Journal of Nonlinear Modeling and Analysis
Volume 6, Number 3, September 2024, 643–668
http://jnma-online.com
DOI:10.12150/jnma.2024.643

Unbiased Grey Polynomial Model Based on Precise Direct Integration Method*

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Abstract During the conversion from difference to differential in the grey polynomial model, the "misplaced replacement" problem will occur. A novel unbiased grey polynomial model, i.e., HUGMP$(1,1,N)$ is presented to overcome the above drawback. Meanwhile, the parameter estimation of HUGMP $(1,1,N)$ is directly constructed by the equivalent relation between the parameter estimation and the recurrence relation of the time response function for GMP$(1,1,N)$ model. The recurrence relation is deduced from the solution of the homogenized differential equations, converted from the whitenization equation of GMP$(1,1,N)$ model by introducing new variables. The simulated values are directly calculated by the precise direct integration method in order to reduce round-off error and improve fitting accuracy. Moreover, it is proved that the proposed unbiased grey polynomial model possesses not only complete coincidence of simulation to non-homogeneous exponential sequence with polynomial time terms, but also multiple transformation consistency. At last, the results of applications verify the effectiveness of the proposed model by comparing with other conventional models.

Keywords Grey system theory, time power, GMP$(1,1,N)$ model, unbiased grey model, precise direct integration method

MSC(2010) 37N40, 65L05.

1. Introduction

Since Professor Deng established grey system theory for small sample and poor information problems, grey forecasting model has been successfully applied in industry, agriculture, energy and so on [1, 2]. As a classical form of grey forecasting model, GM$(1,1)$ model is proposed to fit exponential sequences, but it can't fully fit the sequence even if the original data completely conforms to the homogeneous exponential law, because there exists "misplaced replacement" between parameter estimation and time response of GM$(1,1)$ model. Therefore unbiased GM$(1,1)$ models are presented to eliminate "misplaced replacement" and simulate homogeneous exponential sequences without any bias [3–7], which improve the fitting precision. The study of unbiased grey models becomes an important issue for grey forecasting models. The unbiased grey models are usually constructed by using direct modeling

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*The authors were supported by National Natural Science Foundation of China (No.11971299).

method [3–5], grey derivatives optimization, background values optimization [6,7] and discrete grey models. Among these, direct modeling method is considered as the unity of grey derivative optimization and background value optimization [9], the key of which is to seek the equivalent relation of parameter estimation and time response function in order to overcome the inconsistence from difference to differential in grey models. Moreover, the above three methods don't change the basic form of $\mathrm{GM}(1,1)$ model, but the discrete grey model ($\mathrm{DGM}(1,1)$) is not the precise form of $\mathrm{GM}(1,1)$ model [14], where both the parameter estimation and predicting adopt the difference form. Here, we discuss the unbiased grey models based on direct modeling method for the base form of grey models.

Researchers improve the structure of $\mathrm{GM}(1,1)$ model to construct various grey models to fit different structural data, non-monotonic time series, S-type data, periodic time sequence and other nonlinear data. Then for grey models with simple model structure, non-homogeneous grey model $\mathrm{NGM}(1,1,k)$ model [8,9], Verhulst model [10–12] and grey Bernoulli model [13], their unbiased grey models also have been presented to overcome the "misplaced replacement" problem. The equivalent relations between parameter estimation and time response function of the Verhulst model and grey Bernoulli model are obtained after they are converted to $\mathrm{GM}(1,1)$ model by introducing transformations, and the relation of $\mathrm{NGM}(1,1,k)$ model is easily obtained because the grey action is a linear function. Correspondingly, constructing unbiased grey models for complex grey actions such as, grey Riccati model [15–17], fractional-order grey model [18–20], grey model based on trigonometric function [21], grey model with time power terms [22–24], and grey polynomial model [25–30], is challenging by using direct modeling methods.

Among the complex grey models, grey polynomial model ($\mathrm{GMP}(1,1,N)$) designed by introducing polynomial time terms to the grey action, can describe the original sequence in accordance with a more general trend rather than the special homogeneous or non-homogeneous trend [25]. In order to improve the fitting accuracy and expand its applications for various data, researchers have proposed many improved $\mathrm{GMP}(1,1,N)$ models. Liu et al. put forward extended $\mathrm{GMP}(1,1,N)$ models by introducing the fractional accumulating generation operator and fractional power time terms [26,27]. Li et al. improved $\mathrm{GMP}(1,1,N)$ model by constructing the tuned background coefficient using optimal methods [28]. Wei et al. gave the simulated value of $\mathrm{GMP}(1,1,N)$ model according to the connotation method instead of the whitenization method and proved that the model simulates non-homogeneous exponential sequences without error [29] and proposed a discrete grey polynomial model [30]. But the unbiased $\mathrm{GMP}(1,1,N)$ model based on direct modeling method has not been given yet. Especially, when $N=1$, $\mathrm{GMP}(1,1,N)$ model is simplified to $\mathrm{NGM}(1,1,k)$ model, and its unbiased model has been studied. Unfortunately, for $N>1$, the complexity of polynomial grey action makes it difficult to deduce the recurrence relation of the time response sequence, although the time response function of $\mathrm{GMP}(1,1,N)$ model is easily given. Accordingly, the equivalent relation of parameter estimation and time response sequence can't be built. Thus we propose a novel way to gain the recurrence relation of time response sequence of $\mathrm{GMP}(1,1,N)$ model. The whitenization equation of $\mathrm{GMP}(1,1,N)$ model is firstly transformed to a first order homogeneous differential equations by introducing new variables, then the recurrence relation is deduced by the solution of the homogenized differential equations by the exponential matrix form. Therefore, we obtain the equivalent relation between the parameter estimation formula and the time response recurrence formula to construct the parameter estimation of the unbiased GMP$(1,1,N)$ model. Moreover, the formula of the time response sequence becomes more complicated and round-off error in computation becomes bigger as N increases. So we introduce a highly precise integration method [31] to directly solve the above transformed homogenized differential equations, and to improve the fitting accuracy of unbiased GMP$(1,1,N)$ model.

The remaining paper is organized as follows. In Section 2, an unbiased GMP $(1,1,N)$ model is constructed with direct modeling method, in which parameter estimation and prediction are discussed. In Section 3, the properties of the proposed model are studied. In Section 4, the modeling process of the proposed model is described. In Section 5, we build an unbiased GMP$(1,1,N)$ model to simulate non-homogeneous exponential sequences and nuclear energy consumption, then compare their errors to those of other models. Finally, conclusions are summarized in Section 6.

2. Construction of unbiased GMP(1,1,N) model

Assume that the original sequence is $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$, its first-order accumulated generating sequence is

$$X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n)),$$

where $x^{(1)}(k) = \sum\limits_{i=1}^{k} x^{(0)}(i), k = 1, 2, \cdots, n$.

Definition 2.1. (Definition 2, [25]) The basic form of GMP$(1,1,N)$ model is defined as

$$x^{(0)}(k) + a\frac{x^{(1)}(k-1) + x^{(1)}(k)}{2} = b_0 + \frac{2k-1}{2}b_1 + \cdots + \frac{k^{N+1} - (k-1)^{N+1}}{N+1}b_N$$

and the whitenization equation of GMP$(1,1,N)$ is expressed as

$$\frac{dx^{(1)}(t)}{dt} + ax^{(1)}(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_N t^N, \tag{2.1}$$

where a and b_0 denote the real development coefficient and grey action respectively, $b_1, \cdots, b_N$ are called the time correction terms.

In GMP$(1,1,N)$ model, $a, b_0, b_1, \cdots, b_N$ are estimated from the basic form and play roles as the coefficients of the whitenization equation. Taking the "misplaced replacement" problem into account, we construct the parameter estimation of the unbiased GMP$(1,1,N)$ model using direct modeling method. It is crucial to construct the equivalence relation generated by the exact recurrence relation of time response sequence and the formula of parameter estimation.

2.1. Recurrence relation of the time response sequence

Introducing a new variable $\mathbf{y}(t) = (x^{(1)}(t), 1, t, t^2, \cdots, t^N)^T$, Eq.(2.1) can be transformed into

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}, \tag{2.2}$$

where $\mathbf{A} = \begin{pmatrix} -a & b_0 & b_1 & \cdots & b_{N-1} & b_N \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N & 0 \end{pmatrix}$.

Lemma 2.1. *Assume that $t_k = k(k = 1, 2, \cdots, n)$, and the initial value is $\boldsymbol{y}(t_1)$, the discrete solution $\boldsymbol{y}$ of Eq.(2.2) at t_{k+1} is*

$$\boldsymbol{y}(t_{k+1}) = \boldsymbol{y}(k+1) = e^{\boldsymbol{A}k}\boldsymbol{y}(1) \tag{2.3}$$

if and only if

$$\boldsymbol{y}(k+1) = e^{\boldsymbol{A}}\boldsymbol{y}(k). \tag{2.4}$$

Proof. See Appendix A. □

Theorem 2.1. *The recurrence relation of the time response function for GMP $(1, 1, N)$ model is*

$$x^{(1)}(k+1) = vx^{(1)}(k) + u_0 + u_1 k + u_2 k^2 + \cdots + u_N k^N, \tag{2.5}$$

where v is non-negative, $v = e^{-a}$, $u_t = \sum\limits_{s=t}^{N} P_s^{s-t}\delta_{s-t}b_s(t = 0, 1, \cdots, N)$ with $P_s^{s-t} = \frac{s!}{t!}$, and $\delta_i = \frac{(-1)^{i+1}}{a^{i+1}}[e^{-a} + \sum\limits_{s=0}^{i}(-1)^{s+1}\frac{1}{s!}a^s], i = 0, 1, \cdots, N.$

Proof. Let $\mathbf{b} = (b_0, b_1, b_2, \cdots, b_N)$, $\mathbf{D} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & N & 0 \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} -a & \mathbf{b} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}$.

With operational properties of matrices, it is derived that

$$\mathbf{A}^n = \begin{pmatrix} (-a)^n & \mathbf{b}\sum\limits_{i=0}^{n-1}(-a)^{n-1-i}\mathbf{D}^i \\ \mathbf{0} & \mathbf{D}^n \end{pmatrix}, n \geq 2,$$

and

$$\mathbf{D}^n = \begin{cases} \begin{pmatrix} \mathbf{0}_{n\times(N+1-n)} & \mathbf{0}_{n\times n} \\ \mathbf{T}_{(N+1-n)\times(N+1-n)} & \mathbf{0}_{(N+1-n)\times n} \end{pmatrix}, & n \leq N, \\ \mathbf{0}, & n > N, \end{cases}$$

where $\mathbf{T}_{N+1-n} = \begin{pmatrix} P_n^n & & & \\ & P_{n+1}^n & & \\ & & \ddots & \\ & & & P_N^n \end{pmatrix}$.

So

$$\mathbf{bD}^n = \begin{cases} (P_n^n b_n, P_{n+1}^n b_{n+1}, \cdots, P_N^n b_N, 0, \cdots, 0), & n \le N, \\ \mathbf{0}, & n > N. \end{cases}$$

Substituting them into Taylor series expansion of $e^{\mathbf{A}}$ gives

$$\begin{aligned} e^{\mathbf{A}} &= \mathbf{E} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots + \frac{1}{n!}\mathbf{A}^n + \cdots \\ &= \begin{pmatrix} e^{-a} & \mathbf{b} + \frac{1}{2!}\mathbf{b}(-a + a\mathbf{D}) + \cdots + \frac{1}{n!}\mathbf{b}[(-a)^{n-1} + (-a)^{n-2}\mathbf{D} + \cdots + \mathbf{D}^{n-1}] \\ \mathbf{0} & \mathbf{E} + \mathbf{D} + \frac{1}{2!}\mathbf{D}^2 + \cdots + \frac{1}{n!}\mathbf{D}^n + \cdots \end{pmatrix}. \end{aligned}$$

Here

$$\mathbf{E} + \mathbf{D} + \frac{1}{2!}\mathbf{D}^2 + \cdots + \frac{1}{n!}\mathbf{D}^n + \cdots = \begin{pmatrix} 1 & & & & \\ C_1^1 & 1 & & & \\ C_2^2 & C_2^1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ C_N^N & \cdots & C_N^2 & C_N^1 & 1 \end{pmatrix}. \tag{2.6}$$

For convenience, let $\delta_i = (-1)^{i+1}\frac{1}{a^{i+1}}[e^{-a} + \sum_{s=0}^{i}(-1)^{s+1}\frac{1}{s!}a^s]$. Then

$$\begin{aligned} &\mathbf{b} + \frac{1}{2!}\mathbf{b}(-a + a\mathbf{D}) + \cdots + \frac{1}{n!}\mathbf{b}[-a)^{n-1} + (-a)^{n-2}\mathbf{D} + \cdots + \mathbf{D}^{n-1}] \\ =& -\frac{\mathbf{b}}{a}(e^{-a} - 1) + \frac{\mathbf{bD}}{a^2}(e^{-a} - 1 + a) + \cdots \\ &+ (-1)^{n+1}\frac{\mathbf{bD}^n}{a^{n+1}}[e^{-a} - 1 + \cdots + (-1)^{n+1}\frac{1}{n!}a^n] + \cdots \\ =&\delta_0(b_0, b_1, \cdots, b_N) + \delta_1(b_1, 2b_2, \cdots, Nb_N, 0) + \cdots \\ &+ \delta_{N-1}(P_{N-1}^{N-1}b_{N-1}, P_N^{N-1}b_N, \cdots, 0) + \delta_N(P_N^N b_N, 0, \cdots, 0) \\ =&(\delta_0 b_0 + \delta_1 b_1 + \cdots + \delta_N P_N^N b_N, \delta_0 b_1 + P\delta_1 b_2 + \cdots + P_N^{N-1}b_N, \cdots, \delta_0 b_N) \qquad (2.7) \\ \triangleq&(u_0, u_1, \cdots, u_N). \end{aligned}$$

So substituting Eq.(2.6) and Eq.(2.7) into $e^{\mathbf{A}}$ and using Lemma 2.1, the recurrence relation of Eq.(2.2) is given by

$$\begin{pmatrix} x^{(1)}(k+1) \\ 1 \\ k+1 \\ (k+1)^2 \\ \vdots \\ (k+1)^N \end{pmatrix} = \begin{pmatrix} v & u_0 & u_1 & u_2 & \cdots & u_N \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & C_1^1 & 1 & 0 & \cdots & 0 \\ 0 & C_2^2 & C_2^1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & C_N^N & \cdots & C_N^2 & C_N^1 & 1 \end{pmatrix} \begin{pmatrix} x^{(1)}(k) \\ 1 \\ k \\ k^2 \\ \vdots \\ k^N \end{pmatrix},$$

where $v = e^{-a}$,

$$u_t = \sum_{s=t}^{N} P_s^{s-t} \delta_{s-t} b_s, t = 0, 1, \cdots, N,$$

$$\delta_i = \frac{(-1)^{i+1}}{a^{i+1}} [e^{-a} + \sum_{s=0}^{i} (-1)^{s+1} \frac{1}{s!} a^s], i = 0, 1, \cdots, N.$$

Moreover, it is clear that the last $N+1$ algebra equations are identity by the binomial theorem and the first equation is written as Eq.(2.5). So the theorem is proved. □

2.2. Parameter estimation

Theorem 2.2. *The parameter vector* $\mathbf{u} = (v, u_0, u_1, \cdots, u_N)^T$ *of GMP*$(1,1,N)$ *model can be directly estimated by ordinary least square method*

$$\hat{\mathbf{u}} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{Y}, \tag{2.8}$$

where $\mathbf{B} = \begin{pmatrix} x^{(1)}(1) & 1 & 1 & 1 & \cdots & 1 \\ x^{(1)}(2) & 1 & 2 & 4 & \cdots & 2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{(1)}(n-1) & 1 & n-1 & (n-1)^2 & \cdots & (n-1)^N \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} x^{(1)}(2) \\ x^{(1)}(3) \\ \vdots \\ x^{(1)}(n) \end{pmatrix}$.

Subsequently, by Theorem 2.1, we get the parameter estimation

$$\hat{a} = -ln\hat{v}, \tag{2.9}$$

$$\begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \vdots \\ \hat{b}_N \end{pmatrix} = \begin{pmatrix} \delta_0 & P_1^1\delta_1 & P_2^2\delta_2 & \cdots & P_{N-1}^{N-1}\delta_{N-1} & P_N^N\delta_N \\ & \delta_0 & P_2^1\delta_1 & \cdots & P_{N-1}^{N-2}\delta_{N-2} & P_N^{N-1}\delta_{N-1} \\ & & \delta_0 & P_3^1\delta_1 & \cdots & P_N^{N-1}\delta_{N-2} \\ & & & \ddots & \ddots & \vdots \\ & & & & \delta_0 & P_N^1\delta_1 \\ & & & & & \delta_0 \end{pmatrix}^{-1} \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{pmatrix} \tag{2.10}$$

of unbiased GMP$(1,1,N)$ model to solve the "misplaced replacement" problem illustrated.

2.3. Prediction

The prediction of grey models is usually given by the formula of the time response sequence, but the formula becomes more complicated and round-off error in computation becomes bigger as the power N increases. In this subsection, we introduce a numerical approach (highly precise integration method [31], abbreviated as HPD) to obtain the predicted value with high precision.

Theorem 2.3. *The time response sequence $\hat{\boldsymbol{y}}(k+1)$ of Eq.(2.2) with initial value $\hat{\boldsymbol{y}}(1) = (x^{(1)}(1), 1, 1, \cdots, 1)$ is obtained by HPD method. Accordingly, $\hat{x}^{(1)}(k+1)$ is derived from the first element of $\hat{\boldsymbol{y}}(k+1)$, then the predicted value is*

$$\hat{x}^{(0)}(k+1) = \hat{x}^{(1)}(k+1) - \hat{x}^{(1)}(k), k = 1, 2, \cdots, n-1.$$

Next, the HPD method is introduced in the following, in which it is crucial to compute the transfer matrix $\mathbf{T}(\tau) = e^{\mathbf{A}\tau}$ with high precision.

Algorithm 1 The prediction process of GMP$(1,1,N)$ model.

Input: 1-AGO sequence $X^{(1)}$, polynomial order N, parameter m and matrix

$$\mathbf{A} = \begin{pmatrix} -\hat{a} & \hat{b}_0 & \hat{b}_1 & \cdots & \hat{b}_{N-1} & \hat{b}_N \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N & 0 \end{pmatrix}.$$

Give the initial value $\hat{\mathbf{y}}(1) = (x^{(1)}(1), 1, 1, \cdots, 1)$, time-step $d\tau = \frac{1}{2^m}$ and matrix $\mathbf{S} = d\tau * \mathbf{A}$;
for i=1 to m **do**
  Calculate $\mathbf{S} = 2\mathbf{S} + \mathbf{S} * \mathbf{S}$;
end for
Compute $\mathbf{T} = \mathbf{I} + \mathbf{S}$;
for j=1 to $n-1$ **do**
  Calculate $\hat{\mathbf{y}}(k+1) = \mathbf{A} * \hat{\mathbf{y}}(k)$;
end for
Pick the first row of $\hat{\mathbf{y}}$ and denote it as $\hat{x}^{(1)}$, then calculate $\hat{x}^{(0)}$ by Theorem 2.5;

Output: $\hat{x}^{(0)}$.

Using the above approach, if only the parameter m is chosen properly, the truncation error of $\mathbf{T}(\tau)$,

$$\mathbf{T}(\tau) \approx (\mathbf{I} + \frac{\mathbf{A}\tau}{2^m})^{2^m} \triangleq (\mathbf{I} + \mathbf{S}_0)^{2^m} = [[(\mathbf{I} + \mathbf{S}_0)^2]^2 \cdots]^2 = [(\mathbf{I} + \mathbf{S}_1)^2 \cdots]^2 = \cdots = \mathbf{I} + \mathbf{S}_m$$

is generally negligibly small [32], the same order as the round-off error of computers. Moreover, all small amounts are firstly added up, then finally added to the identity matrix $\mathbf{I}$, that can effectively avoid the round-off error and improve computational accuracy.

The parameters $\hat{a}, \hat{b}_0, \hat{b}_1, \cdots, \hat{b}_N$ of GMP$(1,1,N)$ model are estimated by direct modelling method, and $\hat{x}^{(1)}(k)$ is calculated by HPD, which is called the unbiased grey polynomial model based on highly precise direct integration method(noted as HUGMP$(1,1,N)$ model). If $\hat{x}^{(1)}(k)$ is computed by the solution formula of the whitenization equation, that is called unbiased GMP$(1,1,N)$ model (UGMP$(1,1,N)$).

3. The properties of unbiased GMP(1,1,N) model

In order to verify the validity of the GMP$(1,1,N)$ model, we give the proof of the proposed unbiased GMP$(1,1,N)$ model completely fitting the non-homogeneous exponential sequence with polynomial time terms in Theorem 3.1.

Theorem 3.1. *The unbiased GMP$(1,1,N)$ model fits the following nonhomogeneous sequence completely,*

$$x^{(0)}(k) = \alpha q^k + \beta_0 + \beta_1 k + \cdots + \beta_{N-1} k^{N-1}, k = 1, 2, \cdots, n,$$

where $\alpha, q, \beta_0, \beta_1, \cdots, \beta_{N-1}$ are all constants.

Proof. This proof is completed by the following three steps.

Step 1. Parameter estimation $\hat{\mathbf{v}}, \hat{\mathbf{u}}$.

The accumulated sequence of the original sequence is

$$\begin{aligned} x^{(1)}(k+1) =& q x^{(1)}(k) + \sum_{i=0}^{N-1} \beta_i k^i + \alpha q \\ & + (1-q)[\beta_0 k + \beta_1 \sum_{i=1}^{k} i + \beta_2 \sum_{i=1}^{k} i^2 + \cdots + \beta_{N-1} \sum_{i=1}^{k} i^{N-1}]. \end{aligned}$$

By Lemma 6.3 in Appendix B, it is further simplified to

$$x^{(1)}(k+1) = q x^{(1)}(k) + \alpha q + \sum_{t=0}^{N} F_t k^t,$$

where $F_0 = \sum\limits_{i=0}^{N-1} \beta_i$, $F_N = (1-q)\frac{1}{N}\beta_{N-1}\theta_0$,

$$F_t = \sum_{i=t}^{N-1} \beta_i C_i^t + (1-q) \sum_{i=t-1}^{N-1} \frac{C_{i+1}^{i+1-t}}{i+1} \beta_i \theta_{i+1-t}, \quad t = 1, 2, \cdots, N-1.$$

By Crammer's rule, we have

$$\begin{cases} \hat{v} = q, \\ \hat{u}_0 = \alpha q + F_0 = \alpha q + \sum\limits_{i=0}^{N-1} \beta_i, \\ \hat{u}_t = F_t = \sum_{i=t}^{N-1} \beta_i C_i^t + (1-q) \sum_{s=0}^{N-t} \frac{C_{s+t}^s}{s+t} \beta_{s+t-1} \theta_s, t = 1, 2, \cdots, N-1, \\ \hat{u}_N = F_N = (1-q)\frac{1}{N}\beta_{N-1}\theta_0. \end{cases} \tag{3.1}$$

Step 2. Parameter estimation $\hat{a}, \hat{b}_0, \hat{b}_1, \cdots, \hat{b}_N$ of UGMP$(1,1,N)$ model.

Using Eq.(3.1), we get

$$e^{-\hat{a}} = q, \tag{3.2}$$

$$\sum_{i=0}^{N} P_i^i \delta_i \hat{b}_i = \alpha q + \sum_{i=0}^{N-1} \beta_i, \tag{3.3}$$

$$\sum_{i=t}^{N} P_i^{i-t} \delta_{i-t} \hat{b}_i = \sum_{i=t}^{N-1} \beta_i C_i^t + (1-q) \sum_{s=0}^{N-t} \frac{C_{s+t}^s}{s+t} \beta_{s+t-1} \theta_s, \tag{3.4}$$

$$P_N^0 \delta_0 \hat{b}_N = (1-q) \frac{1}{N} \beta_{N-1} \theta_0. \tag{3.5}$$

From Eq.(3.5), it follows that $\frac{\hat{b}_N}{\hat{a}} = \frac{1}{N} \beta_{N-1} \theta_0$.
From Eq.(3.4), the left is simplified to

$$\begin{aligned}
\sum_{i=t}^{N} P_i^{i-t} \delta_{i-t} \hat{b}_i =& \sum_{i=t}^{N} P_i^{i-t} (-1)^{i-t+1} \frac{1}{\hat{a}^{i-t+1}} [e^{-\hat{a}} + \sum_{s=0}^{i-t} (-1)^{s+1} \frac{\hat{a}^s}{s!}] \hat{b}_i \\
=& (1 - e^{-\hat{a}}) [\sum_{i=t}^{N} P_i^{i-t} (-1)^{i-t} \frac{\hat{b}_i}{\hat{a}^{i-t+1}}] \\
&+ \sum_{i=t+1}^{N} P_i^{i-t} (-1)^{i-t+1} \frac{\hat{b}_i}{\hat{a}^{i-t+1}} \sum_{s=1}^{i-t} (-1)^{s+1} \frac{\hat{a}^s}{s!}.
\end{aligned}$$

Let $\hat{\Delta}_t = \sum\limits_{i=t}^{N} (-1)^{i-t} P_i^{i-t} \frac{\hat{b}_i}{\hat{a}^{i-t+1}}$, then the above is written as

$$\begin{aligned}
\sum_{i=t}^{N} P_i^{i-t} \delta_{i-t} \hat{b}_i =& (1-q) \hat{\Delta}_t + \sum_{i=t+1}^{N} P_i^{i-t} (-1)^{i-t+1} \frac{\hat{b}_i}{\hat{a}^{i-t+1}} \sum_{s=1}^{i-t} (-1)^{s+1} \frac{\hat{a}^s}{s!} \\
=& (1-q) \hat{\Delta}_t + \sum_{s=1}^{N-t} \sum_{i=s+t}^{N} (-1)^{i-t+s} P_i^{i-t} \frac{\hat{b}_i}{s! \hat{a}^{i-t-s+1}} \\
=& (1-q) \hat{\Delta}_t + \sum_{s=1}^{N-t} \hat{\Delta}_{s+t} \frac{P_{i-t}^i}{P_{i-t-s}^i s!} = (1-q) \hat{\Delta}_t + \sum_{s=1}^{N-t} \hat{\Delta}_{s+t} C_{s+t}^t \\
=& (1-q) \hat{\Delta}_t + \sum_{i=t+1}^{N} \hat{\Delta}_i C_i^t.
\end{aligned}$$

So Eq.(3.4) is expressed as

$$(1-q) \hat{\Delta}_t + \sum_{i=t+1}^{N} \hat{\Delta}_i C_i^t = \sum_{i=t}^{N-1} \beta_i C_i^t + (1-q) \sum_{s=0}^{N-t} \frac{C_{s+t}^s}{s+t} \beta_{s+t-1} \theta_s.$$

Firstly, if it is considered as a polynomial of q, we have

$$\hat{\Delta}_t = \sum_{s=0}^{N-t} \beta_{s+t-1} \theta_s \frac{C_{s+t}^s}{s+t}, \tag{3.6}$$

$$\sum_{i=t+1}^{N} C_i^t \Delta_i = \sum_{i=t}^{N-1} \beta_i C_i^t. \tag{3.7}$$

Next, if Eq.(3.6) is true and Eq.(3.7) can also be certified by Eq.(3.6), both Eq.(3.6) and Eq.(3.7) are right. The details are as follows,

$$\begin{aligned}
\sum_{i=t+1}^{N} C_i^t \hat{\Delta}_i &= \sum_{i=t+1}^{N} C_i^t \sum_{s=0}^{N-i} \beta_{s+i-1}\theta_s \frac{C_{s+i}^s}{s+i} \\
&= \sum_{i=t+1}^{N} C_i^t \sum_{h=i-1}^{N-1} \beta_h \theta_{h+1-i} \frac{h+1-i}{h+1} C_{h+1} \\
&= \sum_{h=t}^{N-1} \frac{\beta_h}{h+1} \sum_{i=t+1}^{h+1} C_i^t \theta_{h+1-i} C_{h+1}^{h+1-i} \\
&= \sum_{h=t}^{N-1} \frac{\beta_h}{h+1} \sum_{s=0}^{h-t} C_{s+t+1}^t \theta_{h-s-t} C_{h+1}^{h-s-t} \\
&= \sum_{h=t}^{N-1} \frac{\beta_h C_h^t}{h-t+1} \sum_{s=0}^{h-t} C_{h-t+1}^{h-s-t} \theta_{h-s-t} \\
&= \sum_{h=t}^{N-1} \frac{\beta_h C_h^t}{h-t+1} (h-t+1) \\
&= \sum_{h=t}^{N-1} \beta_h C_h^t.
\end{aligned}$$

Therefore, by Lemma 6.3 in Appendix B, Eq.(3.6) follows that

$$\begin{aligned}
\sum_{t=1}^{N} \hat{\Delta}_t &= \sum_{t=1}^{N} \sum_{s=0}^{N-t} \frac{C_{s+t}^s}{s+t} \beta_{s+t-1}\theta_s = \sum_{s=0}^{N-1} \sum_{t=1}^{N-s} \beta_{s+t-1}\theta_s \frac{C_{s+t}^s}{s+t} \\
&= \sum_{s=0}^{N-1} \sum_{j=s}^{N-1} \beta_j \theta_s \frac{C_{j+1}^s}{j+1} = \sum_{j=0}^{N-1} \frac{\beta_j}{j+1} \sum_{s=0}^{j} C_{j+1}^s \theta_s \\
&= \sum_{j=0}^{N-1} \frac{\beta_j}{j+1}(j+1) = \sum_{j=0}^{N-1} \beta_j.
\end{aligned}$$

From Eq.(3.3), the left is

$$\begin{aligned}
&\sum_{i=0}^{N} P_i^i (-1)^{i+1} \frac{1}{a^{i+1}} [e^{-a} + \sum_{s=0}^{i} (-1)^{s+1} \frac{1}{s!} a^s] \hat{b}_i \\
=&(1-q) \sum_{i=0}^{N} P_i^i (-1)^{i+1} \frac{\hat{b}_i}{a^{i+1}} + \sum_{i=1}^{N} P_i^i (-1)^{i+1} \frac{\hat{b}_i}{a^{i+1}} \sum_{s=1}^{i} (-1)^{s+1} \frac{1}{s!} a^s \\
=&(1-q)\Delta_0 + \sum_{s=1}^{N} \sum_{i=s}^{N} P_i^i (-1)^{i+s} \frac{\hat{b}_i}{a^{i-s+1}} \frac{1}{s!}
\end{aligned}$$

$$
\begin{aligned}
&=(1-q)\Delta_0+\sum_{s=1}^{N}\sum_{t=0}^{N-s}(-1)^t\frac{\hat{b}_{s+t}}{a^{t+1}}P_{s+t}^t\frac{P_{s+t}^{s+t}}{P_{s+t}^t}\frac{1}{s!}\\
&=(1-q)\Delta_0+\sum_{s=1}^{N}\Delta_s.
\end{aligned}
$$

Substituting Eq.(3.6) and Eq.(3.7) to Eq.(3.3), we get

$$
\hat{\Delta}_0=\frac{\alpha q}{1-q}. \tag{3.8}
$$

Step 3. Solve $x^{(0)}(k+1)$.

By Lemma 6.2 in Appendix B, we have

$$
\begin{aligned}
x^{(0)}(k+1)&=x^{(1)}(k+1)-x^{(1)}(k)\\
&=[x^{(1)}(1)-\beta_0-\beta_1-\cdots-\beta_{N-1}-\frac{\alpha q}{1-q}]e^{-(k-1)}(q-1)\\
&+\sum_{t=0}^{N}\Delta_t[(k+1)^t-k^t]\\
&=\alpha q^{k+1}+\sum_{t=1}^{N}\Delta_t[(k+1)^t-k^t]=\alpha q^{k+1}+\sum_{t=1}^{N}\Delta_t\sum_{i=0}^{t-1}C_t^ik^i\\
&=\alpha q^{k+1}+\sum_{i=0}^{N-1}\sum_{t=i+1}^{N}\Delta_tC_t^ik^i=\alpha q^{k+1}+\sum_{i=0}^{N-1}\sum_{t=i}^{N-1}\beta_tC_t^ik^i\\
&=\alpha q^{k+1}+\sum_{i=0}^{N-1}\beta_i(k+1)^i.
\end{aligned}
$$

So the theorem is proven. □

Theorem 3.2. *Assume* $Z^{(0)}=(\rho x^{(0)}(1),\rho x^{(0)}(2),\cdots,\rho x^{(0)}(n))$ *is the multiple transformation of* $X^{(0)}$, *where* $z^{(0)}(k)=\rho x^{(0)}(k)$, $\rho\neq 0$. $\hat{a},\hat{b}_0,\hat{b}_1,\hat{b}_2,\cdots,\hat{b}_N$ *and* $\overline{a},\overline{b}_0,\overline{b}_1,\overline{b}_2,\cdots,\overline{b}_N$ *are the parameter estimations of UGMP*$(1,1,N)$ *models for* $X^{(0)}$ *and* $Z^{(0)}$ *respectively, then*

$$
\hat{a}=\overline{a},\hat{b}_i=\rho\overline{b}_i,i=0,1,2,\cdots,N.
$$

Proof. By Theorem 2.2, the parameter estimation of GMP$(1,1,N)$ model for $X^{(0)}$ is given

$$
\hat{\mathbf{u}}=(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{Y}=\frac{1}{|\mathbf{B}^T\mathbf{B}|}(\mathbf{B}^T\mathbf{B})^*\mathbf{B}^T\mathbf{Y}, \tag{3.9}
$$

and the parameter estimation of $Z^{(0)}$ is

$$
\hat{\overline{\mathbf{u}}}=(\overline{\mathbf{B}}^T\overline{\mathbf{B}})^{-1}\overline{\mathbf{B}}^T\mathbf{Z}=\frac{1}{\rho|\overline{\mathbf{B}}^T\overline{\mathbf{B}}|}(\overline{\mathbf{B}}^T\overline{\mathbf{B}})^*\overline{\mathbf{B}}^T\rho\mathbf{Y}, \tag{3.10}
$$

where $\overline{\mathbf{B}} = \begin{pmatrix} \rho x^{(1)}(1) & \rho x^{(1)}(2) & \cdots & \rho x^{(1)}(n-1) \\ 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & n-1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 2^N & \cdots & (n-1)^N \end{pmatrix}$. With the properties of determinants, we have the relation

$$|\overline{\mathbf{B}}^T\overline{\mathbf{B}}| = \rho^2|\mathbf{B}^T\mathbf{B}|. \tag{3.11}$$

For convenience, the adjoint matrix $(\mathbf{B}^T\mathbf{B})^*$ is noted as $(\mathbf{B}_{ij})_{(N+2)\times(N+2)}$, then

$$(\overline{\mathbf{B}}^T\overline{\mathbf{B}})^* = \begin{pmatrix} \mathbf{B}_{11} & \rho\mathbf{B}_{12} & \cdots & \rho\mathbf{B}_{1,N+2} \\ \rho\mathbf{B}_{21} & \rho^2\mathbf{B}_{22} & \cdots & \rho^2\mathbf{B}_{2,N+2} \\ \vdots & \vdots & \vdots & \vdots \\ \rho\mathbf{B}_{N+2,1} & \rho^2\mathbf{B}_{N+2,2} & \cdots & \rho^2\mathbf{B}_{N+2,N+2} \end{pmatrix}. \tag{3.12}$$

Substituting Eq.(3.11) and Eq.(3.12) to Eq.(3.10), and contrasting with Eq.(3.9), we have

$$\hat{\overline{v}} = \hat{v}, \hat{\overline{u}}_i = \hat{u}_i, \quad i = 0, 1, \cdots, N.$$

Accordingly, by Eq.(2.9) and Eq.(2.10), we obtain

$$\hat{\overline{a}} = \hat{a}, \hat{\overline{b}}_i = \rho\hat{b}_i, \quad i = 0, 1, \cdots, N.$$

$\square$

Theorem 3.3. *Suppose that $Z^{(0)} = (z^{(0)}(1), z^{(0)}(2)), \cdots, z^{(0)}(n))$ is the multiple transformation of $X^{(0)}$, where $z^{(0)}(k) = \rho x^{(0)}(k), k = 1, 2, \cdots, n$, $\rho \neq 0$. The predicted values $\hat{x}^{(0)}(k)$ and $\hat{z}^{(0)}(k)$ of UGMP$(1,1,N)$ models for $X^{(0)}$ and $Z^{(0)}$ satisfy*

$$\hat{z}^{(0)}(k+1) = \rho\hat{x}^{(0)}(k+1), k = 1, 2, \cdots, n-1.$$

Proof. By Lemma 6.2 in Appendix B, we get

$$\hat{x}^{(1)}(k+1) = [x^{(1)}(1) - \sum_{t=0}^{N}\sum_{s=0}^{N-t}(-1)^s P_{s+t}^s \frac{\hat{b}_{s+t}}{\hat{a}^{s+1}}]e^{-\hat{a}k} + \sum_{t=0}^{N}\sum_{s=0}^{N-t}(-1)^s P_{s+t}^s \frac{\hat{b}_{s+t}}{\hat{a}^{s+1}}(k+1)^t$$

and

$$\hat{z}^{(1)}(k+1) = [z^{(1)}(1) - \sum_{t=0}^{N}\sum_{s=0}^{N-t}(-1)^s P_{s+t}^s \frac{\hat{\overline{b}}_{s+t}}{\hat{\overline{a}}^{s+1}}]e^{-\hat{\overline{a}}k} + \sum_{t=0}^{N}\sum_{s=0}^{N-t}(-1)^s P_{s+t}^s \frac{\hat{\overline{b}}_{s+t}}{\hat{\overline{a}}^{s+1}}(k+1)^t.$$

By Theorem 3.2, we have

$$\hat{z}^{(1)}(k+1) = \rho\hat{x}^{(1)}(k+1).$$

So the predict value is

$$\hat{z}^{(0)}(k+1) = \hat{z}^{(1)}(k+1) - \hat{z}^{(1)}(k) = \rho\hat{x}^{(1)}(k+1) - \rho\hat{x}^{(1)}(k) = \rho\hat{x}^{(0)}(k+1), \tag{3.13}$$

where $k = 1, 2, \cdots, n-1$. $\square$

Theorem 3.2 and Theorem 3.3 indicate that the multiple transformation of the original sequence has no influence on modeling performance. Then a suitable number ρ can be selected to reduce the condition number in Eq.(2.8) without changing accuracy.

4. Modeling procedure

As described in the properties of UGMP$(1, 1, N)$ model, the grey polynomial models with different polynomial order N describe different data patterns. Thus, it is important to give a feasible way to select the polynomial order N. In addition, in order to avoid overfitting and ill condition problems, the alternative polynomial orders are always suggested as $E = \{0, 1, 2, 3\}$ in Ref [25].

4.1. The criterion for determining the polynomial order

The following empirical criterion is given based on the sequence $x^{(0)}(k) = \alpha q^k + \beta_0 + \beta_1 k + \cdots + \beta_{N-1} k^{N-1}$.

Definition 4.1. (Definition 3, [25]) Assume that the r order difference sequence of the original sequence is $D^{(r)} = (d^{(r)}(r+1), d^{(r)}(r+2), \cdots, d^{(r)}(n))$, $r \geq 1$. Then the r order stepwise ratio sequence is defined as

$$\delta^{(r)} = (\delta^{(r)}(r+2), \delta^{(r)}(r+3), \cdots, \delta^{(r)}(n)),$$

where $\delta^{(r)}(k) = \frac{d^{(r)}(k)}{d^{(r)}(k-1)}, k = r+2, r+3, \cdots, n$.

Especially, when $r = 0$, $\delta^{(0)}(k) = \frac{x^{(0)}(k)}{x^{(0)}(k-1)}$.

Definition 4.2. (Definition 4, [25]) Assume

$$\delta^{(r)}_{max} = \max_{r+2<k<n} \delta^{(r)}_k, \quad \delta^{(r)}_{min} = \min_{r+2<k<n} \delta^{(r)}_k.$$

Then the degree of grey index law is defined as $\zeta^{(r)} = \delta^{(r)}_{max} - \delta^{(r)}_{min}$ for any non-negative integer r.

From the above definition, if $\zeta^{(r)} = 0$, then $\zeta^{(r+1)} = 0$. The sufficient and necessary condition for the original sequence to be r order homogeneous is $\delta^{(r)} = constant$. Therefore, the previous criterion determines the alternative polynomial orders based on $N = \arg\max_{r\in E} \zeta^{(r)}$. But the sequence usually does not satisfy the form $x^{(0)}(k) = \alpha q^k + \beta_0 + \beta_1 k + \cdots + \beta_{N-1} k^{N-1}$ completely in practice, so the first criterion of selecting polynomial order is usually set according to $\zeta^{(r)} < 10$.

Taking both the fitting performance and the predicting ability into account, the original sequence is divided into two parts, the training data and the testing data. The training set is used to construct the parameter estimations of the proposed model and determine the optimal polynomial order according to the minimal mean absolute percentage error (MAPE),

$$\text{MAPE} = \frac{1}{l} \sum_{k=1}^{l} \left| \frac{x^{(0)}(k) - \hat{x}^{(0)}(k)}{x^{(0)}(k)} \right| \times 100\%.$$

The testing set is used to test the predicting performance of the proposed model.

4.2. Modeling process

According to the above modeling process, the computational steps of HUGMP $(1,1,N)$ can be described in Fig.1.

5. Numerical example and application

To test the performance of HUGMP$(1,1,N)$ model, exponential sequences with polynomial time terms, nuclear energy consumption forecasting and electricity consumption forecasting are considered as case studies.

5.1. Exponential sequences with polynomial time terms

Consider the sequences $x^{(0)}(k) = 2q^k + 8 + 15k - 2k^2, q = 0.5, 1.5, 3, 6, 9, 12, k = 1, 2, \cdots, 6$. Firstly, we choose the polynomial order $N = 3$ according to the stepwise ratio sequences of the raw data with different orders $0 = \zeta^{(3)} < \zeta^{(2)} < \zeta^{(1)}$ in Table 1, then we fit the sequences using the proposed UGMP$(1,1,3)$ model and HUGMP$(1,1,3)$ model. Secondly, Figure 2 shows that the log of MAPE (log_{10}MAPE) of all sequences become stable when the highly precise parameter $m \geq 40$, so we choose $m = 40$ in the following models. Thirdly, Table 2 indicates that the MAPE of HUGMP$(1,1,3)$ model is much lower than that of UGMP$(1,1,3)$ model. It also verifies that HUGMP$(1,1,3)$ model can avoid round-off error and achieve high accuracy. From the listed results in Table 2, we observe that HUGMP$(1,1,3)$ model can break the restriction of q in traditional grey models. Finally, the unbiasedness of the proposed model is verified by highly precise parameter estimation of $\alpha, q, \beta_0, \beta_1, \beta_2$ in Table 3.

Table 1. The stepwise ratio of different sequences.

k	$q=0.5$	$q=1.5$	$q=3$	$q=6$	$q=9$	$q=12$
1	22	24	27	33	39	45
2	30.5	34.5	48	102	192	318
3	35.25	41.75	89	467	1493	3491
4	36.125	46.125	198	2628	13158	41508
5	33.0625	48.1875	519	15585	118131	497697
6	26.03125	48.78125	1484	93338	1062908	5971994
$\zeta^{(0)}$	0.599029	0.425178	1.081567	2.898055	4.074629	4.93259
$\zeta^{(1)}$	5.7959	0.4026	1.0539	0.7110	0.4969	0.3774
$\zeta^{(2)}$	0.0254	0.2495	0.3623	0.0657	0.0275	0.0151
$\zeta^{(3)}$	0	0	0	0	0	0

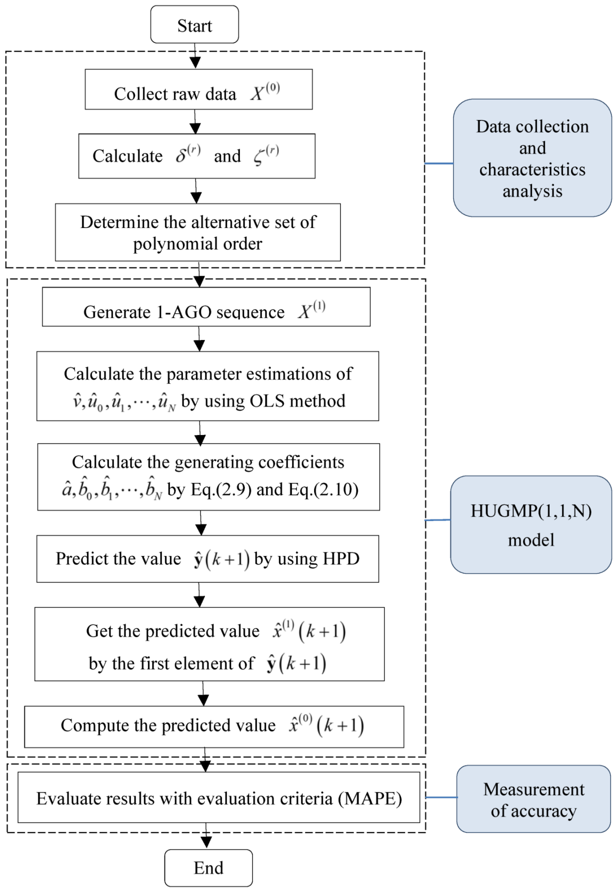

Figure 1. The overall process of the proposed model.

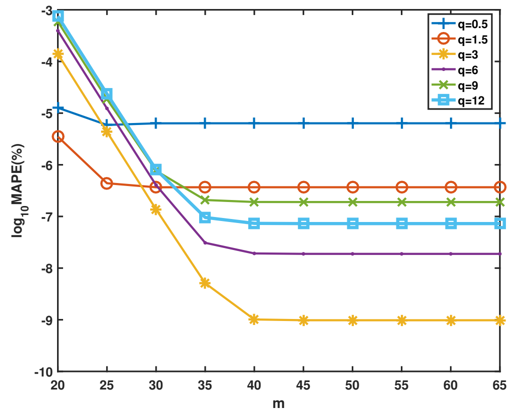

Figure 2. The errors of HUGMP(1,1,3) model with different high precise parameter m.

Table 2. The MAPE errors with different models.

q	GMP(1,1,3)	UGMP(1,1,3)	HUGMP(1,1,3)
0.5	0.54	2.67e-5	**6.36e-6**
1.5	47.79	1.34e-6	**3.67e-7**
3	21.84	2.92e-9	**2.63e-9**
6	52.55	1.25e-7	**1.92e-8**
9	64.76	5.15e-7	**5.34e-9**
12	70.47	6.79e-7	**1.60e-7**

Table 3. The parameter estimations of different series ($m = 40$).

q	α		q		β_0		β_1		β_2	
	$\hat{\alpha}$	APE(%)	$\hat{q}$	APE(%)	$\hat{\beta}_0$	APE(%)	$\hat{\beta}_1$	APE(%)	$\hat{\beta}_2$	APE(%)
0.5	2.0000	1.1806e-4	0.5000	2.4323e-5	8.0000	1.1357e-6	15.0000	5.8676e-6	-2.0000	3.0836e-6
1.5	2.0000	1.4740e-6	1.5000	1.7546e-7	8.0000	1.7352e-7	15.0000	2.7510e-7	-2.0000	3.2188e-7
3	2.0000	3.1176e-9	3.0000	5.0567e-11	8.0000	1.5050e-8	15.0000	7.7616e-9	-2.0000	3.5688e-10
6	2.0000	1.1395e-7	6.0000	2.9902e-12	8.0000	6.8366e-9	15.0000	7.4959e-9	-2.0000	1.6213e-7
9	2.0000	1.8691e-7	9.0000	1.8632e-11	8.0000	9.2949e-7	15.0000	4.4054e-8	-2.0000	1.5079e-7
12	2.0000	2.4723e-6	12.0000	1.6342e-11	8.0000	4.1828e-6	15.0000	1.9735e-6	-2.0000	3.1410e-6

5.2. Nuclear energy consumption

The nuclear energy consumption of China has been discussed with grey Riccati model (GRM) [17]. The data is divided into two groups: the first 15 data from 2001 to 2015 are used to construct models, and the rest from 2016 to 2018 are applied to test the forecasting accuracy of models.

From Table 4, by the previous criterion $\zeta^{(r)} < 10$, the alternative set of polynomial orders is {0,1,3}. Moreover, in order to illustrate the effectiveness of the criterion, we also construct GMP$(1,1,N)$ models and HUGMP$(1,1,N)$ models with $N = 0,1,2,3$, then the results and errors are listed in Table 5 and Table 6, respectively. On the modeling sequence, the MAPE error of HUGMP$(1,1,3)$ model is the smallest, close to that of GMP$(1,1,3)$ model and much smaller than that of other GMP$(1,1,N)$ models, HUGMP$(1,1,N)$ models and GRM model. So we select $N = 3$ as the optimal polynomial order and HUGMP$(1,1,3)$ as the optimal model. Meanwhile, the MAPE of GMP$(1,1,2)$ and HUGMP$(1,1,2)$ are 559.174% and 354.870% respectively, which shows the criterion is effective.

Table 4. The stepwise ratio of the nuclear energy consumption series.

r	0	1	2	3
$\zeta^{(r)}$	0.69	3.96	11.50	6.17

In addition, the MAPE of HUGMP$(1,1,N)$ model is smaller than the MAPE of the GMP$(1,1,N)$ model at the same polynomial order in fitting and prediction. That shows the validity of our proposed unbiased grey model, improving the accuracy of fitting and strengthening the ability of prediction. On the performance of simulation, HUGMP$(1,1,3)$ model is superior to the other models, as shown in Table 6. On the testing sequence, the MAPE of HUGMP$(1,1,3)$ model is a little bigger than HUGMP$(1,1,1)$ model and much less than other models. What is more, all the MAPE of HUGMP$(1,1,3)$ model are below 5% which reveals that the proposed model obtains excellent results and also successfully catches the trend of the nuclear energy consumption of China.

5.3. Electricity consumption

Considering the annual electricity consumption of India, the raw data are collected from BP Statistical Review of World Energy 2019, which was predicted in [26] by fractional grey prediction model (FPGM(1,1,α)). The first 7 data from 2008 to 2014 as the training set are to build the models, and the next 4 data from 2015 to 2018 are used as the testing set to evaluate the predicting performance of models.

Table 7 shows that $\zeta^{(0)} < \zeta^{(1)} < \zeta^{(3)} < 10 < \zeta^{(2)}$, so the alternative polynomial order set is $\{0, 1, 3\}$. But the parameter v in HUGMP(1,1,3) is negative, which isn't satisfied with Theorem 2.5. Therefore, we construct HUGMP(1,1,N) models with $N = 1, 2$. In order to illustrate the applicability and good forecasting performances of HUGMP(1,1,N) model, two benchmark models, including the grey polynomial model (GMP(1,1,N)) and fractional grey prediction model (FGPM(1,1,α)) [26] are established to fit and forecast the electricity in India. Moreover, we list the whitenization equation of these models in Table 8.

The competitive results are listed in Table 9. It is evidently that MAPE of HUGMP(1,1,1) is lower than other benchmark models either in simulation or in prediction, and the changing range of APE is smaller than others. All above indicated the validity of the proposed unbiased polynomial grey models. At the same

Table 5. The results of the nuclear energy consumption of China in different models.

Year	Data	GRM	GMP (1,1,0)	HUGMP (1,1,0)	GMP (1,1,1)	HUGMP (1,1,1)	GMP (1,1,2)	HUGMP (1,1,2)	GMP (1,1,3)	HUGMP (1,1,3)
2001	4.0	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
2002	5.7	8.8675	7.0893	7.1365	2.5280	9.4390	8.5348	8.7335	6.3341	6.1517
2003	9.8	9.4171	7.9914	8.0425	4.2110	9.9176	10.0455	10.1708	8.8629	8.9858
2004	11.4	10.1195	9.0084	9.0636	5.9672	10.5075	11.8773	11.8872	10.7982	11.0184
2005	12.0	10.9320	10.1547	10.2143	7.7998	11.2347	14.2973	14.0868	12.2451	12.4291
2006	12.4	11.8505	11.4470	11.5112	9.7120	12.1311	17.7949	17.1235	13.3243	13.4052
2007	14.1	12.9924	12.9037	12.9726	11.7075	13.2362	23.2673	21.6100	14.1746	14.1425
2008	15.5	14.3672	14.5457	14.6197	13.7897	14.5985	32.3579	28.6076	14.9555	14.8451
2009	15.9	15.9077	16.3967	16.4758	15.9625	16.2778	48.0781	39.9544	15.8506	15.7265
2010	16.7	17.6676	18.4833	18.5676	18.2297	18.3480	75.9455	58.8343	17.0709	17.0092
2011	19.5	19.9067	20.8354	20.9250	20.5956	20.9000	126.070	90.7616	18.8590	18.9256
2012	22.0	22.6708	23.4868	23.5817	23.0644	24.0458	216.977	145.287	21.4940	21.7184
2013	25.3	26.1616	26.4757	26.5756	25.6405	27.9239	382.607	238.954	25.2965	25.6409
2014	30.0	30.7334	29.8448	29.9497	28.3286	32.7046	685.152	400.415	30.6356	30.9574
2015	38.6	37.3061	33.6427	33.7522	31.1337	38.5978	1238.57	679.295	37.9354	37.9442
2016	48.3	46.6249	37.9240	38.0374	34.0608	45.8627	2251.64	1161.55	47.6836	46.8893
2017	56.1	58.9755	42.7500	42.8667	37.1151	54.8184	4106.94	1996.05	60.4411	58.0937
2018	66.6	75.7683	48.1901	48.3091	40.3023	65.8584	7505.45	3440.66	76.8528	71.8714

Table 6. The errors of the nuclear energy consumption of China in different models.

Year	GRM	GMP (1,1,0)	HUGMP (1,1,0)	GMP (1,1,1)	HUGMP (1,1,1)	GMP (1,1,2)	HUGMP (1,1,2)	GMP (1,1,3)	HUGMP (1,1,3)
2001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2002	55.5705	24.3731	25.2011	55.6495	65.5970	49.7327	53.2193	11.1254	7.9243
2003	3.9068	18.4550	17.9335	57.0307	1.1996	2.5053	3.7840	9.5626	8.3081
2004	11.2326	20.9793	20.4947	47.6562	7.8292	4.1868	4.2736	5.2790	3.3472
2005	8.8998	15.3773	14.8805	35.0019	6.3778	19.1438	17.3903	2.0426	3.5756
2006	4.4313	7.6857	7.1680	21.6770	2.1685	43.5075	38.0929	7.4542	8.1064
2007	7.8556	8.4847	7.9954	16.9682	6.1263	65.0164	53.2625	0.5290	0.3011
2008	7.3084	6.1567	5.6795	11.0342	5.8163	108.761	84.5651	3.5127	4.2249
2009	0.0483	3.1242	3.6215	0.3929	2.3761	202.378	151.286	0.3104	1.0912
2010	5.7941	10.6785	16.1833	9.1601	9.8680	354.764	252.301	2.2212	1.8514
2011	2.0854	6.8483	7.3076	5.6185	7.1791	546.514	365.444	3.2870	2.9456
2012	3.0491	6.7583	7.1894	4.8380	9.2993	886.258	560.397	2.3002	1.2800
2013	3.4054	4.6469	5.0420	1.3458	10.3712	1412.28	844.483	0.0138	1.3473
2014	2.4445	0.5172	0.1676	5.5712	9.0152	2183.84	1234.72	2.1186	3.1915
2015	3.3521	12.8426	12.5591	19.3427	0.0056	3108.72	1659.83	1.7219	1.6990
MAPE_{simu}(%)	8.5274	10.4948	9.7615	20.8062	9.5486	599.174	354.870	3.4319	**3.2796**
2016	3.4682	21.4825	21.2476	29.4808	5.0461	4561.78	2304.86	1.2763	2.9207
2017	5.1256	23.7968	23.5887	33.8412	2.2845	7220.75	3458.02	7.7381	3.5538
2018	13.7662	27.6424	27.4637	39.4860	1.1136	11169.4	5066.15	15.3946	7.9150
MAPE_{pred}(%)	7.4534	24.3072	24.1000	34.2693	2.8147	7650.65	3609.68	8.1363	**4.7965**
MAPE_{over}(%)	8.3379	12.9323	12.1513	23.1821	8.4263	1774.42	897.338	4.2160	**3.5324**

Table 7. The stepwise ratio of the electricity consumption sequence.

$\delta^{(r)}$	r=0	r=1	r=2	r=3
$\zeta^{(r)}$	0.0532	1.5392	23.6110	4.2588

time, this illustrates that the proposed model is more suitable for forecasting electricity consumption.

Table 8. The whitenization differential equation of different models.

Model	Whitenization differential equation
FPGM(1,1,α)	$\frac{dx^{0.9642}(t)}{dt} - 0.0678x^{0.9642}(t) = -90.9816t^{0.1596} + 861.5374$
GMP(1,1,0)	$\frac{dx(t)}{dt} - 0.0702x(t) = 792.6196$
HUGMP(1,1,0)	$\frac{dx(t)}{dt} - 0.0702x(t) = 821.5505$
GMP(1,1,1)	$\frac{dx(t)}{dt} - 0.0630x(t) = 7.6630t + 789.3804$
HUGMP(1,1,1)	$\frac{dx(t)}{dt} - 0.0521x(t) = 19.1447t + 784.8909$

Table 9. Comparisons of different models in the simulation and prediction of electricity consumption in India.

Year	Actual	FPGM		GMP(1,1,0)		HUGMP(1,1,0)		GMP(1,1,1)		HUGMP(1,1,1)	
	values	values	APE(%)	values	APE(%)	values	APE(%)	values	APE(%)	values	APE(%)
2008	828.40	828.40	0.0000	828.40	0.0000	828.40	0.0000	828.40	0.0000	828.40	0.0000
2009	879.70	879.69	0.0012	881.36	0.1887	881.80	0.2391	880.43	0.0830	879.42	0.0323
2010	937.50	946.37	0.9466	945.46	0.8491	945.94	0.9003	945.56	0.8602	946.12	0.9190
2011	1034.00	1016.01	1.7399	1014.23	1.9120	1014.74	1.8624	1014.93	1.8441	1016.38	1.7037
2012	1091.80	1089.57	0.2046	1087.99	0.3490	1088.55	0.2977	1088.81	0.2740	1090.41	0.1272
2013	1146.10	1167.70	1.8847	1167.13	1.8349	1167.72	1.8868	1167.49	1.8661	1168.40	1.9457
2014	1262.20	1250.93	0.8926	1252.01	0.8073	1252.66	0.7560	1251.28	0.8652	1250.56	0.9222
MAPE$_{simu}$(%)			0.9448		0.9902		0.9904		0.9654		**0.9417**
2015	1317.30.0	1339.74	1.7034	1343.07	1.9563	1343.77	2.0093	1340.52	1.7625	1337.12	1.5042
2016	1401.70	1434.58	2.3456	1440.76	2.7866	1441.51	2.8399	1435.56	2.4153	1428.30	1.8978
2017	1470.30	1535.92	4.4631	1545.55	5.1180	1546.35	5.1727	1536.77	4.5209	1524.37	3.6772
2018	1561.10	1644.26	5.3268	1657.96	6.2046	1658.83	6.2601	1644.56	5.3465	1625.57	4.1298
MAPE$_{pred}$(%)			3.4598		4.0164		4.0705		3.5114		**2.8023**

6. Conclusion

HUGMP$(1, 1, N)$ model is proposed for solving the "misplaced replacement" problem of GMP$(1, 1, N)$ model. The main conclusions are summarized as follows.

(1) A novel way to obtain the parameter estimation with direct modeling method is proposed, in which the recurrence relation is deduced by the solution of homogenized differential equations transformed from the whitenization equation.

(2) HPD method is used to give the simulated value of unbiased GMP$(1, 1, N)$ model, getting high accuracy and avoiding round-off error in computation.

(3) The properties of UGMP(1,1,N) model and the perfect fitting of nonhomogeneous exponential sequences with polynomial time terms show the unbiasedness of the proposed model and higher accuracy compared with GMP$(1,1,N)$ model.

(4) Two cases, including nuclear energy consumption of China and electricity consumption of India, are illustrated that our proposed model is superior to other benchmark models, GMP(1,1,N) model, fractional grey prediction model (FPGM(1,1,α)) and grey Riccati model (GRM).

The proposed unbiased grey polynomial model based on precise integration method can be applied to other time sequences with nonlinearity and fluctuations. The proposed method for deducing the recurrence relation of the whitenization equation gives a new way to solve the parameter estimation of the unbiased grey model with nonlinear grey action. The high accuracy numerical method (HPD) is used to directly compute the time response function of grey models with complex structures, of which the time response function formula is hard to express. Moreover, the selection of polynomial forms should be explored in order to simplify the structure of grey polynomial model and improve the simulation accuracy.

Acknowledgements

The authors express their sincere gratitude to the editors and reviewers for their helpful comments and suggestions.

Appendix A:

By the solution theory of differential equation, Eq.(2.2) has a unique analytic solution $\mathbf{y}(t) = e^{\mathbf{A}(t-t_1)}\mathbf{y}(t_1)$. Then the discrete solution $\mathbf{y}$ at t_{k+1} is

$$\mathbf{y}(t_{k+1}) = \mathbf{y}(k+1) = e^{\mathbf{A}k}\mathbf{y}(1).$$

Consequently,

$$\mathbf{y}(k+1) = e^{\mathbf{A}k}\mathbf{y}(1) = e^{\mathbf{A}}(e^{\mathbf{A}(k-1)}\mathbf{y}(1)) = e^{\mathbf{A}}\mathbf{y}(k).$$

This implies the necessity of the condition (2.3) for (2.4).

Conversely, assuming that condition (2.3) is satisfied, we have

$$\mathbf{y}(k+1) = e^{\mathbf{A}}\mathbf{y}(k) = e^{\mathbf{A}}e^{\mathbf{A}}\mathbf{y}(k-1) = \cdots = e^{\mathbf{A}k}\mathbf{y}(1).$$

The proof is complete.

Appendix B:

Lemma 6.1. *The discrete time response function of whitening equation is*

$$x^{(1)}(k+1) = e^{-a}x^{(1)}(k) + \sum_{t=0}^{N}[-\Delta_t e^{-a} + \sum_{i=t}^{N} C_i^t \Delta_i]k^t.$$

Proof. By Theorem 2.1,

$$\begin{aligned} x^{(1)}(k+1) &= e^{-a}x^{(1)}(k) + \sum_{t=0}^{N} u_t k^t \\ &= e^{-a}x^{(1)}(k) + \sum_{t=0}^{N}\sum_{s=t}^{N} \delta_s P_{s-t}^{s-t} b_s k^t. \end{aligned}$$

The coefficient u_t is

$$\begin{aligned} u_t &= \sum_{t=0}^{N}\sum_{s=t}^{N} \delta_s P_{s-t}^{s-t} b_s = \sum_{s=0}^{N-t} \delta_s P_{s+t}^{s} b_{s+t} \\ &= \sum_{s=0}^{N-t} [(-1)^{s+1} P_{s+t}^{s} \frac{b_{s+t}}{a^{s+1}} (e^{-a} + \sum_{i=0}^{s} (-1)^{i+1} \frac{a^i}{i!})] \\ &= \sum_{s=0}^{N-t} (-1)^{s+1} P_{s+t}^{s} \frac{b_{s+t}}{a^{s+1}} e^{-a} + \sum_{s=0}^{N-t}\sum_{i=0}^{s} (-1)^{i+s} \frac{a^i}{i! a^{s+1}} P_{s+t}^{s} b_{s+t}. \end{aligned}$$

Assume that $\Delta_t = \sum\limits_{s=0}^{N-t} (-1)^s P_{s+t}^{s} \frac{b_{s+t}}{a^{s+1}}$, u_t is expressed as

$$\begin{aligned} u_t &= \sum_{t=0}^{N} [-\Delta_t e^{-a} + \sum_{s=0}^{N-t}\sum_{i=0}^{s} (-1)^{i+s} \frac{a^{i-s-1}}{i!} P_{s+t}^{s} b_{s+t}] \\ &= -e^{-a}\sum_{t=0}^{N} \Delta_t + \sum_{i=0}^{N-t}\sum_{s=i}^{N-t} (-1)^{i+s} \frac{a^{i-s-1}}{i!} P_{s+t}^{s} b_{s+t}] \\ &= -e^{-a}\sum_{t=0}^{N} \Delta_t + \sum_{i=0}^{N-t}\sum_{h=0}^{N-t-i} (-1)^{h} \frac{a^{-h-1}}{i!} P_{h+i+t}^{h+i} b_{h+i+t}] \\ &= -e^{-a}\sum_{t=0}^{N} \Delta_t + \sum_{i=0}^{N-t} \Delta_{t+i} \frac{1}{i!} \frac{P_{h+t+i}^{h+i}}{P_{h+t+i}^{h}} \\ &= -e^{-a}\sum_{t=0}^{N} \Delta_t + \sum_{i=0}^{N-t} \Delta_{t+i} C_{t+i}^{t} \\ &= -e^{-a}\sum_{t=0}^{N} \Delta_t + \sum_{i=t}^{N} \Delta_i C_i^t. \end{aligned}$$

Hence, the discrete time response function of whitenization equation is

$$x^{(1)}(k+1) = e^{-a}x^{(1)}(k) + \sum_{t=0}^{N} [-\Delta_t e^{-a} + \sum_{i=t}^{N} C_i^t \Delta_i] k^t.$$

□

Lemma 6.2. *Discrete time response function of whitenization equation (2.1) is*

$$x^{(1)}(k+1) = e^{-a}x^{(1)}(k) + \sum_{t=0}^{N} [-\Delta_t e^{-a} + \sum_{i=t}^{N} C_i^t \Delta_i] k^t \tag{6.1}$$

DOI https://doi.org/10.12150/jnma.2024.643 | Generated on 2025-03-11 05:36:12

if and only if

$$x^{(1)}(k+1) = [x^{(1)}(1) - \sum_{t=0}^{N} \Delta_t] e^{-ak} + \sum_{t=0}^{N} \Delta_t (k+1)^t, \tag{6.2}$$

where $\Delta_t = \sum\limits_{s=0}^{N-t} (-1)^s P_{s+t}^s \frac{b_{s+t}}{a^{s+1}}$.

Proof.

$$\begin{aligned}
x^{(1)}(k+1) &= [x^{(1)}(1) - \sum_{t=0}^{N} \Delta_t] e^{-ak} + \sum_{t=0}^{N} \Delta_t (k+1)^t \\
&= e^{-a} x^{(1)}(k) - e^{-a} \sum_{t=0}^{N} \Delta_t k^t + \sum_{t=0}^{N} \Delta_t (k+1)^t \\
&= e^{-a} x^{(1)}(k) - e^{-a} \sum_{t=0}^{N} \Delta_t k^t + \sum_{t=0}^{N} \Delta_t \sum_{i=0}^{t} C_t^i k^i \\
&= e^{-a} x^{(1)}(k) - e^{-a} \sum_{t=0}^{N} \Delta_t k^t + \sum_{t=0}^{N} \sum_{i=t}^{N} \Delta_i C_i^t k^t \\
&= e^{-a} x^{(1)}(k) + \sum_{t=0}^{N} [-\Delta_t e^{-a} + \sum_{i=t}^{N} C_i^t \Delta_i] k^t,
\end{aligned}$$

this implies the necessity of the condition (6.2) for (6.1).

Conversely, when $k = 2$, the conclusion is right

$$\begin{aligned}
x^{(1)}(2) &= e^{-a} x^{(1)}(1) + \sum_{t=0}^{N} [-\Delta_t e^{-a} + \sum_{i=t}^{N} C_i^t \Delta_i] \\
&= [x^{(1)}(1) - \sum_{t=0}^{N} \Delta_t] e^{-a} + \sum_{t=0}^{N} \sum_{i=t}^{N} \Delta_i C_i^t \\
&= [x^{(1)}(1) - \sum_{t=0}^{N} \Delta_t] e^{-a} + \sum_{t=0}^{N} \Delta_t 2^t.
\end{aligned}$$

If $k \leq s$, the conclusion (6.2) is right. When $k = s+1$,

$$\begin{aligned}
x^{(1)}(s+2) &= e^{-a} x^{(1)}(s+1) + \sum_{t=0}^{N} [-\Delta_t e^{-a} + \sum_{i=t}^{N} C_i^t \Delta_i] (s+1)^t \\
&= e^{-a} [x^{(1)}(1) - \sum_{t=0}^{N} \Delta_t] e^{-as} - e^{-a} \sum_{t=0}^{N} \Delta_t (s+1)^t \\
&\quad + \sum_{t=0}^{N} \sum_{i=t}^{N} \Delta_i C_i^t (s+1)^t + e^{-a} \sum_{t=0}^{N} \Delta_t (s+1)^t \\
&= [x^{(1)}(1) - \sum_{t=0}^{N} \Delta_t] e^{-a(s+1)} + \sum_{t=0}^{N} \Delta_t (s+2)^t.
\end{aligned}$$

The proof is complete. □

Lemma 6.3. *For arbitrary t,*

$$\sum_{i=1}^{k} i^t = \frac{1}{t+1}\sum_{s=0}^{t} C_{t+1}^{s}\theta_s k^{t+1-s},$$

where $\theta_i, i = 0, 1, \cdots, t$ are absolute constants.

Especially, when $k = 1$, by Lemma 6.3, we get

$$\sum_{i=1}^{1} i^t = \frac{1}{t+1}\sum_{s=0}^{t} C_{t+1}^{s}\theta_s 1^{t+1-s},$$

that is

$$\sum_{s=0}^{t} C_{t+1}^{s}\theta_s = t+1. \tag{6.3}$$

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