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On Nodal Solutions of the Schrödinger-Poisson System with a Cubic Term*

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Abstract In this paper, we consider the following Schrödinger-Poisson system with a cubic term

$$\begin{cases} -\Delta u + V(|x|)u + \lambda\phi u = |u|^2 u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{0.1}$$

where $\lambda > 0$ and the radial function $V(x)$ is an external potential. By taking advantage of the Gersgorin disc theorem and Miranda theorem, via the variational method and blow up analysis, we prove that for each positive integer k, problem (0.1) admits a radial nodal solution $U_{k,4}^{\lambda}$ that changes sign exactly k times. Furthermore, the energy of $U_{k,4}^{\lambda}$ is strictly increasing in k and the asymptotic behavior of $U_{k,4}^{\lambda}$ as $\lambda \to 0_+$ is established. These results extend the existing ones from the super-cubic case in [17] to the cubic case.

Keywords Schrödinger-Poisson system, nodal solutions, Gersgorin disc theorem, Miranda theorem, blow-up analysis

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1. Introduction

In the last decades, the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = |u|^{p-1} u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

has attracted much research attention due to its deep physical backgrounds and mathematical challenges. Here $\lambda > 0, 1 < p < 5$ and V represents external potential function. From a physical point of view, system (1.1) comes from semiconductor theory and is used to simulate the evolution of electronic ensemble in semiconductor

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crystals, see [4, 20] for instance. In mathematical contents, the appearance of the nonlocal term $\lambda\phi u$ causes some mathematical difficulties and makes the study of (1.1) interesting. As we know, there are many existence results in the literature on the solutions of (1.1), such as ground state solutions [3, 15], bound state solutions [1, 15, 22], positive solutions [5, 21], non-radial solutions [9], and semiclassical state solutions [14]. For more related problems, one can refer to [6, 27] and references therein.

Recently, some researchers have shown interest in the existence and properties of nodal solutions (or sign-changing solutions) to (1.1). When the nonlinearity $|u|^{p-2}u$ satisfies the super-cubic growth condition that $p \in (3,5)$, via the Nehari manifold method, Wang-Zhou [23] studied the existence of a least energy nodal solution of (1.1) which changes sign only once. Later, the existence of infinitely many radial nodal solutions of (1.1) with any prescribed number of nodal domains was proved by Kim-Seok [17] via the variational method and gluing method for $p \in (3,5)$, see also [13] for a dynamical method. For the more general nonlinearity $f(u)$ satisfying super-cubic condition, one can see [2, 7, 8, 10, 16] for instance. For the cubic case $p = 3$, Zhong-Tang [28] investigated the existence and asymptotical behaviors of a least energy nodal solution with exactly two nodal domains to (1.1) by the Nehari manifold method. Later, Sun-Wu [22] extended this result to the sub-cubic case $p \in (1,3)$. Furthermore, Liu-Wang-Zhang [18] obtained infinitely many sign-changing solutions for $p \in (2,3]$ by using the perturbation method and the invariant subsets of descending flow. In [14], Ianni-Vaira obtained infinitely many nonradial sign-changing solutions in the semiclassical limit for $p \in (1,3]$ by using the Lyapunov-Schmit reduction method. For more related results and details, one can refer to [11, 25, 26]. From the above discussions, we see that $p = 3$ is a critical value. So a natural question arises that whether equation (1.1) with $p = 3$ admits radial nodal solutions with a prescribed number of nodal domains. In this paper, we shall give a confirmative answer to the following cubic case $p = 3$ of (1.1), that is,

$$\begin{cases} -\Delta u + V(|x|)u + \lambda\phi u = |u|^2 u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and V satisfies

(V) $V(|x|) \in \mathbb{C}([0,+\infty),\mathbb{R})$ is bounded from below by a positive constant V_0.

As is well known, equation (1.2) is equivalent to

$$-\Delta u + V(|x|)u + \lambda\phi_u u = |u|^2 u \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

with $\phi_u(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}dy$, which has a variational structure. Let

$$H_V = \{u \in H^1(\mathbb{R}^3) : u(x) = u(|x|),\ \int_{\mathbb{R}^3} V(|x|)u^2 < +\infty\}$$

be endowed with the norm $\|u\|_{H_V} = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)|u|^2)dx\right)^{\frac{1}{2}}$. Then its energy functional $I_{\lambda,4} : H_V \to \mathbb{R}$ is

$$I_{\lambda,4}(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(|x|)u^2)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \frac{1}{4}\int_{\mathbb{R}^3}|u|^4.$$

We denote the usual Nehari manifold by $\mathcal{N} := \{u \in H_V \backslash \{0\} : \langle I'_{\lambda,4}(u), u\rangle = 0\}$ and the ground state solution of (1.2) by $U_{0,4}$, which is obtained in [3] and satisfies

$$m := \inf_{u\in\mathcal{N}} I_{\lambda,4}(u) = I_{\lambda,4}(U_{0,4}) > 0. \tag{1.4}$$

Now we are ready to illustrate our main results. First we give the existence result.

Theorem 1.1. *For any positive integer k, problem* (1.2) *admits a radial nodal solution $U_{k,4}$ which has exactly $k+1$ nodal domains.*

We point out that the dynamical method used in [13] is not available here, because it is difficult to analyze the number of nodes when $V \not\equiv constant$. At the meanwhile, all the techniques concerning the super-cubic case used in [17], are also no longer valid, because the cubic term $|u|^2u$ has a complicated competitive relationship with the 3−homogeneous term $\phi_u u$ in the sense that $\phi_{tu}tu = t^3\phi_u u$ for any $t \in \mathbb{R}$. Hence some novel ideas are necessary. By taking advantage of the Gersgorin disc theorem and Miranda theorem, Theorem 1.1 is proved via variational method together with a limit procedure.

The next result shows that the energy of $U_{k,4}$ obtained in Theorem 1.1 increases as the number of nodes.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the energy of $U_{k,4}$ is strictly increasing with k, namely,*

$$I_{\lambda,4}(U_{k+1,4}) > I_{\lambda,4}(U_{k,4}), \quad \forall k \in \mathbb{N}_+.$$

Moreover, $I_{\lambda,4}(U_{k,4}) > (k+1)I_{\lambda,4}(U_{0,4})$.

Obviously, $U_{k,4}$ obtained in Theorem 1.1 depends on λ. We shall sometimes denote $U_{k,4}$ by $U^{\lambda}_{k,4}$ to emphasize this dependence. The following result shows the convergence property of $U^{\lambda}_{k,4}$ as $\lambda \to 0_+$.

Theorem 1.3. *Under the assumptions of Theorem 1.1, for any sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \to 0_+$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}_{n\geq 1}$, such that $U^{\lambda_n}_{k,4}$ converges to $U^0_{k,4}$ strongly in H_V as $n \to \infty$, where $U^0_{k,4}$ is a least energy radial nodal solution having exactly $k+1$ nodal domains to the following equation*

$$-\Delta u + V(|x|)u = |u|^2u. \tag{1.5}$$

The contribution of this paper are twofold: on one hand, our results extend and complement the previous results in [13] via the variational method. On the other hand, this paper partially solves the open problem proposed in [17]. We emphasize that for the case $p < 3$, the existence of such sign-changing solutions of (1.1) with any prescribed number of nodes is still open.

This paper is organized as follows. In Section 2, we give a variational framework of problem (1.2), and in Section 3, we give some properties of the Nehari type set. In Section 4, we prove Theorem 1.1 by the limit approach. In Section 5, the energy comparison and asymptotic behaviors are obtained.

2. Preliminaries

In this section, we give some notations and useful lemmas. For each $k \in \mathbb{N}_+$, we define

$$\Gamma_k = \left\{ \mathbf{r}_k := (r_1, \cdots, r_k) \in (\mathbb{R}_{>0})^k : 0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := +\infty \right\}, \tag{2.1}$$

and for each $\mathbf{r}_k \in \Gamma_k$, we denote by

$$\begin{aligned} B_1^{\mathbf{r}_k} &:= \left\{ x \in \mathbb{R}^3 : |x| < r_1 \right\}, \\ B_i^{\mathbf{r}_k} &:= \left\{ x \in \mathbb{R}^3 : r_{i-1} < |x| < r_i \right\}, i = 2, \cdots, k, \\ B_{k+1}^{\mathbf{r}_k} &:= \left\{ x \in \mathbb{R}^3 : |x| > r_k \right\}. \end{aligned}$$

Clearly, $B_1^{\mathbf{r}_k}$ is a ball, $B_2^{\mathbf{r}_k}, \cdots, B_k^{\mathbf{r}_k}$ are annulus and $B_{k+1}^{\mathbf{r}_k}$ is the complement of a ball. Moreover, $\mathbb{R}^3 = \overline{\cup_{i=1}^{k+1} B_i^{\mathbf{r}_k}}$. For $u \in H_V$, we denote by $u_i = u\chi_{B_i^{\mathbf{r}_k}}$, where $\chi_{B_i^{\mathbf{r}_k}}$ is the characteristic function on $B_i^{\mathbf{r}_k}$. We define the infimum level

$$c_{k,4} := \inf_{u \in \mathcal{N}_{k,4}} I_{\lambda,4}(u) \tag{2.2}$$

constrained on the Nehari set

$$\mathcal{N}_{k,4} = \{u \in H_V : \text{there exists } \mathbf{r}_k \text{ s.t. } u_i \neq 0 \text{ in } B_i^{\mathbf{r}_k},\ \langle I'_{\lambda,4}(u), u_i \rangle = 0,\ i = 1, \cdots, k+1\}. \tag{2.3}$$

In order to study $\mathcal{N}_{k,4}$, we set

$$H_i^{\mathbf{r}_k} := \left\{ u \in H_0^1(B_i^{\mathbf{r}_k}) : u(x) = u(|x|), u(x) = 0,\ x \in \partial B_i^{\mathbf{r}_k} \right\}$$

with the norm $\|u\|_i := \|u\|_{H_i^{\mathbf{r}_k}} = \left(\int_{B_i^{\mathbf{r}_k}} (|\nabla u|^2 + V(|x|)u^2) dx \right)^{\frac{1}{2}}$, and define a product space

$$\mathcal{H}_k^{\mathbf{r}_k} = H_1^{\mathbf{r}_k} \times \cdots \times H_{k+1}^{\mathbf{r}_k}. \tag{2.4}$$

Next we introduce an auxiliary function $E_{\lambda,4} : \mathcal{H}_k^{\mathbf{r}_k} \to \mathbb{R}$ related to $I_{\lambda,4}$,

$$E_{\lambda,4}(u_1, \cdots, u_{k+1}) = \sum_{i=1}^{k+1} \left(\frac{1}{2}\|u_i\|_i^2 + \frac{\lambda}{4} \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \int_{B_j^{\mathbf{r}_k}} \frac{u_j^2(y) u_i^2(x)}{4\pi|x-y|} dy dx - \frac{1}{4} \int_{B_i^{\mathbf{r}_k}} u_i^4 dx \right), \tag{2.5}$$

which satisfies

$$E_{\lambda,4}(u_1, \cdots, u_{k+1}) = I_{\lambda,4}(\sum_{i=1}^{k+1} u_i). \tag{2.6}$$

Then

$$\langle \partial_{u_i} E_{\lambda,4}(u_1, \cdots, u_{k+1}), u_i \rangle = \|u_i\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} u_i^4,$$

and the Nehari type set for $E_{\lambda,4}$ is

$$\mathcal{M}_{k,4}^{\mathbf{r}_k} := \{(u_1, \cdots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k} : u_i \neq 0, \langle \partial_{u_i} E_{\lambda,4}(u_1, \cdots, u_{k+1}), u_i \rangle = 0, i = 1, \cdots, k+1\}. \tag{2.7}$$

Obviously, if $(u_1, \cdots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ is a critical point of $E_{\lambda,4}$, then each u_i satisfies the following system

$$\begin{cases} -\Delta u_i + V(|x|)u_i + \sum_{j=1}^{k+1} \lambda \phi_{u_j} u_i = |u_i|^2 u_i \quad \text{in } B_i^{\mathbf{r}_k}, \quad 1 \le i \le k+1, \\ u_i = 0 \quad \text{on } \partial B_i^{\mathbf{r}_k}. \end{cases} \tag{2.8}$$

In the following, we list the Miranda theorem and a variant of the Gersgorin disc theorem, which will play an important role in our proofs.

Lemma 2.1. (Miranda Theorem, [19]) *Let*

$$D = \left\{ x := (x_1, \cdots, x_n) \in \mathbb{R}^3 : |x_i| < L, \quad \forall 1 \le i \le n \right\}.$$

Suppose that the mapping $H = (h_1, \cdots, h_n) : \overline{D} \to \mathbb{R}^3$ *is continuous on* $\overline{D}$ *satisfying*

$$H(x) \ne \theta, \quad \forall\ x \in \partial D$$

and

(i) $h_i(x_1, \cdots, x_{i-1}, -L, x_{i+1}, \cdots, x_n) \ge 0$ *for* $1 \le i \le n$,

(ii) $h_i(x_1, \cdots, x_{i-1}, L, x_{i+1}, \cdots, x_n) \le 0$ *for* $1 \le i \le n$,

where $\theta := (0, \cdots, 0)$. *Then* $H(x) = \theta$ *has a solution in* D.

Lemma 2.2. (Lemma 2.3, a variant of the Gersgorin disc theorem, [12]) *For any* $a_{ij} = a_{ji} > 0$ *with* $i \ne j \in \{1, \cdots, n\}$ *and* $s_i > 0$ *with* $i = 1, \cdots, n$, *define the matrix* $B := (b_{ij})_{n\times n}$ *by*

$$b_{ij} = \begin{cases} -\sum_{l\ne i} \dfrac{s_l a_{il}}{s_i} & i = j, \\ a_{ij} > 0 & i \ne j. \end{cases}$$

Then the real symmetric matrix $(b_{ij})_{n\times n}$ *is non-positive definite.*

Lemma 2.3. (Lemma 2.3, [24]) *If* $f \in C^1(\mathbb{R}^n, \mathbb{R})$ *is a strictly concave function and has a critical point* $(s_1, \cdots, s_n) \in \mathbb{R}^n$, *then* $(s_1, \cdots, s_n)$ *is the unique critical point of* f *in* $\mathbb{R}^n$.

3. Properties of the Nehari type set

In this section, we prove some properties of the Nehari type set $\mathcal{M}_{k,4}^{\mathbf{r}_k}$ and $\mathcal{N}_{k,4}$. Before the proof of Theorem 1.1, we first establish the framework of the following equation for the super-cubic case

$$\begin{cases} -\Delta u + V(|x|)u + \lambda\phi u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{3.1}$$

For $\mathbf{r}_k \in \Gamma_k$ and $p \in (4, 6)$, we introduce the energy functional $I_{\lambda,p} : H_V \to \mathbb{R}$ associated with (3.1) by

$$I_{\lambda,p}(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(|x|)u^2) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p,$$

and $E_{\lambda,p}:\mathcal{H}_k^{\mathbf{r}_k}\to\mathbb{R}$ by

$$E_{\lambda,p}(u_1,\cdots,u_{k+1}):=\sum_{i=1}^{k+1}\left(\frac{1}{2}\|u_i\|_i^2+\frac{\lambda}{4}\sum_{j=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{u_j^2(y)u_i^2(x)}{4\pi|x-y|}dydx-\frac{1}{p}\int_{B_i^{\mathbf{r}_k}}|u_i|^p\right).$$

Similarly, we define

$$\begin{aligned}\mathcal{N}_{k,p}=&\{u\in H_V:\text{there exists }\mathbf{r}_k\text{ s.t. }u_i\neq 0\text{ in }B_i^{\mathbf{r}_k},\\&\langle I'_{\lambda,p}(u),u_i\rangle=0,\ i=1,\cdots,k+1\big\},\\ \mathcal{M}_{k,p}^{\mathbf{r}_k}:=&\{(u_1,\cdots,u_{k+1})\in\mathcal{H}_k^{\mathbf{r}_k}:u_i\neq 0,\\&\langle\partial_{u_i}E_{\lambda,p}(u_1,\cdots,u_{k+1}),u_i\rangle=0,\,i=1,\cdots,k+1\}\end{aligned}$$

and

$$c_{k,p}:=\inf_{u\in\mathcal{N}_{k,p}}I_{\lambda,p}(u). \tag{3.2}$$

Obviously, $\mathcal{N}_{k,p},\mathcal{M}_{k,p}^{\mathbf{r}_k}$ are consistent with $\mathcal{N}_{k,4},\mathcal{M}_{k,4}^{\mathbf{r}_k}$ at $p=4$. Moreover, any critical point of $E_{\lambda,p}$ satisfies the following system

$$\begin{cases}-\Delta u_i+V(|x|)u_i+\displaystyle\sum_{j=1}^{k+1}\lambda\phi_{u_j}u_i=|u_i|^{p-2}u_i\quad\text{in }B_i^{\mathbf{r}_k},\quad 1\le i\le k+1,\\ u_i=0\quad\text{on }\partial B_i^{\mathbf{r}_k}.\end{cases}\tag{3.3}$$

For each $(u_1,\cdots,u_{k+1})\in\mathcal{H}_k^{\mathbf{r}_k}$, let $G_p^u:(\mathbb{R}_{\ge 0})^{k+1}\to\mathbb{R}$ be defined as

$$\begin{aligned}G_p^u(s_1,\cdots,s_{k+1}):=&E_{\lambda,p}(s_1u_1,\cdots,s_{k+1}u_{k+1})\\=&\sum_{i=1}^{k+1}\left(\frac{1}{2}s_i^2\|u_i\|_i^2+\frac{\lambda s_i^2}{4}\sum_{j=1}^{k+1}s_j^2\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}|u_i|^2-\frac{s_i^p}{p}\int_{B_i^{\mathbf{r}_k}}|u_i|^p\right).\end{aligned}\tag{3.4}$$

Proposition 3.1. (Proposition 3.1, Lemma 3.3, [17]) *For each $k\in\mathbb{N}^+$ and $p\in(4,6)$, the following results hold true:*

(i) for any $\boldsymbol{r}_k\in\Gamma_k$ and $(u_1,\cdots,u_{k+1})\in\mathcal{H}_k^{\boldsymbol{r}_k}$ with $u_i\neq 0$, there exists a unique maximum point $(s_1,\cdots,s_{k+1})\in(\mathbb{R}_{>0})^{k+1}$ of G_p^u in $(\mathbb{R}_{\ge 0})^{k+1}$ such that

$$(s_1u_1,\cdots,s_{k+1}u_{k+1})\in\mathcal{M}_{k,p}^{\boldsymbol{r}_k},$$

(ii) equation (3.1) admits a radial nodal solution $U_{k,p}\in H_V$ with exactly k nodes $0<r_1<\cdots<r_k<+\infty$ such that

$$I_{\lambda,p}(U_{k,p})=c_{k,p}.$$

By virtue of Proposition 3.1, we shall prove the following result.

Lemma 3.1. *For each $\boldsymbol{r}_k\in\Gamma_k$, the set $\mathcal{M}_{k,4}^{\boldsymbol{r}_k}\neq\emptyset$, which is defined in (2.7).*

Proof. The proof is similar to Lemma 3.2 in [10] with a slight modification. For the completeness, we give the sketch of the proof.

For each $\mathbf{r}_k = (r_1, \cdots, r_{k+1}) \in \Gamma_k$, we take $(\psi_1, \cdots, \psi_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ with $\psi_i \neq 0$ such that $\min_i \left\{ \frac{\|\nabla \psi_i\|^2_{L^2(B_i^{\mathbf{r}_k})}}{\|\psi_i\|^4_{L^4(B_i^{\mathbf{r}_k})}} \right\} > 1$. Then there is $\delta_0 > 0$ such that

$$1 < \delta_0^2 < \min\{\|\nabla \psi_i\|^2_{L^2(B_i^{\mathbf{r}_k})} / \|\psi_i\|^4_{L^4(B_i^{\mathbf{r}_k})}\}, 1 \leq i \leq k+1. \tag{3.5}$$

We define

$$v_i^{\delta_0}(x) = \delta_0^2 \psi_i(r_{i-1} + \delta_0(|x| - r_{i-1})).$$

Clearly, $supp(\psi_i) \subset B_i^{\mathbf{r}_k}$ and $supp(v_i^{\delta_0}) \subset \{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_{i-1} + (r_i - r_{i-1})/\delta_0\} \subset B_i^{\mathbf{r}_k}$. Moreover,

$$\begin{aligned}
&\|v_i^{\delta_0}\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{\mathbb{R}^3} \phi_{v_j^{\delta_0}} |v_i^{\delta_0}|^2 - \int_{\mathbb{R}^3} |v_i^{\delta_0}|^4 \\
=&\delta_0^3 \|\nabla \psi_i\|^2_{L^2(B_i^{\mathbf{r}_k})} + \delta_0 \int_{B_i^{\mathbf{r}_k}} V(\frac{|x| - r_{i-1}}{\delta_0} + r_{i-1}) |\psi_i|^2 \\
&+ \lambda \delta_0^3 \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{\psi_j} \psi_i^2 - \delta_0^5 \int_{B_i^{\mathbf{r}_k}} |\psi_i|^4 \\
=&: h_i(\delta_0).
\end{aligned}$$

Obviously, by (3.5) and the condition $(\mathbf{V})$, $h_i(\delta_0) \geq \delta_0^3 \|\nabla \psi_i\|^2_{L^2(B_i^{\mathbf{r}_k})} - \delta_0^5 \int_{B_i^{\mathbf{r}_k}} |\psi_i|^4 > 0$. Then $h_i(\delta) > 0$ for any $\delta \in (0, \delta_0)$. Moreover, by virtue of the condition $\int_{B_i^{\mathbf{r}_k}} V(|x|)|\psi_i|^2 < +\infty$, a direct computation gives that $h_i(\delta) \to -\infty$ as $\delta \to +\infty$. Thus there is $\delta_i \in (\delta_0, +\infty)$ such that $h_i(\delta_i) = 0$. Let $\delta_{max} = \max\{\delta_1, \cdots, \delta_{k+1}\}$. Then $h_i(\delta_{max}) \leq h_i(\delta_i) \leq 0$.

Now, we set

$$w_i(x) := v_i^{\delta_{\max}}(x). \tag{3.6}$$

Then $w_i(x) = \delta_{\max}^2 \psi_i(r_{i-1} + \delta_{\max}(|x| - r_{i-1}))$ and

$$\begin{aligned}
&supp(w_i) \subset \left\{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_{i-1} + (r_i - r_{i-1})/\delta_{\max}\right\} \subset B_i^{\mathbf{r}_k}, \\
&(w_1, \cdots, w_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k} \quad \text{with } w_i \neq 0.
\end{aligned}$$

We claim that there exists $(t_{1,4}, \cdots, t_{k+1,4}) \in (\mathbb{R}_{>0})^{k+1}$ such that

$$(t_{1,4} w_1, \cdots, t_{k+1,4} w_{k+1}) \in \mathcal{M}_{k,4}^{\mathbf{r}_k}. \tag{3.7}$$

Indeed, by Proposition 3.1 (i), there exists a unique global maximum point $(t_{1,p}, \cdots, t_{k+1,p}) \in (\mathbb{R}_{>0})^{k+1}$ of G_p^w such that

$$t_{i,p}^2 \|w_i\|_i^2 + \lambda \sum_{j=1}^{k+1} t_{i,p}^2 t_{j,p}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{w_j} |w_i|^2 - t_{i,p}^p \int_{B_i^{\mathbf{r}_k}} |w_i|^p = 0, \quad \forall 1 \leq i \leq k+1. \tag{3.8}$$

We assert that $(t_{1,p}, \cdots, t_{k+1,p})$ is bounded for $p \to 4_+$. Suppose on the contrary that there is $i_p \in \{1, \cdots, k+1\}$ such that $t_{i_p,p} \to +\infty$ as $p \to 4_+$. Then it follows from (3.6) and (3.8) that

$$
\begin{aligned}
0 &= t_{i_p,p}^{2-p}\|w_{i_p}\|_{i_p}^2 + \lambda \sum_{j=1}^{k+1} \frac{t_{j,p}^2}{t_{i_p,p}^2} t_{i_p,p}^{4-p} \int_{B_{i_p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i_p}^{\mathbf{r}_k}} |w_{i_p}|^p \\
&\le t_{i_p,p}^{2-p}\|w_{i_p}\|_{i_p}^2 + \lambda \sum_{j=1}^{k+1} \int_{B_{i_p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i_p}^{\mathbf{r}_k}} |w_{i_p}|^p \\
&\to \lambda \sum_{j=1}^{k+1} \int_{B_{i_p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i_p}^{\mathbf{r}_k}} |w_{i_p}|^4 \quad \text{as } p \to 4_+ \\
&= \lambda \delta_{max}^3 \sum_{j=1}^{k+1} \int_{B_{i_p}^{\mathbf{r}_k}} \phi_{\psi_j}\psi_{i_p}^2 - \delta_{max}^5 \int_{B_{i_p}^{\mathbf{r}_k}} |\psi_{i_p}|^4 \\
&= h_{i_p}(\delta_{max}) - \delta_{max}^3\|\nabla\psi_{i_p}\|_{L^2(B_{i_p}^{\mathbf{r}_k})}^2 - \delta_{max} \int_{B_{i_p}^{\mathbf{r}_k}} |\psi_{i_p}|^2 \\
&< 0,
\end{aligned}
\tag{3.9}
$$

which leads to a contradiction. Thus the assertion follows.

Then there is $(t_{1,4}, \cdots, t_{k+1,4}) \in (\mathbb{R}_{\ge 0})^{k+1}$ and a sequence $\{p_n\}_n$ such that

$$(t_{1,p_n}, \cdots, t_{k+1,p_n}) \to (t_{1,4}, \cdots, t_{k+1,4}) \text{ as } p_n \to 4_+.$$

By the continuity of G_p^w and the fact that $(t_{1,p}, \cdots, t_{k+1,p})$ is the global maximum point of G_p^w, $(t_{1,4}, \cdots, t_{k+1,4})$ is also a global maximum point of G_4^w and thus

$$t_{i,4}^2\|w_i\|_i^2 + \lambda \sum_{j=1}^{k+1} t_{i,4}^2 t_{j,4}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{w_j} w_i^2 = t_{i,4}^4 \int_{B_i^{\mathbf{r}_k}} |w_i|^4. \tag{3.10}$$

Next, we prove $(t_{1,4}, \cdots, t_{k+1,4}) \in (\mathbb{R}_{>0})^{k+1}$. Indeed, suppose on the contrary that there is $i_0 \in \{1, \cdots, k+1\}$ such that $(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4})$ is the global maximum point of G_4^w in $(\mathbb{R}_{\ge 0})^{k+1}$. Since

$$
\begin{aligned}
&G_4^w(t_{1,4}, \cdots, t_{i_0-1,4}, \mu, t_{i_0+1,4}, \cdots, t_{k+1,4}) \\
&= G_4^w(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4}) \\
&\quad + \frac{\mu^2}{2}\|w_{i_0}\|_{i_0}^2 + \frac{\lambda\mu^4}{4}\int \phi_{w_{i_0}} w_{i_0}^2 + \frac{\lambda\mu^2}{4} \sum_{j \ne i_0} t_{j,4}^2 \int \phi_{w_j} w_{i_0}^2 - \mu^4 \int |w_{i_0}|^4 \\
&= G_4^w(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4}) + \theta(\mu),
\end{aligned}
$$

where $\theta(\mu) := \frac{\mu^2}{2}\|w_{i_0}\|_{i_0}^2 + \frac{\lambda\mu^4}{4}\int \phi_{w_{i_0}} w_{i_0}^2 + \frac{\lambda\mu^2}{4}\sum_{j\ne i_0} t_{j,4}^2 \int \phi_{w_j} w_{i_0}^2 - \mu^4 \int |w_{i_0}|^4 > 0$ if μ is sufficiently small, it leads to a contradiction. Thus $t_{i,4} > 0$ for all $1 \le i \le k+1$.

Therefore, the claim (3.7) follows due to (3.10) and $(t_{1,4}, \cdots, t_{k+1,4}) \in (\mathbb{R}_{>0})^{k+1}$. So $\mathcal{M}_{k,4}^{\mathbf{r}_k} \ne \emptyset$ and the proof is completed. □

Lemma 3.2. *If* $(u_1, \cdots, u_{k+1}) \in \mathcal{M}_{k,4}^{\mathbf{r}_k}$*, then for any* $(b_1, \cdots, b_{k+1}) \in (\mathbb{R}_{\ge 0})^{k+1} \backslash (1, \cdots, 1)$,

$$E_{\lambda,4}(b_1 u_1, \cdots, b_{k+1} u_{k+1}) < E_{\lambda,4}(u_1, \cdots, u_{k+1}).$$

Proof. For $(u_1,\cdots,u_{k+1})\in\mathcal{M}_{k,4}^{\mathbf{r}_k}$ and $(b_1,\cdots,b_{k+1})\in(\mathbb{R}_{\geq 0})^{k+1}\backslash(1,\cdots,1)$, it follows that

$$
\begin{aligned}
&E_{\lambda,4}(b_1u_1,\cdots,b_{k+1}u_{k+1})\\
=&E_{\lambda,4}(b_1u_1,\cdots,b_{k+1}u_{k+1})-\sum_{i=1}^{k+1}\frac{b_i^4}{4}\langle\partial_{u_i}E_{\lambda,4}(u_1,\cdots,u_{k+1}),u_i\rangle\\
=&\sum_{i=1}^{k+1}\left(\frac{b_i^2}{2}\|u_i\|_i^2+\frac{\lambda b_i^2}{4}\sum_{j=1}^{k+1}b_j^2\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2-\int_{B_i^{\mathbf{r}_k}}\frac{b_i^4}{4}u_i^4\right)\\
&-\sum_{i=1}^{k+1}\frac{b_i^4}{4}\left(\|u_i\|_i^2+\lambda\sum_{j=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2-\int_{B_i^{\mathbf{r}_k}}u_i^4\right)\\
=&\sum_{i=1}^{k+1}\left((\frac{b_i^2}{2}-\frac{b_i^4}{4})\|u_i\|_i^2\right)+\lambda\sum_{i,j=1}^{k+1}\left(\frac{b_i^2b_j^2-b_i^4}{4}+\frac{b_i^2b_j^2-b_j^4}{4}\right)\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2\\
<&\sum_{i=1}^{k+1}\left(\frac{1}{4}\|u_i\|_i^2\right)-\frac{\lambda}{4}\sum_{i,j=1}^{k+1}(b_i^2-b_j^2)^2\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2\leq\sum_{i=1}^{k+1}\left(\frac{1}{4}\|u_i\|_i^2\right)\\
=&E_{\lambda,4}(u_1,\cdots,u_{k+1})-\sum_{i=1}^{k+1}\frac{1}{4}\langle\partial_{u_i}E_{\lambda,4}(u_1,\cdots,u_{k+1}),u_i\rangle\\
=&E_{\lambda,4}(u_1,\cdots,u_{k+1}).
\end{aligned}
$$

The proof is completed. □

By using Lemma 3.1, we prove that $\mathcal{N}_{k,4}$ is non-empty.

Lemma 3.3. *There hold $\mathcal{N}_{k,4}\neq\emptyset$ and $0<c_{k,4}<+\infty$, where $c_{k,4}$ and $\mathcal{N}_{k,4}$ are defined in* (2.2) *and* (2.3)*, respectively.*

Proof. By Lemma 3.1, we can take $(v_1,\cdots,v_{k+1})\in\mathcal{M}_{k,4}^{\mathbf{r}_k}$. Then by (2.6), $\langle I'_{\lambda,4}(\sum_{i=1}^{k+1}v_i),v_i\rangle=\langle\partial_{u_i}E_{\lambda,4}(v_1,\cdots,v_{k+1}),v_i\rangle=0$. So $\sum_{i=1}^{k+1}v_i\in\mathcal{N}_{k,4}$. Moreover, since $\mathcal{N}_{k,4}\subset\mathcal{N}$, it follows from (1.4) that

$$0<m:=\inf_{u\in\mathcal{N}}I_{\lambda,4}(u)\leq\inf_{u\in\mathcal{N}_{k,4}}I_{\lambda,4}(u)=c_{k,4}\leq I_{\lambda,4}(\sum_{i=1}^{k+1}v_i)<+\infty.$$

The proof is completed. □

4. Proof of Theorem 1.1

With the help of Proposition 3.1, we are going to prove Theorem 1.1 by the limit approach and blow up analysis in this section.

Proof of Theorem 1.1. According to Theorem 1.1 in [17], for each $k\in\mathbb{N}^+$ and $p\in(4,6)$, there exists $\mathbf{r}_k\in\Gamma_k$ and a radial nodal solution $\mathbf{u}_{k,p}:=(u_{1,p},\cdots,u_{k+1,p})\in\mathcal{H}_k^{\mathbf{r}_k}\backslash\{0\}$ of (3.3) such that

$$I_{\lambda,p}\left(\sum_{i=1}^{k+1}u_{i,p}\right)=E_{\lambda,p}(\mathbf{u}_{k,p})=c_{k,p}.$$

Moreover, $U_{k,p} := \sum_{i=1}^{k+1} u_{i,p}$ is a radial nodal solution having exactly k nodes of equation (3.1). Then we shall finish our proof by four steps.

Step 1. Prove

$$\limsup_{p\to 4_+} c_{k,p} \le c_{k,4} < +\infty. \tag{4.1}$$

Indeed, for any $(w_{1,4},\cdots,w_{k+1,4}) \in \mathcal{M}^{\mathbf{r}_k}_{k,4}$, it follows from Proposition 3.1 (i) that for each $p\in(4,6)$, there exists a unique $k+1$ tuple $(m_{1,p},\cdots,m_{k+1,p}) \in (\mathbb{R}_{>0})^{k+1}$ such that $(m_{1,p}w_{1,4},\cdots,m_{k+1,p}w_{k+1,4}) \in \mathcal{M}^{\mathbf{r}_k}_{k,p}$, that is,

$$m_{i,p}^2\|w_{i,4}\|_i^2 + \lambda\sum_{j=1}^{k+1} m_{i,p}^2 m_{j,p}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{w_{j,4}} w_{i,4}^2 - m_{i,p}^p \int_{B_i^{\mathbf{r}_k}} |w_{i,4}|^p = 0,\ \forall 1\le i\le k+1. \tag{4.2}$$

We assert that $(m_{1,p},\cdots,m_{k+1,p})$ is bounded for $p\to 4_+$. In fact, we argue it by contradiction. Suppose on the contrary that for each p, there is $i_p\in\{1,\cdots,k+1\}$ such that

$$m_{i_p,p} := \max_{j=1,\cdots,k+1}\{m_{j,p}\} \to +\infty \quad \text{as } p\to 4_+.$$

Then it follows from (4.2) that

$$\begin{aligned} 0 &= m_{i_p,p}^{-2}\|w_{i_p}\|_{i_p}^2 + \lambda\sum_{j=1}^{k+1}\frac{m_{j,p}^2}{m_{i_p,p}^2}\int_{B_{i_p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - m_{i_p,p}^{p-4}\int_{B_{i_p}^{\mathbf{r}_k}} |w_{i_p}|^p \\ &\le m_{i_p,p}^{-2}\|w_{i_p}\|_{i_p}^2 + \lambda\sum_{j=1}^{k+1}\int_{B_{i_p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - m_{i_p,p}^{p-4}\int_{B_{i_p}^{\mathbf{r}_k}} |w_{i_p}|^p \\ &\to \lambda\sum_{j=1}^{k+1}\int_{B_{i_p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i_p}^{\mathbf{r}_k}} |w_{i_p}|^4 < 0 \quad \text{as } p\to 4_+, \end{aligned} \tag{4.3}$$

which leads to a contradiction. Thus the assertion is proved.

By the assertion above, there exists $(m_{1,4},\cdots,m_{k+1,4}) \in (\mathbb{R}_{\ge 0})^{k+1}$ and a sequence
$\{(m_{1,p_n},\cdots,m_{k+1,p_n})\}$ such that

$$(m_{1,p_n},\cdots,m_{k+1,p_n}) \to (m_{1,4},\cdots,m_{k+1,4}) \quad \text{as} \quad p_n\to 4_+.$$

Since (4.2) implies $\lim_{n\to\infty} m_{i,p_n}^{p_n-2} \ge \lim_{n\to\infty} \frac{\|w_{i,4}\|_i^2}{\int_{B_i^{\mathbf{r}_k}}|w_{i,4}|^{p_n}} = \frac{\|w_{i,4}\|_i^2}{\int_{B_i^{\mathbf{r}_k}}|w_{i,4}|^4} > 0$, this shows

$$(m_{1,4},\cdots,m_{k+1,4}) \in (\mathbb{R}_{>0})^{k+1}.$$

Then taking $p_n\to 4$, it follows from (4.2) that

$$m_{i,4}^2\|w_{i,4}\|_i^2 + \lambda\sum_{j=1}^{k+1} m_{i,4}^2 m_{j,4}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{w_{j,4}} w_{i,4}^2 - m_{i,4}^4 \int_{B_i^{\mathbf{r}_k}} |w_{i,4}|^4 = 0,\ \forall 1\le i\le k+1. \tag{4.4}$$

Next, we prove

$$(m_{1,4},\cdots,m_{k+1,4}) = (1,\cdots,1). \tag{4.5}$$

In fact, let $h:(\mathbb{R}_{>0})^{k+1}\to\mathbb{R}$ be defined by

$$\begin{aligned} h(a_1,\cdots,a_{k+1}):=&E_{\lambda,4}(a_1^{\frac14}w_{1,4},\cdots,a_{k+1}^{\frac14}w_{k+1,4})\\ =&\sum_{i=1}^{k+1}\Bigg(\frac{a_i^{\frac12}}{2}\|w_{i,4}\|_i^2+\frac{\lambda a_i}{4}\int_{B_i^{\mathbf{r}_k}}\int_{B_i^{\mathbf{r}_k}}\frac{w_{i,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\\ &+\frac{\lambda}{4}\sum_{j\neq i}^{k+1}a_i^{\frac12}a_j^{\frac12}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{w_{j,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\\ &-\frac{a_i}{4}\sum_{i=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}|w_{i,4}|^4dx\Bigg). \end{aligned}$$

By some direct calculations, we get that

$$\begin{aligned} h_{a_i}(a_1,\cdots,a_{k+1})=&\frac14 a_i^{-\frac12}\|w_{i,4}\|_i^2+\frac{\lambda}{4}\int_{B_i^{\mathbf{r}_k}}\int_{B_i^{\mathbf{r}_k}}\frac{w_{i,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\\ &+\frac{\lambda}{4}\sum_{j\neq i}^{k+1}\left(a_i^{-\frac12}a_j^{\frac12}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{w_{j,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\right)-\frac14\int_{B_i^{\mathbf{r}_k}}|w_{i,4}|^4dx, \end{aligned}$$

and

$$\begin{aligned} &h_{a_ia_i}(a_1,\cdots,a_{k+1})=-\frac18 a_i^{-\frac32}\|w_{i,4}\|_i^2-\frac{\lambda}{8}\sum_{j\neq i}^{k+1}\left(a_i^{-\frac32}a_j^{\frac12}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{w_{j,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\right),\\ &h_{a_ia_j}(a_1,\cdots,a_{k+1})=\frac{\lambda}{8}a_j^{-\frac12}a_i^{-\frac12}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{w_{j,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx. \end{aligned}$$

For simplicity, we denote by

$$\begin{aligned} &B_{ii}=-\frac18 a_i^{-\frac32}\|w_{i,4}\|_i^2,\quad C_{ii}=-\sum_{j\neq i}^{k+1}\frac{a_j}{a_i}\left(\frac{\lambda}{8}a_i^{-\frac12}a_j^{-\frac12}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{w_{j,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\right),\\ &B_{ij}=0,\qquad C_{ij}=\frac{\lambda}{8}a_j^{-\frac12}a_i^{-\frac12}\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{w_{j,4}^2(y)w_{i,4}^2(x)}{4\pi|x-y|}dydx\quad\text{if } i\neq j. \end{aligned}$$

Then

$$A_{ij}:=h_{a_ia_j}(a_1,\cdots,a_{k+1})=B_{ij}+C_{ij}$$

According to Lemma 2.2, the matrix $(C_{ij})_{(k+1,k+1)}$ is non-positive definite. This together with the fact that B_{ij} is negative definite, $(A_{ij})_{(k+1)\times(k+1)}$ is negative definite. So h is a strictly concave function in $(\mathbb{R}_{>0})^{k+1}$. Note from $(w_{1,4},\cdots,w_{k+1,4})\in\mathcal{M}_{k,4}^{\mathbf{r}_k}$ that $(1,\cdots,1)$ is a critical point of h, and from (4.4) that $(m_{1,4}^4,\cdots,m_{k+1,4}^4)$ is a critical point of h. Then (4.5) follows from Lemma 2.3 immediately.

Thus by (3.2) and (4.5), it follows that

$$\limsup_{p\to 4_+}c_{k,p}\le\limsup_{p\to 4_+}I_{\lambda,p}(\sum_{i=1}^{k+1}m_{i,p}w_{i,4})=I_{\lambda,4}(\sum_{i=1}^{k+1}w_{i,4}).$$

Since the choice of $(w_{1,4},\cdots,w_{k+1,4})\in\mathcal{M}_{k,4}^{\mathbf{r}_k}$ is arbitrary, it follows immediately that

$$\limsup_{p\to 4_+} c_{k,p}\le c_{k,4}<+\infty.$$

Step 2. Prove that there is $U_{k,4}\in H_V$ such that

$$U_{k,p}\to U_{k,4}\neq 0 \text{ strongly in } H_V \text{ as } p\to 4_+. \tag{4.6}$$

In fact, by (4.4) and Proposition 3.1, we have

$$\begin{aligned}
c_{k,p}&=I_{\lambda,p}(U_{k,p})-\frac{1}{p}\langle I'_{\lambda,p}(U_{k,p}),U_{k,p}\rangle\\
&=(\frac{1}{2}-\frac{1}{p})\|U_{k,p}\|^2_{H_V}+(\frac{1}{4}-\frac{1}{p})\int_{\mathbb{R}^3}\phi_{U_{k,p}}U_{k,p}^2\\
&\ge(\frac{1}{2}-\frac{1}{p})\|U_{k,p}\|^2_{H_V},
\end{aligned}$$

which gives that $\|U_{k,p}\|_{H_V}$ is bounded for $p\to 4_+$. Then there exists a sequence $\{U_{k,p_n}\}_{n\ge1}$ and some $U_{k,4}\in H_V$ such that $U_{k,p_n}\rightharpoonup U_{k,4}$ in H_V as $p_n\to 4_+$. Moreover, by the compactly embedding theorem,

$$\int_{\mathbb{R}^3}\phi_{U_{k,p_n}}U_{k,p_n}^2\to\int_{\mathbb{R}^3}\phi_{U_{k,4}}U_{k,4}^2 \text{ and } \int_{\mathbb{R}^3}\phi_{U_{k,p_n}}U_{k,p_n}U_{k,4}\to\int_{\mathbb{R}^3}\phi_{U_{k,4}}U_{k,4}^2 \text{ as } p_n\to 4_+.$$

This, combined with the fact that U_{k,p_n} is a solution of (3.1), yields immediately that

$$\begin{aligned}
0&=\lim_{n\to+\infty}\langle I'_{\lambda,p_n}(U_{k,p_n}),U_{k,p_n}-U_{k,4}\rangle\\
&=\lim_{n\to+\infty}\int_{\mathbb{R}^3}\nabla U_{k,p_n}\nabla(U_{k,p_n}-U_{k,4})+\int_{\mathbb{R}^3}V(|x|)U_{k,p_n}(U_{k,p_n}-U_{k,4})\\
&+\lambda\int_{\mathbb{R}^3}\phi_{U_{k,p_n}}U_{k,p_n}^2-\lambda\int_{\mathbb{R}^3}\phi_{U_{k,p_n}}U_{k,p_n}U_{k,4}\\
&-\int_{\mathbb{R}^3}|U_{k,p_n}|^{p_n-2}U_{k,p_n}^2-\int_{\mathbb{R}^3}|U_{k,p_n}|^{p_n-2}U_{k,p_n}U_{k,4}\\
&\to\lim_{n\to\infty}\big(\|U_{k,p_n}\|^2_{H_V}-\|U_{k,4}\|^2_{H_V}\big)\ge 0 \quad\text{as}\quad n\to\infty,
\end{aligned}$$

due to $\liminf_{n\to\infty}\|U_{k,p_n}\|^2_{H_V}\ge\|U_{k,4}\|^2_{H_V}$. Hence, $U_{k,p_n}\to U_{k,4}$ strongly in H_V as $p_n\to 4_+$. Besides , it follows from $\|U_{k,p_n}\|^2_{H_V}\le\int_{\mathbb{R}^3}|U_{k,p_n}|^{p_n}\le C\|U_{k,p_n}\|^{p_n}_{H_V}$ that $\liminf_{n\to\infty}\|U_{k,p_n}\|_{H_V}>0$. Thus

$$U_{k,4}\neq 0.$$

Therefore, (4.6) follows and $U_{k,4}$ is a nontrivial weak solution of (1.3). Then by the standard elliptic regularity theory, $U_{k,4}\in C^2(\mathbb{R}^3)$ and then $U_{k,4}$ can be viewed as a radial nodal function which has at most $k+1$ components, because U_{k,p_n} has exactly k nodal domains. So we may assume that $U_{k,4}=\sum_{i=1}^{k+1}u_{i,4}\neq 0$ with k nodes $\mathbf{r}_{k,4}:=(r_{1,4},\cdots,r_{k,4})$, where $u_{i,4}=\chi_{B_i^{\mathbf{r}_k}}U_{k,4}$.

Step 3. Prove $u_{i,4}\neq 0$ for all $1\le i\le k+1$.

We prove it by contradiction. If NOT, there are two cases that occur: either

case 1: $r_{k,p_n} \to +\infty$ as $p_n \to 4_+$, or
case 2: there exists a subsequence $p_n \to 4_+$ as $n \to +\infty$ and $i_0 \in \{1, \cdots, k+1\}$ such that

$$\begin{aligned} \text{either} \quad & \liminf_{n\to\infty} \|u_{i_0,p_n}\|_{i_0}^2 \neq 0 \quad \text{and} \quad \liminf_{n\to\infty} \|u_{i_0+1,p_n}\|_{i_0+1}^2 = 0, \\ \text{or} \quad & \liminf_{n\to\infty} \|u_{i_0,p_n}\|_{i_0}^2 = 0 \quad \text{and} \quad \liminf_{n\to\infty} \|u_{i_0+1,p_n}\|_{i_0+1}^2 \neq 0. \end{aligned} \tag{4.7}$$

If **case 1** happens, by the Strauss inequality, there exists a constant $C > 0$ such that $|u(x)| \leq C\frac{\|u(x)\|_{H_V}}{|x|}$ in $\mathbb{R}^3$ for any $u \in H_V$. Then

$$\begin{aligned} \|u_{k+1,p_n}\|_{k+1}^2 &\leq \int_{B_{k+1}^{\mathbf{r}_{k,p_n}}} |u_{k+1,p_n}|^{p_n} dx \\ &\leq C \int_{B_{k+1}^{\mathbf{r}_{k,p_n}}} \frac{\|u_{k+1,p_n}\|_{k+1}^{p_n}}{|x|^{p_n}} dx \\ &= C r_{k,p_n}^{3-p_n} \|u_{k+1,p_n}\|_{k+1}^{p_n}. \end{aligned}$$

This shows that $\|u_{k+1,p_n}\|_{k+1} \to \infty$ as $n \to \infty$, which contradicts with the boundness of $\{U_{k,p_n}\}$.

If **case 2** happens, we consider the latter situation in (4.7), while the former situation can be settled by similar arguments. Without loss of generality, we may assume $u_{i_0,p_n} < 0$ in $B_{i_0}^{\mathbf{r}_{k,p_n}}$ and $u_{i_0+1,p_n} > 0$ in $B_{i_0+1}^{\mathbf{r}_{k,p_n}}$. For the convenience, we denote by $\Omega^{p_n} = \overline{B_{i_0}^{\mathbf{r}_{k,p_n}} \cup B_{i_0+1}^{\mathbf{r}_{k,p_n}}}$ and set

$$\Omega^4 := \lim_{n\to\infty} \Omega^{p_n} = \lim_{n\to\infty} \overline{B_{i_0}^{\mathbf{r}_{k,p_n}} \cup B_{i_0+1}^{\mathbf{r}_{k,p_n}}}.$$

Let $v_{p_n} = \frac{u_{i_0,p_n}}{\|u_{i_0,p_n}\|_{i_0}}$. Obviously v_{p_n} is bounded in H_V, and there exists $v_4 \in H_V$ such that $v_{p_n} \rightharpoonup v_4$ in H_V. Then by the compactly embedding theorem, it follows from $\|u_{i_0,p_n}\|_{i_0}^2 + \int_{\mathbb{R}^3} \phi_{u_{i_0,p_n}} u_{i_0,p_n}^2 \leq \int_{\mathbb{R}^3} |u_{i_0,p_n}|^{p_n}$ that

$$1 + \frac{\int_{\mathbb{R}^3} \phi_{u_{i_0,p_n}} u_{i_0,p_n}^2}{\|u_{i_0,p_n}\|_{i_0}^2} \leq \int_{\mathbb{R}^3} u_{i_0,p_n}^{p_n-2} v_{p_n}^2 \to \int_{\mathbb{R}^3} u_{i_0,4}^2 v_4^2 \quad \text{as} \quad p_n \to 4_+.$$

This implies that $v_4 \neq 0$ and thereby the set $\{x \in \mathbb{R}^3 : v_4(x) < 0\} \neq \emptyset$. Since $\{x \in \mathbb{R}^3 : v_{p_n} < 0\} \subset \{x \in \mathbb{R}^3 : u_{i_0,p_n} \leq 0\}$ for all p_n, we have

$$\emptyset \neq \{x \in \mathbb{R}^3 : v_4(x) < 0\} \subset \{x \in \mathbb{R}^3 : u_{i_0,4}(x) \leq 0\}. \tag{4.8}$$

On the other hand, (4.7) implies that $M_{i_0,4}(x) := u_{i_0,4}(x) + u_{i_0+1,4}(x) \geq 0$ in Ω^4, and the strong convergence $U_{k,p_n} \to U_{k,4}$ in H_V as $p_n \to 4_+$ shows that $M_{i_0,4}$ satisfies

$$\begin{cases} -\Delta M_{i_0,4} + V(|x|)M_{i_0,4} + \lambda \left(\displaystyle\int_{\mathbb{R}^3} \frac{U_{k,4}^2(y)}{4\pi|x-y|} dy \right) M_{i_0,4} = |M_{i_0,4}|^2 M_{i_0,4}, \quad \text{in } \Omega^4, \\ M_{i_0,4} = 0, \quad \text{on } \partial\Omega^4. \end{cases} \tag{4.9}$$

By the classical elliptic regularity theory and the strong maximum principle, we obtain

$$M_{i_0,4}(x) > 0 \quad \text{in} \quad \Omega^4,$$

which leads to $\{x\in\mathbb{R}^3 : u_{i_0,4}\le 0\}=\emptyset$. Obviously it contradicts with (4.8). Hence, the claim follows immediately and thereby $u_{i,4}\neq 0$ for all $1\le i\le k+1$.

Step 4. Prove that $U_{k.4}$ changes sign exactly k times and $c_{k,4}=I_{\lambda,4}(U_{k,4})$. Indeed, since $U_{k.4}$ is a solution of (1.3), by the classical regularity arguments and the strong maximum principle, we have $u_{i,4}<0$ or $u_{i,4}>0$ in $B_i^{\mathbf{r}_k}$. Thus $U_{k,4}$ changes sign exactly k times. Moreover, by (2.7) and (4.4), it follows that

$$\begin{aligned}
c_{k,4} &\ge \limsup_{n\to\infty} I_{\lambda,p_n}(U_{k,p_n})\\
&=\limsup_{n\to\infty}\left((\frac{1}{2}-\frac{1}{p_n})\|U_{k,p_n}\|_{H_V}^2+(\frac{1}{4}-\frac{1}{p_n})\int_{\mathbb{R}^3}\phi_{U_{k,p_n}}U_{k,p_n}^2\right)\\
&=\frac{1}{4}\|U_{k,4}\|_{H_V}^2=I_{\lambda,4}(U_{k,4})-\frac{1}{4}\langle I'_{\lambda,4}(U_{k,4}),U_{k,4}\rangle=I_{\lambda,4}(U_{k,4})\ge c_{k,4}.
\end{aligned}\tag{4.10}$$

Thus $I_{\lambda,4}(U_{k,4})=c_{k,4}$ and the proof is completed. □

5. Proofs of Theorems 1.2 and 1.3

In this section, we investigate the energy comparison and the convergence properties of the radial nodal solutions obtained in Theorem 1.1.

Proof of Theorem 1.2. According to Theorem 1.1, there exists $\bar{\mathbf{r}}_{k+1}=(\bar r_1,\cdots,\bar r_{k+1})\in\Gamma_{k+1}$ and a solution

$$U_{k+1}:=w_1^{\bar{\mathbf{r}}_{k+1}}+\cdots+w_{k+2}^{\bar{\mathbf{r}}_{k+1}}$$

of (1.2), which changes sign exactly $k+1$ times.

We first prove

$$I_{\lambda,4}(U_{k+1})>I_{\lambda,4}(U_k),\quad \forall k\in\mathbb{N}_+.$$

In fact, observe that $\sum_{i=2}^{k+2}n_i w_i^{\bar{\mathbf{r}}_{k+1}}\in\mathcal{N}_{k,4}$ if and only if

$$\begin{aligned}
0&=n_i^2\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2+\sum_{j=2}^{k+2}n_i^2n_j^2\lambda\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\int_{B_j^{\bar{\mathbf{r}}_{k+1}}}\frac{|w_j^{\bar{\mathbf{r}}_{k+1}}(y)|^2}{4\pi|x-y|}|w_i^{\bar{\mathbf{r}}_{k+1}}(x)|^2dydx\\
&\quad -n_i^4\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}|w_i^{\bar{\mathbf{r}}_{k+1}}|^4dx\\
&=:N_i(n_2,\cdots,n_{k+2}),\quad i=2,\cdots k+2.
\end{aligned}\tag{5.1}$$

Note that there exists some $\delta\in(0,1)$ small enough such that $N_i(\delta,\cdots,\delta)>0$ for all $i=2,\cdots,k+2$, and that

$$N_i(1,\cdots,1)<0,\quad \forall i=2,\cdots,k+2,$$

because U_{k+1} is a nodal solution of (1.2) satisfying

$$\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2+\sum_{j=1}^{k+2}\lambda\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\int_{B_j^{\bar{\mathbf{r}}_{k+1}}}\frac{|w_j^{\bar{\mathbf{r}}_{k+1}}(y)|^2}{4\pi|x-y|}|w_i^{\bar{\mathbf{r}}_{k+1}}(x)|^2dydx-\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}|w_i^{\bar{\mathbf{r}}_{k+1}}|^4dx=0.$$

Then we deduce from (5.1) that

$$N_i(n_2,\cdots,n_{i-1},\delta,n_{i+1},\cdots,n_{k+2})>0,\quad \forall n_j\in[\delta,1],j\neq i,$$

$$N_i(n_2,\cdots,n_{i-1},1,n_{i+1},\cdots,n_{k+2})<0,\quad \forall n_j\in[\delta,1], j\neq i.$$

By Lemma 2.1,there exists some $\tilde{\mathbf{s}}:=(\tilde{s}_2,\cdots,\tilde{s}_{k+2})\in P^1_\delta$ such that

$$(N_2(\tilde{\mathbf{s}}),\cdots,N_{k+2}(\tilde{\mathbf{s}}))=0,$$

where $P^1_\delta:=\{(n_2,\cdots,n_{k+2})\in(\mathbb{R}_{>0})^{k+1}:\delta<n_j<1,\forall j=2,\cdots,k+2\}$. This implies $\sum_{i=2}^{k+2}\tilde{s}_i w_i^{\bar{\mathbf{r}}_{k+1}}\in\mathcal{N}_{k,4}$ and thus

$$I_{\lambda,4}(\sum_{i=2}^{k+2}\tilde{s}_i w_i^{\bar{\mathbf{r}}_{k+1}})>I_{\lambda,4}(U_k).$$

Note that

$$\begin{aligned}&I_{\lambda,4}(\textstyle\sum_{i=2}^{k+2}\tilde{s}_i w_i^{\bar{\mathbf{r}}_{k+1}})=E_{\lambda,4}(0,\tilde{s}_2 w_2^{\bar{\mathbf{r}}_{k+1}},\cdots,\tilde{s}_{k+2}w_{k+1}^{\bar{\mathbf{r}}_{k+1}}),\\&I_{\lambda,4}(U_{k+1})=I_{\lambda,4}(\textstyle\sum_{i=1}^{k+2}w_i^{\bar{\mathbf{r}}_{k+1}})=E_{\lambda,4}(w_1^{\bar{\mathbf{r}}_{k+1}},\cdots,w_{k+2}^{\bar{\mathbf{r}}_{k+1}}).\end{aligned}$$

Since Lemma 3.2 gives

$$E_{\lambda,4}(w_1^{\bar{\mathbf{r}}_{k+1}},\cdots,w_{k+2}^{\bar{\mathbf{r}}_{k+1}})>E_{\lambda,4}(0,\tilde{s}_2 w_2^{\bar{\mathbf{r}}_{k+1}},\cdots,\tilde{s}_{k+2}w_{k+1}^{\bar{\mathbf{r}}_{k+1}}),$$

we can deduce from the above inequalities easily that $I_{\lambda,4}(U_{k+1})>I_{\lambda,4}(U_k)$.

Next, we prove $I_{\lambda,4}(U_{k+1})>(k+2)I_{\lambda,4}(U_0)$. In fact, $\langle I'_{\lambda,4}(U_{k+1}),w_i^{\bar{\mathbf{r}}_{k+1}}\rangle=0$ gives

$$\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2+\lambda\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\frac{|w_i^{\bar{\mathbf{r}}_{k+1}}(y)|^2}{4\pi|x-y|}|w_i^{\bar{\mathbf{r}}_{k+1}}(x)|^2dydx-\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}|w_i^{\bar{\mathbf{r}}_{k+1}}|^4dx<0.$$

Note that there exists a small $\bar{\delta}>0$ such that for all i,

$$\bar{\delta}^2\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2+\bar{\delta}^4\lambda\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\frac{|w_i^{\bar{\mathbf{r}}_{k+1}}(y)|^2}{4\pi|x-y|}|w_i^{\bar{\mathbf{r}}_{k+1}}(x)|^2dydx-\bar{\delta}^4\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}|w_i^{\bar{\mathbf{r}}_{k+1}}|^4dx>0.$$

Then for each i, there exists $\bar{\delta}_i\in(\bar{\delta},1)$ such that

$$\bar{\delta}_i^2\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2+\lambda\bar{\delta}_i^4\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}\frac{|w_i^{\bar{\mathbf{r}}_{k+1}}(y)|^2}{4\pi|x-y|}|w_i^{\bar{\mathbf{r}}_{k+1}}(x)|^2dydx-\bar{\delta}_i^4\int_{B_i^{\bar{\mathbf{r}}_{k+1}}}|w_i^{\bar{\mathbf{r}}_{k+1}}|^4dx=0,$$

which shows $\bar{\delta}_i w_i^{\bar{\mathbf{r}}_{k+1}}\in\mathcal{N}$. Hence, $I_{\lambda,4}(\bar{\delta}_i w_i^{\bar{\mathbf{r}}_{k+1}})\geq I_{\lambda,4}(U_0)$ and thus

$$\begin{aligned}(k+2)I_{\lambda,4}(U_0)\leq&\sum_{i=1}^{k+2}I_{\lambda,4}(\bar{\delta}_i w_i^{\bar{\mathbf{r}}_{k+1}})=\sum_{i=1}^{k+2}\left(I_{\lambda,4}(\bar{\delta}_i w_i^{\bar{\mathbf{r}}_{k+1}})-\frac{1}{4}\langle I'_{\lambda,4}(\bar{\delta}_i w_i^{\bar{\mathbf{r}}_{k+1}}),\bar{\delta}_i w_i^{\bar{\mathbf{r}}_{k+1}}\rangle\right)\\=&\sum_{i=1}^{k+2}\frac{1}{4}\bar{\delta}_i^2\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2<\sum_{i=1}^{k+2}\frac{1}{4}\|w_i^{\bar{\mathbf{r}}_{k+1}}\|_i^2=I_{\lambda,4}(\sum_{i=1}^{k+2}w_i^{\bar{\mathbf{r}}_{k+1}})\\&-\frac{1}{4}\langle I'_{\lambda,4}(\sum_{i=1}^{k+2}w_i^{\bar{\mathbf{r}}_{k+1}}),w_i^{\bar{\mathbf{r}}_{k+1}}\rangle\\=&I_{\lambda,4}(\sum_{i=1}^{k+2}w_i^{\bar{\mathbf{r}}_{k+1}})=I_{\lambda,4}(U_{k+1}).\end{aligned}$$

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The proof is completed. □

Proof of Theorem 1.3. For $\lambda > 0$, let $U_{k,4}^{\lambda} \in H_V$ be the radial nodal solution of (1.2) obtained in Theorem 1.1 which changes sign exactly k times. We divide the whole proof into three steps.

Step 1. We claim that for any sequence $\{\lambda_n\}$ with $\lambda_n \to 0_+$ as $n \to \infty$, $\{U_{k,4}^{\lambda_n}\}_{n\geq 1}$ is bounded in H_V.

In fact, we take $\mathbf{r}_k \in \Gamma_k$ and $(\psi_1, \cdots, \psi_{k+1}) \in \mathcal{M}_{k,4}^{\mathbf{r}_k}$ with $\psi_i \neq 0$ such that

$$\|\psi_i\|_i^2 + \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{\psi_j} \psi_i^2 - \int_{B_i^{\mathbf{r}_k}} \psi_i^4 = 0.$$

For $\lambda \in (0, 1]$, we define $g_i^{\lambda} : (\mathbb{R}_{>0})^{k+1} \to \mathbb{R}$ by

$$g_i^{\lambda}(a_1, \cdots, a_{k+1}) = a_i^2 \|\psi_i\|_i^2 + \lambda \sum_{j=1}^{k+1} a_i^2 a_j^2 \int_{B_i^{\mathbf{r}_k}} \phi_{\psi_j} \psi_i^2 - \int_{B_i^{\mathbf{r}_k}} a_i^4 \psi_i^4.$$

Obviously, there is $\delta > 0$ small enough such that for all $\lambda \in (0, 1]$,

$$\begin{aligned} &g_i^{\lambda}(\delta, \cdots, \delta) \geq g_i^0(\delta, \cdots, \delta) > 0, \\ &g_i^{\lambda}(1, \cdots, 1) \leq 0. \end{aligned}$$

Some direct computations give

$$\begin{aligned} &g_i^{\lambda}(a_1, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_{k+1}) < 0, \quad \forall \delta \leq a_j \leq 1, j \neq i, \\ &g_i^{\lambda}(a_1, \cdots, a_{i-1}, \delta, a_{i+1}, \cdots, a_{k+1}) > 0, \quad \forall \delta \leq a_j \leq 1, j \neq i. \end{aligned}$$

Let $D_{\delta}^1 = \{(a_1, \cdots, a_{k+1}) \in (\mathbb{R}_{>0})^{k+1} : \delta \leq a_i \leq 1\}$. Then by Lemma 2.1, there exists $(\bar{a}_1(\lambda), \cdots, \bar{a}_{k+1}(\lambda)) \in D_{\delta}^1$ such that

$$g_i^{\lambda}(\bar{a}_1(\lambda), \cdots, \bar{a}_{k+1}(\lambda)) = 0, \forall 1 \leq i \leq k+1,$$

which implies

$$(\bar{\psi}_1, \cdots, \bar{\psi}_{k+1}) := (\bar{a}_1(\lambda)\psi_1, \cdots, \bar{a}_{k+1}(\lambda)\psi_{k+1}) \in \mathcal{M}_{k,4}^{\mathbf{r}_k}, \quad \forall \lambda \in (0, 1].$$

Thus, for any $\lambda \in (0, 1]$, we have

$$\begin{aligned} I_{\lambda,4}(U_{k,4}^{\lambda}) &\leq E_{\lambda,4}(\bar{\psi}_1, \cdots, \bar{\psi}_{k+1}) \\ &= E_{\lambda,4}(\bar{\psi}_1, \cdots, \bar{\psi}_{k+1}) - \frac{1}{4}\langle \partial_{\bar{\psi}_i} E_{\lambda,4}(\bar{\psi}_1, \cdots, \bar{\psi}_{k+1}), \bar{\psi}_i \rangle \\ &= \frac{1}{4}\sum_{i=1}^{k+1} \|\bar{\psi}_i\|_i^2 = \frac{1}{4}\sum_{i=1}^{k+1} \|a_i(\lambda)\psi_i\|_i^2 \\ &\leq \frac{1}{4}\sum_{i=1}^{k+1} \|\psi_i\|_i^2 := C_0, \end{aligned} \tag{5.2}$$

where $C_0 > 0$ and $\bar{a}_i(\lambda) \leq 1$ are used. Hence

$$C_0 \geq I_{\lambda,4}(U_{k,4}^{\lambda}) = I_{\lambda,4}(U_{k,4}^{\lambda}) - \frac{1}{4}\langle I'_{\lambda,4}(U_{k,4}^{\lambda}), U_{k,4}^{\lambda} \rangle = \frac{1}{4}\|U_{k,4}^{\lambda}\|_{H_V}^2.$$

Thus $\{U_{k,4}^{\lambda}\}$ is bounded for $\lambda\in(0,1]$ in H_V and the claim is true. Step 1 is finished.

Step 2. Up to a subsequence, there exists $U_{k,4}^0$ such that $U_{k,4}^{\lambda_n}\rightharpoonup U_{k,4}^0$ weakly in H_V as $n\to\infty$. Then $U_{k,4}^0$ is a weak solution of (1.5), due to the fact that $U_{k,4}^{\lambda_n}$ is a solution of (3.1). By the compactly embedding theorem $H_V\hookrightarrow L^q(\mathbb{R}^3)$ for $2<q<6$, we deduce that

$$\begin{aligned}&\|U_{k,4}^{\lambda_n}-U_{k,4}^0\|_{H_V}^2\\ =&\langle I'_{\lambda_n,4}(U_{k,4}^{\lambda_n})-I'_{0,4}(U_{k,4}^0),U_{k,4}^{\lambda_n}-U_{k,4}^0\rangle\\ &-\lambda_n\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|U_{k,4}^{\lambda_n}(y)|^2}{4\pi|x-y|}U_{k,4}^{\lambda_n}(x)(U_{k,4}^{\lambda_n}(x)-U_{k,4}^0(x))dydx\\ &+\int_{\mathbb{R}^3}(U_{k,4}^{\lambda_n})^3(U_{k,4}^{\lambda_n}-U_{k,4}^0)-\int_{\mathbb{R}^3}(U_{k,4}^0)^3(U_{k,4}^{\lambda_n}-U_{k,4}^0)\to 0,\quad \text{as } n\to\infty.\end{aligned}$$

So $U_{k,4}^{\lambda_n}\to U_{k,4}^0$ strongly in H_V as $n\to\infty$. Similar arguments could give $(U_{k,4}^{\lambda_n})_i\to(U_{k,4}^0)_i$ strongly in H_V.

Notice from $\langle I'_{\lambda_n,4}(U_{k,4}^{\lambda_n}),(U_{k,4}^{\lambda_n})_i\rangle=0$ that

$$\liminf_{n\to+\infty}\|(U_{k,4}^{\lambda_n})_i\|_i>0.$$

This result, together with strong convergence, shows that $(U_{k,4}^{\lambda_n})_i\neq 0$. Moreover, by the standard elliptic regularity theory and strong maximum principle, we know that $(U_{k,4}^0)_i$ has a constant sign. Thus, $U_{k,4}^0$ is a radial solution of (1.5) with exactly $k+1$ nodal domains.

Step 3. Let $\bar{\mathbf{v}}_k=\sum_{i=1}^{k+1}v_i$ be a least energy radial nodal solution of (1.5). Notice that

$$\begin{aligned}0=&b_{i,n}^2\|v_i\|_i^2+\sum_{j=1}^{k+1}\lambda_n b_{i,n}^2b_{j,n}^2\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{v_j^2(y)}{4\pi|x-y|}v_i^2(x)dydx-b_{i,n}^4\int_{B_i^{\mathbf{r}_k}}|v_i|^4dx\\ =:&\,k_i^n(b_{1,n},\cdots,b_{k+1,n}),\end{aligned}\tag{5.3}$$

if and only if $\sum_{i+1}^{k+1}b_{i,n}v_i\in\mathcal{N}_{k,4,\lambda_n}$, where $\mathcal{N}_{k,4,\lambda_n}$ is defined as in (2.3) for $\lambda=\lambda_n$. Since $\langle I'_{0,4}(\bar{\mathbf{v}}_k),v_i\rangle=0$, we know

$$\begin{aligned}k_i^n(1,\cdots,1)=&\|v_i\|_i^2+\sum_{i=1}^{k+1}\lambda_n\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{v_j^2(y)}{4\pi|x-y|}v_i^2(x)dydx-\int_{B_i^{\mathbf{r}_k}}|v_i|^4dx\\ >&\|v_i\|_i^2-\int_{B_i^{\mathbf{r}_k}}|v_i|^4dx=0.\end{aligned}\tag{5.4}$$

Moreover, for any $A>1$,

$$\begin{aligned}A^2\|v_i\|_i^2-\int_{B_i^{\mathbf{r}_k}}A^4|v_i|^4dx&=A^2\left(\|v_i\|_i^2-\int_{B_i^{\mathbf{r}_k}}A^2v_i^4dx\right)\\ &=A^2\left(\int_{B_i^{\mathbf{r}_k}}v_i^4-\int_{B_i^{\mathbf{r}_k}}A^2v_i^4dx\right)\\ &=A^2(1-A^2)\int_{B_i^{\mathbf{r}_k}}v_i^4dx<0.\end{aligned}$$

Then there exists a large $N > 0$ such that for any $n \geq N$, there holds

$$k_i^n(A,\cdots,A)=A^2\|v_i\|_i^2+\sum_{j=1}^{k+1}\lambda_n A^4\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}}\frac{v_j^2(y)}{4\pi|x-y|}v_i^2(x)dydx-\int_{B_i^{\mathbf{r}_k}}A^4|v_i|^4dx<0. \tag{5.5}$$

Let $A = 1 + \frac{1}{m}$ and λ_{n_m} be chosen small enough satisfying (5.5). Then by Lemma 2.1, (5.4) and (5.5), there exists

$$(b_{1,n_m},\cdots,b_{k+1,n_m}) \in D_1^{1+\frac{1}{m}} := \left\{(s_1,\cdots,s_{k+1}) \in (\mathbb{R}_{>0})^{k+1} : 1 \leq s_i \leq 1+\frac{1}{m}\right\}$$

such that

$$k_i^{n_m}(b_{1,n_m},\cdots,b_{k+1,n_m}) = 0.$$

So $\sum_{i+1}^{k+1} b_{i,n_m}v_i \in \mathcal{N}_{k,4,\lambda_{n_m}}$. Clearly, $(b_{1,n_m},\cdots,b_{k+1,n_m}) \to (1,\cdots,1)$ and $\lambda_{n_m} \to 0$ as $m \to +\infty$.

Therefore,

$$\begin{aligned} I_{0,4}(\tilde{\mathbf{v}}_k) \leq I_{0,4}(U_{k,4}^0) &= \lim_{m\to+\infty} I_{\lambda_{n_m},4}(U_{k,4}^{\lambda_{n_m}}) \\ &\leq \lim_{m\to+\infty} I_{\lambda_{n_m},4}(\sum_{i=1}^{k+1} b_{i,n_m}v_i) = I_{0,4}(\sum_{i=1}^{k+1} v_i) = I_{0,4}(\tilde{\mathbf{v}}_k). \end{aligned} \tag{5.6}$$

Here $U_{k,4}^0$ is a least energy nodal solution of (1.2) among all the radial nodal solutions having exactly $k+1$ nodal domains. The proof is completed. □

References

[1] Ambrosetti A and Ruiz D. *Multiple bound states for the Schrödinger-Poisson problem*, Communications in Contemporary Mathematics, 2008, 10(3), 391–404.

[2] Alves C, Souto M and Soares S. *A sign-changing solution for the Schrödinger-Poisson equation in* $\mathbb{R}^3$, The Rocky Mountain Journal of Mathematics, 2017, 47(1), 1–25.

[3] Azzollini A and Pomponio A. *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, Journal of Mathematical Analysis and Applications, 2008, 345(1), 90–108.

[4] Benci V and Fortunato D. *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topological Methods in Nonlinear Analysis, 1998, 11(2), 283–293.

[5] Cerami G and Vaira G. *Positive solutions for some non-autonomous Schrödinger-Poisson systems*, Journal of Differential Equations, 2010, 248(3), 521–543.

[6] Che G and Chen H. *Existence and multiplicity of solutions for Kirchhoff-Schrödinger-Poisson system with critical growth*, International Journal of Mathematics, 2022, 33(1), 2250008.

[7] Chen S and Tang X. *Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in* $\mathbb{R}^3$, Zeitschrift für Angewandte Mathematik und Physik, 2016, 67, 18.

[8] Cheng K and Wang L. *Nodal solutions for the Kirchhoff-Schrödinger-Poisson system in* $\mathbb{R}^3$, AIMS Mathematics, 2022, 7(9), 16787–16810.

[9] D'Avenia P. *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Advanced Nonlinear Studies, 2002, 2(2), 177–192.

[10] Guo H, Tang R and Wang T. *Nodal solutions for the Schrödinger-Poisson system with an asymptotically cubic term*, Mathematical Methods in the Applied Sciences, 2022, 1–23.

[11] Guo H and Wang T. *A note on sign-changing solutions for the Schrödinger-Poisson system*, Electronic Research Archive, 2020, 28(1), 195–203.

[12] Guo H and Wu D. *Nodal solutions for the Schrdinger-Poisson equations with convolution terms*, Nonlinear Analysis, 2020, 196, 111781.

[13] Ianni I. *Sign-changing radial solutions for the Schrödinger-Poisson-Slater problem*, Topological Methods in Nonlinear Analysis, 2013, 41(2), 365–385.

[14] Ianni I and Vaira G. *Non-radial sign-changing solutions for the Schrödinger-Poisson problem in the semiclassical limit*, NoDEA. Nonlinear Difffferential Equations and Applications, 2015, 22(4), 741–776.

[15] lanni I and Ruiz D. *Ground and bound states for a static Schrodinger-Posson-Slater problem*, Communications In Contemporary Mathematics, 2012, 14(1), 1250003.

[16] Ji C, Fang F and Zhang B. *Least energy sign-changing solutions for the nonlinear Schrodinger-Poisson system*, Electronic Journal of Difffferential Equations, 2017, 2017, 1–13.

[17] Kim S and Seok J. *On nodal solutions of the nonlinear Schrödinger-Poisson equations*, Communications in Contemporary Mathematics, 2012, 14(6), 1250041.

[18] Liu Z, Wang Z and Zhang J. *Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system*, Annali di Matematica Pura ed Applicata, 2015, 195(3), 775–794.

[19] Miranda C. *Un'osservazione su un teorema di Brouwer*, Bollettino della Unione Matematica Italiana, 1940, 3(9), 5–7.

[20] Sanchez O and Soler J. L*ong-time dynamics of the Schrödinger-Poisson-Slater system*, Journal of Statistical Physics, 2004, 114(1), 179–204.

[21] Siciliano G. *Multiple positive solutions for a Schrödinger-Poisson-Slater system*, Journal of Mathematical Analysis and Applications, 2010, 365(1), 288–299.

[22] Sun J and Wu T. *Bound state nodal solutions for the non-autonomous Schrödinger-Poisson system in* $\mathbb{R}^3$, Journal of Difffferential Equations, 2020, 268(11), 7121–7163.

[23] Wang Z and Zhou H. *Sign-changing solutions for the nonlinear Schrödinger-Poisson system in* $\mathbb{R}^3$, Calculus of Variations and Partial Difffferential Equations, 2015, 52(3), 927–943.

[24] Wang T, Yang Y and Guo H. *Multiple nodal solutions of the Kirchhoff-type problem with a cubic term*, Advances In Nonlinear Analysis, 2022, 11(1), 1030–1047.

[25] Yu S, Zhang Z and Yuan R. *Sign-changing solutions for a Schrödinger-Kirchhoff-Poisson system with 4-sublinear growth nonlinearity*, Electronic Journal of Qualitative Theory of Diffferential Equations, 2021, 2021(86), 1–21.

[26] Zhang H, Liu Z, Tang C and Zhang J. *Existence and multiplicity of sign-changing solutions for quasilinear Schrödinger equations with sub-cubic nonlinearity*, arXiv preprint arXiv, 2021, 2109, 08810.

[27] Zhao L and Zhao F. *On the existence of solutions for the Schrödinger-Poisson equations*, Journal of Mathematical Analysis and Applications, 2008, 346(1), 155–169.

[28] Zhong X and Tang C. *Ground state sign-changing solutions for a Schrödinger-Poisson system with a 3-linear growth nonlinearity*, Journal of Mathematical Analysis and Applications, 2017, 455(2), 1956–1974.