lemmas 2.3 and 2.4 in [9] show that $r(\mathbb{T}_2)<1$, and hence, $r(\widetilde{DT}_1(\theta_{d_u,r},0))<1$. To obtain the desired result, we need to discuss the spectral radius of $DT_1(\theta_{d_u,r},0)$. Assume that $(h,k)\in E$ is the corresponding eigenfunction associated to λ, where λ is an eigenvalue of $DT_1(\theta_{d_u,r},0)$. Then we have

$$\begin{cases}(-\Delta+M)^{-1}\left[\left(\frac{1}{d_u}(r-2\theta_{d_u,r})+M\right)h-\frac{1}{d_u}\frac{\theta_{d_u,r}(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}k\right]=\lambda h, & x\in\Omega,\\ (-\Delta+M)^{-1}\left(\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}+M\right)k=\lambda k, & x\in\Omega,\\ h=k=0, & x\in\partial\Omega.\end{cases}$$

If $k\not\equiv 0$ in Ω, then λ is an eigenvalue of $\mathbb{T}_2$, so $|\lambda|<1$. If $k\equiv 0$ in Ω, then λ must be an eigenvalue of $\mathbb{T}_1$, and hence, we derive from Lemmas 2.3 and 2.4 in [9] that $r(\mathbb{T}_1)<1$ since $\lambda_1(d_u,2\theta_{d_u,r}-r)>0$, which means that $|\lambda|<1$. As a result, $r(DT_1(\theta_{d_u,r},0))<1$. Therefore, it follows from Proposition 2 in [10] that $\mathrm{index}_W(T_1,(\theta_{d_u,r},0))=1$. □

Lemma 2.5. *Assume that* $c\alpha r<1$, $\lambda_1(d_u,-r)<0$ *and* $\lambda_1(d_v+\beta,-m)<0$. *If* $\lambda_1(d_u,(1+\alpha\theta_{d_v+\beta,m})\theta_{d_v+\beta,m}-r)<0$ *(resp.* >0*), then* $\mathrm{index}_W(T_1,(0,(d_v+\beta)\theta_{d_v+\beta,m}))=0$ *(resp.* $=1$*).*

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

Proof. The proof is similar to that of Lemma 2.4, so we omit it. □

Proof of Theorem 1.2. In the case $\lambda_1(d_v+\beta,-m)<0$. Assume that (2.3) has no non-negative solutions other than $(0,0)$, $(0,(d_v+\beta)\theta_{d_v+\beta,m})$ and $(\theta_{d_u,r},0)$. Thus, $(0,0)$, $(0,(d_v+\beta)\theta_{d_v+\beta,m})$ and $(\theta_{d_u,r},0)$ are only fixed points of T_1. In view of Lemma 2.3, $\text{index}_W(T_1,(0,0))=0$. Moreover, since

$$\lambda_1\left(1,-\frac{(m+c\theta_{d_u,r})(1+\gamma\theta_{d_u,r})}{d_v+d_v\gamma\theta_{d_u,r}+\beta}\right)<\lambda_1\left(1,-\frac{m}{d_v+\beta}\right)=\frac{\lambda_1(d_v+\beta,-m)}{d_v+\beta}<0,$$

Lemma 2.4 gives $\text{index}_W(T_1,(\theta_{d_u,r},0))=0$. Hence, when

$$\lambda_1(d_u,(1+\alpha\theta_{d_v+\beta,m})\theta_{d_v+\beta,m}-r)<0,$$

it follows from the excision property that

$$\begin{aligned}1&=\deg_W(I-T_1,\text{int }O)\\&=\text{index}_W(T_1,(0,0))+\text{index}_W(T_1,(0,(d_v+\beta)\theta_{d_v+\beta,m}))\\&\quad+\text{index}_W(T_1,(\theta_{d_u,r},0))\\&=0,\end{aligned}$$

a contradiction. This completes the proof of part (1).

In the case $\lambda_1(d_v+\beta,-m)>0$. We repeat the same argument as the above. Note that $(0,(d_v+\beta)\theta_{d_v+\beta,m})$ does not exist for $\lambda_1(d_v+\beta,-m)>0$. Assume that (2.3) has no non-negative solutions except for $(0,0)$ and $(\theta_{d_u,r},0)$. Thus, $(0,0)$ and $(\theta_{d_u,r},0)$ are only fixed points of T_1. Hence, the excision property of fixed point index gives

$$\begin{aligned}1&=\deg_W(I-T_1,\text{int }O)\\&=\text{index}_W(T_1,(0,0))+\text{index}_W(T_1,(\theta_{d_u,r},0))\\&=0,\end{aligned}$$

a contradiction. This completes the proof of part (2).

In the case $\lambda_1(d_v+\beta,-m)=0$. Select a sequence $\{m_i\}$ such that $\lambda_1(d_v+\beta,-m_i)\neq 0$ and $\lambda_1(d_v+\beta,-m_\infty)=0$, where $\lim_{i\to\infty}m_i=m_\infty$. Then we may assume (u_i,V_i) is a positive solution of (2.3) with $m=m_i$. Moreover, by using a procedure similar to that in the proof of Lemma 2.1, we can prove that $0\leqslant u_i(x)\leqslant r$ and $0\leqslant V_i(x)\leqslant (d_v+\beta)\frac{m_i+cr}{1-c\alpha r}$ for all $x\in\Omega$. Hence, it follows from the regularity theory for elliptic operators that we may assume $\lim_{i\to\infty}(u_i,V_i)=(u_\infty,V_\infty)$ in $C^1(\overline{\Omega})\times C^1(\overline{\Omega})$, where (u_∞,V_∞) is a non-negative solution of (2.3) with $m=m_\infty$. We next prove by contradiction that (u_∞,V_∞) is a positive solution of (2.3) with $m=m_\infty$.

Assume that $u_\infty\equiv 0$ in Ω. Let $\widehat{u}_i=u_i/\|u_i\|_\infty$. Then $\widehat{u}_i$ satisfies

$$\begin{cases}-d_u\Delta\widehat{u}_i=\widehat{u}_i\left(r-u_i-\left(1+\alpha\frac{(1+\gamma u_i)V_i}{d_v+d_v\gamma u_i+\beta}\right)\frac{(1+\gamma u_i)V_i}{d_v+d_v\gamma u_i+\beta}\right), & x\in\Omega,\\ \widehat{u}_i=0, & x\in\partial\Omega.\end{cases}$$

The elliptic regularity theory enables us to extract a subsequence such that $\widehat{u}_i\to$

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

$\widehat{u}_\infty \geq (\not\equiv) 0$ in $C^1(\overline{\Omega})$. Then $(\widehat{u}_\infty, V_\infty)$ satisfies

$$\begin{cases} -d_u \Delta \widehat{u}_\infty = \widehat{u}_\infty \left(r - \left(1 + \alpha \frac{V_\infty}{d_v+\beta}\right) \frac{V_\infty}{d_v+\beta} \right), & x \in \Omega, \\ -\Delta V_\infty = \frac{V_\infty}{d_v+\beta} \left(m_\infty - \frac{V_\infty}{d_v+\beta} \right), & x \in \Omega, \\ \widehat{u}_\infty = V_\infty = 0, & x \in \partial\Omega. \end{cases}$$

Since $\lambda_1(d_v+\beta, -m_\infty) = 0$, we deduce from the equation for V_∞ that $V_\infty \equiv 0$ in Ω. Thus, we have $\widehat{u}_\infty \equiv 0$ in Ω since $\lambda_1(d_u, -r) < 0$. This is a contradiction to $\widehat{u}_\infty \not\equiv 0$, and hence, $u_\infty \not\equiv 0$ in Ω. Furthermore, the strong maximum principle ensures that $u_\infty > 0$ in Ω. On the other hand, we assume $V_\infty \equiv 0$ in Ω. Let $\widehat{V}_i = V_i / \|V_i\|_\infty$. Thus, we have

$$\begin{cases} -d_u \Delta u_\infty = r u_\infty - u_\infty^2, & x \in \Omega, \\ -\Delta \widehat{V}_\infty = \frac{(1+\gamma u_\infty)\widehat{V}_\infty}{d_v + d_v \gamma u_\infty + \beta} \left(m_\infty + c u_\infty \right), & x \in \Omega, \\ u_\infty = \widehat{V}_\infty = 0, & x \in \partial\Omega. \end{cases}$$

Here $\widehat{V}_\infty \geq (\not\equiv) 0$ is a limit function of a subsequence of $\{\widehat{V}_i\}$ in $C^1(\overline{\Omega})$. Since $u_\infty > 0$ in Ω, we deduce from the equation for u_∞ that $u_\infty = \theta_{d_u,r}$. Moreover, it follows from the Krein-Rutman theorem that

$$\lambda_1 \left(1, -\frac{(m_\infty + c\theta_{d_u,r})(1 + \gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta} \right) = 0.$$

Consequently,

$$\begin{aligned} \frac{\lambda_1(d_v+\beta, -m_\infty)}{d_v+\beta} &= \lambda_1 \left(1, -\frac{m_\infty}{d_v+\beta} \right) \\ &> \lambda_1 \left(1, -\frac{(m_\infty + c\theta_{d_u,r})(1 + \gamma\theta_{d_u,r})}{d_v + d_v\gamma\theta_{d_u,r} + \beta} \right) \\ &= 0, \end{aligned}$$

and hence, $\lambda_1(d_v+\beta, -m_\infty) > 0$. This contradicts our assumption to $\lambda_1(d_v + \beta, -m_\infty) = 0$. Hence, the strong maximum principle ensures that $V_\infty > 0$ in Ω. The desired result is obtained.

Acknowledgements

We gratefully acknowledge the constructive comments by an anonymous referee on an earlier version of this paper.

References

[1] M.T. Alves and F.M. Hilker, *Hunting cooperation and Allee effects in predators*, Journal of Theoretical Biology, 2017, 419, 13–22.

[2] E.N. Dancer, *On positive solutions of some pairs of differential equations*, Transactions of American Mathematical Society, 1984, 284(2), 729–743.

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS

[3] S. Dey, M. Banerjee and S. Ghorai, *Bifurcation analysis and spatio-temporal patterns of a prey-predator model with hunting cooperation*, International Journal of Bifurcation and Chaos, 2020, 32(11), 2250173.

[4] T. Kadota and K. Kuto, *Positive steady states for a prey-predator model with some nonlinear diffusion terms*, Journal of Mathematical Analysis and Applications, 2006, 323, 1387–1401.

[5] K. Kuto, *A strongly coupled diffusion effect on the stationary solution set of a prey-predator model*, Advances in Differential Equations, 2007, 12(2), 145–172.

[6] K. Kuto and Y. Yamada, *Coexistence problem for a prey-predator model with density-dependent diffusion*, Nonlinear Analysis, 2009, 71, e2223–e2232.

[7] K. Kuto and Y. Yamada, *Limiting characterization of stationary solutions for a prey-predator model with nonlinear diffusion of fractional type*, Differential and Integral Equations, 2009, 22(7-8), 725–752.

[8] J. López-Gómez and R. Pardo, *Existence and uniqueness of coexistence states for the predator-prey model with diffusion*, Differential and Integral Equations, 1993, 6(5), 1025–1031.

[9] L. Li, *Coexistence theorems of steady-states for predator-prey interacting systems*, Transactions of American Mathematical Society, 1988, 305(1), 143–166.

[10] K. Nakashima and Y. Yamada, *Positive steady states for prey-predator models with cross-diffusion*, Advances in Differential Equations, 1996, 1(6), 1099–1122.

[11] A. Okubo and L.A. Levin, *Diffusion and Ecological Problems: Modern Perspective*, second edition, Interdisciplinary Applied Mathematics, vol. 14, Springer-Verlag, New York, 2001.

[12] S. Pal, N. Pal, S. Samanta and J. Chattopadhyay, *Effect of hunting cooperation and fear in a predator-prey model*, Ecological Complexity, 2019, 39, 100770.

[13] M. Potier-Ferry, *The linearization principle for the stability of solutions of quasilinear parabolic equations-I*, Archive for Rational Mechanics and Analysis, 1981, 77, 301–320.

[14] K. Ryu and W. Ko, *Asymptotic behavior of positive solutions to a predator-prey elliptic system with strong hunting cooperation in predators*, Physica A: Statistical Mechanics and its Applications, 2019, 531, 121726.

[15] K. Vishwakarma and M. Sen, *Influence of Allee effect in prey and hunting cooperation in predator with Holling type-III functional response*, Journal of Applied Mathematics and Computing, 2022, 68, 249–269.

[16] Y. Yamada, *Stability of steady-states for prey-predator diffusion equations with homogeneous Dirichlet conditions*, SIAM Journal on Mathematical Analysis, 1990, 21(2), 327–345.

[17] J. Zhang and W.N. Zhang, *Dynamics of a predator-prey model with hunting cooperation and allee effects in predators*, International Journal of Bifurcation and Chaos, 2020, 30, 2050199.

DOI https://doi.org/10.12150/jnma.2024.612 | Generated on 2024-12-27 20:17:17
OPEN ACCESS