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Limit Cycles for a Class of Continuous-Discontinuous Piecewise Differential Systems

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Abstract During this century, an increasing interest appeared in studying the planar piecewise differential systems. This is due to their numerous applications for modelling many natural phenomena. For understanding the dynamics of the planar differential systems we must control the existence or non-existence of periodic orbits and limit cycles. So many papers have been published studying the existence or non-existence of periodic orbits and limit cycles for continuous or discontinuous piecewise differential systems. But until now very few papers have studied the periodic orbits and limit cycles of piecewise differential systems where two differential systems of the piecewise differential system are continuous and discontinuous respectively. We study the periodic orbits and limit cycles of the planar continuous–discontinuous piecewise differential systems separated by two parallel straight lines, such that either in one of these straight lines the piecewise differential system is continuous and in the other one discontinuous. In two pieces of these piecewise differential systems there are arbitrary Hamiltonian systems of degree two and in the third piece there is an arbitrary Hamiltonian system of degree one forming the continuous-discontinuous piecewise differential systems. We determine the limit cycles of these piecewise differential systems by considering two cases. In the first the Hamiltonian system of degree one can be in the middle of the three zones, and in the second it is on one side of the three zones.

Keywords Limit cycles, Hamiltonian system, continuous-discontinuous piecewise linear differential systems, first integrals

MSC(2010) Primary 34C05, 34A34.

1. Introduction

Poincaré's works started the qualitative study of the differential systems instead of finding exact or approximative solutions of themselves. With him also appeared the notion of the limit cycles which became one of the most important objects for understanding the dynamics of the differential systems in the plane, see [27].

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The concept of a limit cycle is a concept whose importance is not hidden from any researcher in the area of the differential systems in dimension two and related fields. But in general, to determine the existence or absence of a limit cycle is not an easy task. See for instance the Hilbert's 16th problem [10,12,14].

At the beginning of the 1930s the limit cycles started to be studied in the continuous and discontinuous piecewise differential systems due to their importance in many mechanical and electrical applications. For more information on their past and present applications, see the books [1,6,28] and the survey [24]. The continuous piecewise differential systems have been studied for several authors, see for instance [4,9,17,22,23], and for the discontinuous ones see without being exhaustive [2,3,5, 9,11,13,15,16,18–21,25,26].

In this paper we consider continuous-discontinuous piecewise differential systems separated by two parallel straight lines. These systems have quadratic Hamiltonian systems in two different regions and a linear Hamiltonian system in the third region, and we want to study the existence and non–existence of limit cycles, and in the case of the existence of limit cycles, we also want to find their maximum number of limit cycles that these continuous-discontinuous piecewise differential systems can exhibit. See a previous paper on continuous-discontinuous piecewise differential systems separated by two parallel straight lines in [19]. Now we need to take into account two cases: the Hamiltonian system of degree one may be located either in the center of the three zones, or in a lateral zone.

Here we shall study the periodic orbits and the limit cycles which intersect exactly at two points, both parallel straight lines of the continuous-discontinuous piecewise differential systems.

Doing a rescaling of the plane variables and a rotation of themselves, if necessary, we can assume without loss of generality that the two parallel straight lines are $x=-1$ and $x=1$. Thus we shall study the continuous-discontinuous piecewise differential systems of the form

$$(\dot{x},\dot{y})=\begin{cases}\left(-\dfrac{\partial H_i}{\partial y},\dfrac{\partial H_i}{\partial x}\right) \text{ if } x\geq 1,\\ \left(-\dfrac{\partial H_j}{\partial y},\dfrac{\partial H_j}{\partial x}\right) \text{ if } -1\leq x\leq 1,\\ \left(-\dfrac{\partial H_2}{\partial y},\dfrac{\partial H_2}{\partial x}\right) \text{ if } x\leq -1,\end{cases} \tag{1.1}$$

where $H_2=H_2(x,y)$ is an arbitrary polynomial of degree 3, $H_i=H_i(x,y)$, $H_j=H_j(x,y)$ are arbitrary polynomials of degree 3 and 2 for $i=3$, $j=1$, or $H_i=H_i(x,y)$, $H_j=H_j(x,y)$ are arbitrary polynomials of degree 2 or 3 for $i=1$, $j=3$. In the straight line $x=-1$ the piecewise differential system is continuous and in $x=1$ discontinuous.

This paper studies the existence or non-existence of periodic orbits and limit cycles that such kinds of continuous-discontinuous piecewise differential systems can exhibit. And in the case of the existence of limit cycles we determine their maximum numbers.

In what follows we give explicitly the Hamiltonian systems which form the continuous-discontinuous piecewise differential systems (1.1)

$$\begin{aligned} \dot{x} &= -\alpha_2 - \alpha_4 x - 2\alpha_5 y, \\ \dot{y} &= \alpha_1 + 2\alpha_3 x + \alpha_4 y, \end{aligned} \tag{1.2}$$

$$\begin{aligned} \dot{x} &= -\beta_2 - \beta_7 x^2 - \beta_4 x - 2\beta_8 xy - 3\beta_9 y^2 - 2\beta_5 y, \\ \dot{y} &= \beta_1 + 3\beta_6 x^2 + 2\beta_3 x + 2\beta_7 xy + \beta_8 y^2 + \beta_4 y, \end{aligned} \tag{1.3}$$

$$\begin{aligned} \dot{x} &= -\gamma_2 - \gamma_7 x^2 - \gamma_4 x - 2\gamma_8 xy - 3\gamma_9 y^2 - 2\gamma_5 y, \\ \dot{y} &= \gamma_1 + 3\gamma_6 x^2 + 2\gamma_3 x + 2\gamma_7 xy + \gamma_8 y^2 + \gamma_4 y. \end{aligned} \tag{1.4}$$

Of course, the Hamiltonians of the Hamiltonian systems (1.2), (1.3) and (1.4) are

$$H_1(x,y) = \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2,$$

$$H_2(x,y) = \beta_1 x + \beta_2 y + \beta_3 x^2 + \beta_4 xy + \beta_5 y^2 + \beta_6 x^3 + \beta_7 x^2 y + \beta_8 xy^2 + \beta_9 y^3,$$

$$H_3(x,y) = \gamma_1 x + \gamma_2 y + \gamma_3 x^2 + \gamma_4 xy + \gamma_5 y^2 + \gamma_6 x^3 + \gamma_7 x^2 y + \gamma_8 xy^2 + \gamma_9 y^3,$$

respectively.

Our main result when the Hamiltonian system of degree one is in the middle of the three zones is the following.

Theorem 1.1. *A continuous–discontinuous piecewise differential system* (1.1) *such that in the straight line $x = -1$ the system is continuous and in the straight line $x = 1$ discontinuous, formed by the Hamiltonian systems* (1.3) *in the region $x \le -1$,* (1.2) *in the region $-1 \le x \le 1$ and* (1.4) *in the region $x \ge 1$, can have at most one limit cycle. And there are such piecewise differential systems having one limit cycle, see Figure 1.*

Theorem 1.1 is proved in section 2.

Our main result when the Hamiltonian system of degree one is not in the middle of the three zones is the following.

Theorem 1.2. *A continuous–discontinuous piecewise differential system* (1.1) *such that in the straight line $x = -1$ the system is continuous and in the straight line $x = 1$ discontinuous, formed by the Hamiltonian systems* (1.3) *in the region $x \le -1$,* (1.4) *in the region $-1 \le x \le 1$ and* (1.2) *in the region $x \ge 1$, can have at most one limit cycle. And there are such piecewise differential systems having one limit cycle, see Figure 2.*

Theorem 1.2 is proved in section 3.

2. Proof of Theorem 1.1

Under the assumptions of Theorem 1.1 the Hamiltonian system of degree one is in the middle of the three regions. If a continuous-discontinuous piecewise differential system (1.1) has a periodic orbit intersecting the two straight lines of separation in

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the points $(-1,y_1)$, $(-1,y_2)$, $(1,y_3)$ and $(1,y_4)$ with $y_1>y_2$ and $y_3>y_4$, these four points must satisfy the following polynomial system of 8 equations and 8 unknowns

$$\begin{aligned} &H_2(-1,y_1)=H_2(-1,y_2)=k_1,\ H_1(-1,y_1)=H_1(1,y_3)=k_2,\\ &H_1(-1,y_2)=H_1(1,y_4)=k_3,\quad H_3(1,y_3)=H_3(1,y_4)=k_4, \end{aligned} \tag{2.1}$$

where k_i, $i\in\{1,2,3,4\}$ are constants.

Proof. [Proof of Theorem 1.1] In order that the piecewise differential system (1.1) is continuous on the straight line $x=-1$, the Hamiltonian systems (1.3) and (1.2) must coincide on $x=-1$. This implies that $\beta_2=(2\alpha_2-\alpha_4+\beta_4)/2$, $\beta_5=\alpha_5$, $\beta_6=(\alpha_1-2\alpha_3-\beta_1+2\beta_3)/3$, $\beta_7=(\beta_4-\alpha_4)/2$, $\beta_8=0$, and $\beta_9=0$. Then the polynomial system (2.1) becomes

$$\begin{aligned} &(-\alpha_1+2\alpha_3-2\beta_1+\beta_3-3k_1+3y_1(\alpha_2-\alpha_4+\alpha_5y_1))/3=0,\\ &(-\alpha_1+2\alpha_3-2\beta_1+\beta_3-3k_1+3y_2(\alpha_2-\alpha_4+\alpha_5y_2))/3=0,\\ &-\alpha_1+\alpha_3-k_2+y_1(\alpha_2-\alpha_4+\alpha_5y_1)=0,\\ &\alpha_1+\alpha_3-k_2+y_3(\alpha_2+\alpha_4+\alpha_5y_3)=0,\\ &-\alpha_1+\alpha_3-k_3+y_2(\alpha_2-\alpha_4+\alpha_5y_2)=0,\\ &\alpha_1+\alpha_3-k_3+y_4(\alpha_2+\alpha_4+\alpha_5y_4)=0,\\ &\gamma_1+\gamma_3+\gamma_6-k_4+y_3(\gamma_2+\gamma_4+\gamma_7+y_3(\gamma_5+\gamma_8+\gamma_9y_3))=0,\\ &\gamma_1+\gamma_3+\gamma_6-k_4+y_4(\gamma_2+\gamma_4+\gamma_7+y_4(\gamma_5+\gamma_8+\gamma_9y_4))=0. \end{aligned} \tag{2.2}$$

From these equations we get eliminating the constants k_i for $i=1,2,3,4$ the next four equations

$$\begin{aligned} &(y_1-y_2)(\alpha_2-\alpha_4+\alpha_5(y_1+y_2))=0,\\ &-2\alpha_1+\alpha_5y_1^2+(\alpha_2-\alpha_4)y_1-y_3(\alpha_2+\alpha_4+\alpha_5y_3)=0,\\ &-2\alpha_1+\alpha_5y_2^2+(\alpha_2-\alpha_4)y_2-y_4(\alpha_2+\alpha_4+\alpha_5y_4)=0,\\ &(y_3-y_4)(\gamma_2+\gamma_4+\gamma_7+(\gamma_5+\gamma_8)(y_3+y_4))+\gamma_9(y_3^3-y_4^3)=0. \end{aligned} \tag{2.3}$$

Note that if $y_1=y_2$ or $y_3=y_4$. Then the solutions of system (2.2) cannot provide the periodic orbits of the piecewise differential systems that we are studying. Now the rest of the proof is divided into four cases.

Case 1: $\alpha_5\neq0$ and $\gamma_9\neq0$. We solve (2.2) and (2.3) with respect to y_1, y_2, y_3, y_4, k_1, k_2, k_3 and k_4, and taking into account that $y_1>y_2$ and $y_3>y_4$ the unique solution is

$$\begin{aligned} &k_1=-\tfrac{(\alpha_2+\alpha_4)^2}{\alpha_5}+\tfrac{(\gamma_5+\gamma_8)(\alpha_2+\alpha_4)}{\gamma_9}+\tfrac{5\alpha_1+2\alpha_3-2\beta_1+\beta_3}{3}-\tfrac{\alpha_5(\gamma_2+\gamma_4+\gamma_7)}{\gamma_9},\\ &k_2=-\tfrac{(\alpha_2+\alpha_4)^2}{\alpha_5}+\alpha_1+\alpha_3+\tfrac{(\alpha_2+\alpha_4)(\gamma_5+\gamma_8)-\alpha_5(\gamma_2+\gamma_4+\gamma_7)}{\gamma_9},\\ &k_3=-\tfrac{(\alpha_2+\alpha_4)^2}{\alpha_5}+\alpha_1+\alpha_3+\tfrac{(\alpha_2+\alpha_4)(\gamma_5+\gamma_8)-\alpha_5(\gamma_2+\gamma_4+\gamma_7)}{\gamma_9},\quad k_4=\tfrac{\mathcal{T}}{\alpha_5^3\gamma_9},\\ &y_1=\tfrac{\sqrt{-\gamma_9\mathcal{R}}-\alpha_2\gamma_9+\alpha_4\gamma_9}{2\alpha_5\gamma_9},\quad y_2=-\tfrac{\sqrt{-\gamma_9\mathcal{R}}+\alpha_2\gamma_9-\alpha_4\gamma_9}{2\alpha_5\gamma_9},\\ &y_3=\tfrac{\sqrt{\mathcal{S}}-(\alpha_2+\alpha_4)\alpha_5\gamma_9}{2\alpha_5^2\gamma_9},\quad y_4=-\tfrac{\sqrt{\mathcal{S}}+(\alpha_2+\alpha_4)\alpha_5\gamma_9}{2\alpha_5^2\gamma_9}, \end{aligned}$$

if $-\gamma_9\mathcal{R} \geq 0$ and $\mathcal{S} \geq 0$, where
$\mathcal{R} = 4(\gamma_2+\gamma_4+\gamma_7)\alpha_5^2 - 4(\alpha_2+\alpha_4)(\gamma_5+\gamma_8)\alpha_5 - 8\alpha_1\gamma_9\alpha_5 + (3\alpha_2+\alpha_4)(\alpha_2+3\alpha_4)\gamma_9$,
$\mathcal{S} = \alpha_5^2\gamma_9(-3\gamma_9(\alpha_2+\alpha_4)^2 + 4\alpha_5(\gamma_5+\gamma_8)(\alpha_2+\alpha_4) - 4\alpha_5^2(\gamma_2+\gamma_4+\gamma_7))$, and
$\mathcal{T} = \gamma_9^2(\alpha_2+\alpha_4)^3 - 2\alpha_5(\gamma_5+\gamma_8)\gamma_9(\alpha_2+\alpha_4)^2 + \alpha_5^2((\gamma_5+\gamma_8)^2 + (\gamma_2+\gamma_4+\gamma_7)\gamma_9)(\alpha_2+\alpha_4) + \alpha_5^3((\gamma_1+\gamma_3+\gamma_6)\gamma_9 - (\gamma_2+\gamma_4+\gamma_7)(\gamma_5+\gamma_8))$.

Hence in this case the piecewise differential systems have at most one limit cycle.
Case 2: $\alpha_5 = 0$ and $\gamma_9 \neq 0$. From system (2.1) we obtain

$$\begin{aligned}
H_2(-1,y_1) - H_2(-1,y_2) &= \alpha_2 - \alpha_4 = 0,\\
H_1(-1,y_1) - H_1(1,y_3) &= -2\alpha_1 + \alpha_2 y_1 - \alpha_2 y_3 - \alpha_4 y_1 - \alpha_4 y_3 = 0,\\
H_1(-1,y_2) - H_1(1,y_4) &= -2\alpha_1 + \alpha_2 y_2 - \alpha_2 y_4 - \alpha_4 y_2 - \alpha_4 y_4 = 0,\\
H_3(1,y_3) - H_3(1,y_4) &= \gamma_2 + \gamma_4 + \gamma_7 + \gamma_9 y_3^2 + \gamma_5 y_3 + \gamma_8 y_3 + \gamma_9 y_4 y_3 + \gamma_5 y_4 + \gamma_8 y_4\\
&\quad + \gamma_9 y_4^2 = 0.
\end{aligned}$$

Since $\alpha_2 = \alpha_4$, from the second and third equation we get that $y_3 = y_4 = -\alpha_1/\alpha_4$. Then in the piecewise differential systems there are no periodic orbits, and consequently no limit cycles.
Case 3: $\gamma_9 = 0$ and $\alpha_5 \neq 0$. Therefore system (2.1) becomes

$$\begin{aligned}
H_2(-1,y_1) - H_2(-1,y_2) &= \alpha_2 - \alpha_4 + \alpha_5 y_1 + \alpha_5 y_2 = 0,\\
H_1(-1,y_1) - H_1(1,y_3) &= -2\alpha_1 + \alpha_5 y_1^2 + \alpha_2 y_1 - \alpha_4 y_1 - \alpha_2 y_3 - \alpha_4 y_3 - \alpha_5 y_3^2 = 0,\\
H_1(-1,y_2) - H_1(1,y_4) &= -2\alpha_1 + \alpha_5 y_2^2 + \alpha_2 y_2 - \alpha_4 y_2 - \alpha_2 y_4 - \alpha_4 y_4 - \alpha_5 y_4^2 = 0,\\
H_3(1,y_3) - H_3(1,y_4) &= \gamma_2 + \gamma_4 + \gamma_7 + \gamma_5 y_3 + \gamma_5 y_4 + \gamma_8 y_3 + \gamma_8 y_4 = 0.
\end{aligned}$$

So $y_3 = y_4 = -(\gamma_2+\gamma_4+\gamma_7)/(2(\gamma_5+\gamma_8))$, and consequently no periodic orbits if $\gamma_5 \neq -\gamma_8$.
If $\gamma_5 = -\gamma_8$, the last system becomes

$$\begin{aligned}
&\alpha_2 - \alpha_4 + \alpha_5 y_1 + \alpha_5 y_2 = 0,\\
&-2\alpha_1 + \alpha_5 y_1^2 + \alpha_2 y_1 - \alpha_4 y_1 - \alpha_2 y_3 - \alpha_4 y_3 - \alpha_5 y_3^2 = 0,\\
&-2\alpha_1 + \alpha_5 y_2^2 + \alpha_2 y_2 - \alpha_4 y_2 - \alpha_2 y_4 - \alpha_4 y_4 - \alpha_5 y_4^2 = 0,\\
&\gamma_2 + \gamma_4 + \gamma_7 = 0.
\end{aligned}$$

Therefore $\gamma_2 = -\gamma_4 - \gamma_7$. Now replacing the parameters γ_9, γ_5 and γ_2 into system (2.2) we obtain the following two solutions

$$\begin{aligned}
&k_1 = \pm\tfrac{1}{3}\left(-\alpha_1 + 2\alpha_3 - 2\beta_1 + \beta_3 + 3y_1\left(\alpha_2 - \alpha_4 + \alpha_5 y_1\right)\right),\\
&k_2 = -\alpha_1 + \alpha_3 + y_1\left(\alpha_2 - \alpha_4 + \alpha_5 y_1\right), && y_2 = -\tfrac{\alpha_2 - \alpha_4 + \alpha_5 y_1}{\alpha_5},\\
&k_3 = -\alpha_1 + \alpha_3 + y_1\left(\alpha_2 - \alpha_4 + \alpha_5 y_1\right), && y_3 = -\tfrac{\alpha_2 + \alpha_4 + \sqrt{\mathcal{R}_1}}{2\alpha_5},\\
&k_4 = \gamma_1 + \gamma_3 + \gamma_6, && y_4 = -\tfrac{\alpha_2 + \alpha_4 - \sqrt{\mathcal{R}_1}}{2\alpha_5},
\end{aligned}$$

if $\mathcal{R}_1$ and $\mathcal{R}_2 > 0$, where
$\mathcal{R}_1 = (\alpha_2+\alpha_4)^2 - 8\alpha_1\alpha_5 + 4\alpha_5 y_1(\alpha_2 - \alpha_4 + \alpha_5 y_1)$, and

$\mathcal{R}_2 = \alpha_2^2 - 8\alpha_1\alpha_5 + 2\alpha_2(\alpha_4 + 2\alpha_5 y_1) + (\alpha_4 - 2\alpha_5 y_1)^2$.

So in this case if piecewise differential systems have a solution, they have a continuum of solutions, which would produce a continuum of periodic orbits, then no limit cycle can exhibit.

Case 4: $\gamma_9 = 0$ and $\alpha_5 = 0$. From system (2.1) we obtain

$$\begin{aligned}
&H_2(-1, y_1) - H_2(-1, y_2) = \alpha_2 - \alpha_4 = 0,\\
&H_1(-1, y_1) - H_1(1, y_3) = -2\alpha_1 + \alpha_2 y_1 - \alpha_2 y_3 - \alpha_4 y_1 - \alpha_4 y_3 = 0,\\
&H_1(-1, y_2) - H_1(1, y_4) = -2\alpha_1 + \alpha_2 y_2 - \alpha_2 y_4 - \alpha_4 y_2 - \alpha_4 y_4 = 0,\\
&H_3(1, y_3) - H_3(1, y_4) = \gamma_2 + \gamma_4 + \gamma_7 + \gamma_5 y_3 + \gamma_8 y_3 + \gamma_5 y_4 + \gamma_8 y_4 = 0.
\end{aligned}$$

Since $\alpha_2 = \alpha_4$, from the second and third equations we get that $y_3 = y_4 = -\alpha_1/\alpha_4$. Then in the piecewise differential systems there are no periodic orbits, and consequently no limit cycles.

In order to complete the proof of Theorem 1.1, we provide a piecewise differential system having the Hamiltonian system of degree one in the middle of the three zones exhibiting one limit cycle. In the previous **case 1** we choose the piecewise differential system formed by

$$\begin{aligned}
&\dot{x} = -2x - 2y + 1, \quad \dot{y} = 6x + 2y, && \text{in } -1 < x < 1,\\
&\dot{x} = -x^2 - 4x - 2y, \quad \dot{y} = -3x^2 + 2xy + 4x + 4y + 1, && \text{in } x < -1,\\
&\dot{x} = 2xy + 3y^2, \quad \dot{y} = 3x^2 + 2x - y^2 + 1, && \text{in } x > 1,
\end{aligned}$$

with first integrals

$$\begin{aligned}
&H_1(x, y) = 3x^2 + 2xy + y^2 - y,\\
&H_2(x, y) = -x^3 + x^2 y + 2x^2 + 4xy + x + y^2,\\
&H_3(x, y) = x^3 + x^2 - xy^2 + x - y^3,
\end{aligned}$$

respectively. Then the corresponding piecewise differential system satisfies that the points of intersection with the straight line $x = -1$ have $y_1 = 3$ and $y_2 = 0$, while the points of intersection with the straight line $x = 1$ have $y_3 = 0$ and $y_4 = -1$. Therefore the limit cycle of this piecewise differential system is drawn in Figure 1. □

3. Proof of Theorem 1.2

In this case the Hamiltonian system of degree one is on the right of the region. The four points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 > y_4$, must satisfy the following polynomial system

$$\begin{aligned}
&H_2(-1, y_1) = H_2(-1, y_2) = k_1,\ H_3(-1, y_1) = H_3(1, y_3) = k_2,\\
&H_3(-1, y_2) = H_3(1, y_4) = k_3, \quad H_1(1, y_3) = H_1(1, y_4) = k_4,
\end{aligned} \tag{3.1}$$

where k_i, $i \in \{1, 2, 3, 4\}$ are constants.

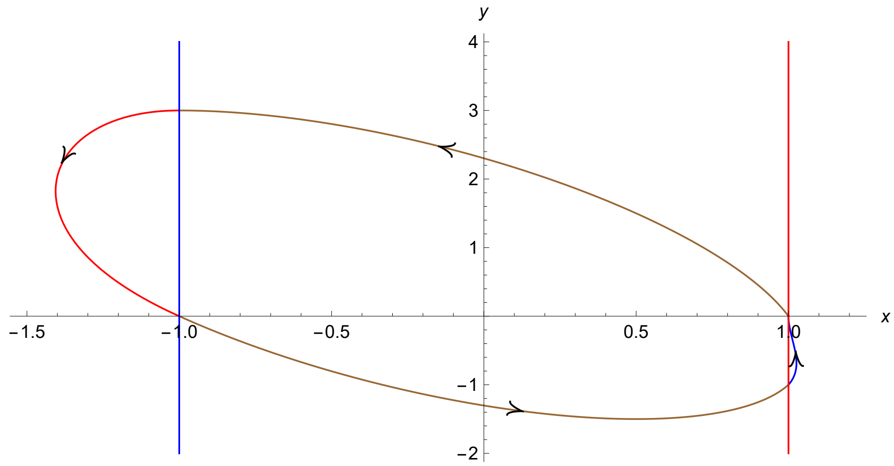

Figure 1. A limit cycle under the assumptions of Theorem 1.1.

Proof. [Proof of Theorem 1.2] The Hamiltonian systems (1.3) and (1.4) must coincide on $x = -1$ for the piecewise differential system (1.1) to be continuous on that line. This implies that $\beta_2 = (2\alpha_2 - \alpha_4 + \beta_4)/2$, $\beta_5 = \alpha_5$, $\beta_6 = (\alpha_1 - 2\alpha_3 - \beta_1 + 2\beta_3)/3$, $\beta_7 = (\beta_4 - \alpha_4)/2$, $\beta_8 = 0$, $\beta_9 = 0$. Then the polynomial system (3.1) becomes

$$\begin{aligned}
&-k_1 - \beta_1 + \beta_3 - \beta_6 + y_1(\beta_2 - \beta_4 + \beta_7 + y_1(\beta_5 - \beta_8 + y_1\beta_9)) = 0,\\
&-k_1 - \beta_1 + \beta_3 - \beta_6 + y_2(\beta_2 - \beta_4 + \beta_7 + y_2(\beta_5 - \beta_8 + y_2\beta_9)) = 0,\\
&(-3k_2 - \beta_1 + 2\beta_3 - 3\beta_6 + 3y_1(\beta_2 - \beta_4 + \beta_7 + y_1(\beta_5 - \beta_8 + y_1\beta_9))\\
&\quad -2\gamma_1 + \gamma_3)/3 = 0,\\
&(-3k_2 + \beta_1 - 2\beta_3 + 3\beta_6 + 2\gamma_1 + 5\gamma_3 + 3y_3(\beta_2 - \beta_4 + \beta_7 + y_3(\beta_5 + \beta_8\\
&\quad +y_3\beta_9) + 2\gamma_4))/3 = 0,\\
&(-3k_3 - \beta_1 + 2\beta_3 - 3\beta_6 + 3y_2(\beta_2 - \beta_4 + \beta_7 + y_2(\beta_5 - \beta_8 + y_2\beta_9))\\
&\quad -2\gamma_1 + \gamma_3)/3 = 0,\\
&(-3k_3 + \beta_1 - 2\beta_3 + 3\beta_6 + 2\gamma_1 + 5\gamma_3 + 3y_4(\beta_2 - \beta_4 + \beta_7 + y_4(\beta_5 + \beta_8\\
&\quad +y_4\beta_9) + 2\gamma_4))/3 = 0,\\
&-k_4 + \alpha_1 + \alpha_3 + y_3(\alpha_2 + \alpha_4 + y_3\alpha_5) = 0,\\
&-k_4 + \alpha_1 + \alpha_3 + y_4(\alpha_2 + \alpha_4 + y_4\alpha_5) = 0.
\end{aligned} \tag{3.2}$$

Eliminating the constants k_i for $i = 1, 2, 3, 4$ yields the following four equations.

$$
\begin{aligned}
&(y_1 - y_2)(\beta_2 - \beta_4 + \beta_7 + (y_1 + y_2)(\beta_5 - \beta_8)) + (y_1^3 - y_2^3)\beta_9 = 0,\\
&\beta_9 y_1^3 + (\beta_5 - \beta_8)y_1^2 + (\beta_2 - \beta_4 + \beta_7)y_1 + 4\beta_3/3 - 2\beta_6 - 4(\gamma_1 + \gamma_3)/3 - y_3(\beta_2 - \beta_4\\
&\qquad + \beta_7 + y_3(\beta_5 + \beta_8 + y_3\beta_9) + 2\gamma_4) - 2\beta_1/3 = 0,\\
&\beta_9 y_2^3 + (\beta_5 - \beta_8)y_2^2 + (\beta_2 - \beta_4 + \beta_7)y_2 + 4\beta_3/3 - 2\beta_6 - 4(\gamma_1 + \gamma_3)/3 - y_4(\beta_2 - \beta_4\\
&\qquad + \beta_7 + y_4(\beta_5 + \beta_8 + y_4\beta_9) + 2\gamma_4) - 2\beta_1/3 = 0,\\
&(y_3 - y_4)(\alpha_2 + \alpha_4 + (y_3 + y_4)\alpha_5) = 0.
\end{aligned}
$$

The periodic orbits we are considering cannot be provided if $y_1 = y_2$ and $y_3 = y_4$. The study of the solutions of system (3.2) is divided into five cases.

Case 1: $\alpha_5 \neq 0$, $\beta_9 \neq 0$ and $A_0 \neq 0$. Here $A_0 = \mathcal{R}_0 + \sqrt{\mathcal{S}_0}$, where
$\mathcal{R}_0 = 486\beta_1\beta_9^5\alpha_5^9 - 972\beta_3\beta_9^5\alpha_5^9 + 1458\beta_6\beta_9^5\alpha_5^9 - 486\beta_2\beta_5\beta_9^4\alpha_5^9 + 486\beta_4\beta_5\beta_9^4\alpha_5^9 - 486\beta_5\beta_7\beta_9^4\alpha_5^9 - 972\beta_2\beta_8\beta_9^4\alpha_5^9 + 972\beta_4\beta_8\beta_9^4\alpha_5^9 - 972\beta_7\beta_8\beta_9^4\alpha_5^9 - 54\beta_5^3\beta_9^3\alpha_5^9 + 54\beta_8^3\beta_9^3\alpha_5^9 - 162\beta_5\beta_8^2\beta_9^3\alpha_5^9 + 162\beta_5^2\beta_8\beta_9^3\alpha_5^9 + 972\beta_9^5\gamma_1\alpha_5^9 + 972\beta_9^5\gamma_3\alpha_5^9 - 1458\beta_5\beta_9^4\gamma_4\alpha_5^9 - 1458\beta_8\beta_9^4\gamma_4\alpha_5^9 + 729\alpha_2\beta_2\beta_9^5\alpha_5^8 + 729\alpha_4\beta_2\beta_9^5\alpha_5^8 - 729\alpha_2\beta_4\beta_9^5\alpha_5^8 - 729\alpha_4\beta_4\beta_9^5\alpha_5^8 + 729\alpha_2\beta_7\beta_9^5\alpha_5^8 + 729\alpha_4\beta_7\beta_9^5\alpha_5^8 + 729\alpha_2\beta_5^2\beta_9^4\alpha_5^8 + 729\alpha_4\beta_5^2\beta_9^4\alpha_5^8 + 729\alpha_2\beta_8^2\beta_9^4\alpha_5^8 + 729\alpha_4\beta_8^2\beta_9^4\alpha_5^8 + 1458\alpha_2\beta_5\beta_8\beta_9^4\alpha_5^8 + 1458\alpha_4\beta_5\beta_8\beta_9^4\alpha_5^8 + 1458\alpha_2\beta_9^5\gamma_4\alpha_5^8 + 1458\alpha_4\beta_9^5\gamma_4\alpha_5^8 - 1458\alpha_2^2\beta_5\beta_9^5\alpha_5^7 - 1458\alpha_4^2\beta_5\beta_9^5\alpha_5^7 - 2916\alpha_2\alpha_4\beta_5\beta_9^5\alpha_5^7 - 1458\alpha_2^2\beta_8\beta_9^5\alpha_5^7 - 1458\alpha_4^2\beta_8\beta_9^5\alpha_5^7 - 2916\alpha_2\alpha_4\beta_8\beta_9^5\alpha_5^7 + 729\alpha_2^3\beta_9^6\alpha_5^6 + 729\alpha_4^3\beta_9^6\alpha_5^6 + 2187\alpha_2\alpha_4^2\beta_9^6\alpha_5^6 + 2187\alpha_2^2\alpha_4\beta_9^6\alpha_5^6$, and
$\mathcal{S}_0 = 4(27\alpha_5^3\beta_9^2(\beta_2\beta_9\alpha_5^3 - \beta_4\beta_9\alpha_5^3 + \beta_7\beta_9\alpha_5^3) - 9(\alpha_5^3\beta_5\beta_9 - \alpha_5^3\beta_8\beta_9)^2)^3 + (486\beta_1\beta_9^5\alpha_5^9 - 972\beta_3\beta_9^5\alpha_5^9 + 1458\beta_6\beta_9^5\alpha_5^9 - 486\beta_2\beta_5\beta_9^4\alpha_5^9 + 486\beta_4\beta_5\beta_9^4\alpha_5^9 - 486\beta_5\beta_7\beta_9^4\alpha_5^9 - 972\beta_2\beta_8\beta_9^4\alpha_5^9 + 972\beta_4\beta_8\beta_9^4\alpha_5^9 - 972\beta_7\beta_8\beta_9^4\alpha_5^9 - 54\beta_5^3\beta_9^3\alpha_5^9 + 54\beta_8^3\beta_9^3\alpha_5^9 - 162\beta_5\beta_8^2\beta_9^3\alpha_5^9 + 162\beta_5^2\beta_8\beta_9^3\alpha_5^9 + 972\beta_9^5\gamma_1\alpha_5^9 + 972\beta_9^5\gamma_3\alpha_5^9 - 1458\beta_5\beta_9^4\gamma_4\alpha_5^9 - 1458\beta_8\beta_9^4\gamma_4\alpha_5^9 + 729\alpha_2\beta_2\beta_9^5\alpha_5^8 + 729\alpha_4\beta_2\beta_9^5\alpha_5^8 - 729\alpha_2\beta_4\beta_9^5\alpha_5^8 - 729\alpha_4\beta_4\beta_9^5\alpha_5^8 + 729\alpha_2\beta_7\beta_9^5\alpha_5^8 + 729\alpha_4\beta_7\beta_9^5\alpha_5^8 + 729\alpha_2\beta_5^2\beta_9^4\alpha_5^8 + 729\alpha_4\beta_5^2\beta_9^4\alpha_5^8 + 729\alpha_2\beta_8^2\beta_9^4\alpha_5^8 + 729\alpha_4\beta_8^2\beta_9^4\alpha_5^8 + 1458\alpha_2\beta_5\beta_8\beta_9^4\alpha_5^8 + 1458\alpha_4\beta_5\beta_8\beta_9^4\alpha_5^8 + 1458\alpha_2\beta_9^5\gamma_4\alpha_5^8 + 1458\alpha_4\beta_9^5\gamma_4\alpha_5^8 - 1458\alpha_2^2\beta_5\beta_9^5\alpha_5^7 - 1458\alpha_4^2\beta_5\beta_9^5\alpha_5^7 - 2916\alpha_2\alpha_4\beta_5\beta_9^5\alpha_5^7 - 1458\alpha_2^2\beta_8\beta_9^5\alpha_5^7 - 1458\alpha_4^2\beta_8\beta_9^5\alpha_5^7 - 2916\alpha_2\alpha_4\beta_8\beta_9^5\alpha_5^7 + 729\alpha_2^3\beta_9^6\alpha_5^6 + 729\alpha_4^3\beta_9^6\alpha_5^6 + 2187\alpha_2\alpha_4^2\beta_9^6\alpha_5^6 + 2187\alpha_2^2\alpha_4\beta_9^6\alpha_5^6)^2$.

If $A_0 = 0$ then $\mathcal{R}_0 = -\sqrt{\mathcal{S}_0}$, squaring both sides, we obtain

$$
{\mathcal{R}_0}^2 - \mathcal{S}_0 = -2916\alpha_5^{18}\beta_9^6\left(-\beta_5^2 + 2\beta_8\beta_5 - \beta_8^2 + 3\beta_2\beta_9 - 3\beta_4\beta_9 + 3\beta_7\beta_9\right)^3,
$$

which is equivalent to

$$
\alpha_5 = 0,\ \beta_9 = 0 \text{ or } \beta_7 = (\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3\beta_2\beta_9 + 3\beta_4\beta_9)/(3\beta_9).
$$

Then in this case we have that

$$
\alpha_5 \neq 0,\ \beta_9 \neq 0 \text{ and } \beta_7 \neq (\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3\beta_2\beta_9 + 3\beta_4\beta_9)/(3\beta_9).
$$

Now we solve system (3.2) with respect to y_1, y_2, y_3, y_4, k_1, k_2, k_3 and k_4, and taking into account that $y_1 > y_2$ and $y_3 > y_4$, we get at most 4 real solutions, which all have one of the two following pairs for (y_3, y_4)

$$
(y_{3_1}, y_{4_1}) = \left(-\frac{(\alpha_2 + \alpha_4)\,\alpha_5\beta_9 + \sqrt{B_1}}{2\alpha_5^2\beta_9}, \frac{\sqrt{B_0} - (\alpha_2 + \alpha_4)\,\alpha_5\beta_9}{2\alpha_5^2\beta_9}\right),
$$

or

$$
(y_{3_2}, y_{4_2}) = \left(\frac{\sqrt{B_1} - (\alpha_2 + \alpha_4)\,\alpha_5\beta_9}{2\alpha_5^2\beta_9}, -\frac{(\alpha_2 + \alpha_4)\,\alpha_5\beta_9 + \sqrt{B_0}}{2\alpha_5^2\beta_9}\right),
$$

if B_0 and B_1 are positive, where
$B_0 = -\alpha_5^2\beta_9(3\alpha_2^2\beta_9 - 4\alpha_5\alpha_2\beta_5 - 4\alpha_5\alpha_2\beta_8 + 6\alpha_4\alpha_2\beta_9 + 4\alpha_5^2\beta_2 - 4\alpha_5^2\beta_4 - 4\alpha_4\alpha_5\beta_5 + 4\alpha_5^2\beta_7 - 4\alpha_4\alpha_5\beta_8 + 3\alpha_4^2\beta_9 + 8\alpha_5^2\gamma_4)$, and
$B_1 = (\alpha_2\alpha_5\beta_9 + \alpha_4\alpha_5\beta_9)^2 - 4\alpha_5^2\beta_9(\alpha_2^2\beta_9 - \alpha_5\alpha_2\beta_5 - \alpha_5\alpha_2\beta_8 + 2\alpha_4\alpha_2\beta_9 + \alpha_5^2\beta_2 - \alpha_5^2\beta_4 - \alpha_4\alpha_5\beta_5 + \alpha_5^2\beta_7 - \alpha_4\alpha_5\beta_8 + \alpha_4^2\beta_9 + 2\alpha_5^2\gamma_4)$.

Note that all the pairs (y_1, y_2) are functions of A_0, A_1, α_5, β_2, β_4, β_5, β_7, β_8, and β_9. Then the 4 real solutions of system (3.2) are

$$s_{1,2} = \begin{cases} y_1 = \frac{\sqrt[3]{A_0}}{9\sqrt[3]{2}\alpha_5^3\beta_9^2} + \frac{\sqrt[3]{2}\alpha_5^3\left(\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3(\beta_2 - \beta_4 + \beta_7)\beta_9\right)}{\sqrt[3]{A_0}} + \frac{\beta_8 - \beta_5}{3\beta_9}, \\ y_2 = \frac{1}{36}\left(-\frac{2^{2/3}\sqrt[3]{A_0}}{\alpha_5^3\beta_9^2} - \frac{18\sqrt[3]{2}\alpha_5^3\left(\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3(\beta_2 - \beta_4 + \beta_7)\beta_9\right)}{\sqrt[3]{A_0}} \right. \\ \qquad \left. - \frac{6\left(3\sqrt{A_1} + 2\beta_5 - 2\beta_8\right)}{\beta_9}\right), \\ \text{either } (y_3, y_4) = (y_{3_1}, y_{4_1}), \text{ or } (y_3, y_4) = (y_{3_2}, y_{4_2}), \end{cases}$$

$$s_{3,4} = \begin{cases} y_1 = \frac{\sqrt[3]{A_0}}{9\sqrt[3]{2}\alpha_5^3\beta_9^2} + \frac{\sqrt[3]{2}\alpha_5^3\left(\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3(\beta_2 - \beta_4 + \beta_7)\beta_9\right)}{\sqrt[3]{A_0}} + \frac{\beta_8 - \beta_5}{3\beta_9}, \\ y_2 = \frac{1}{36}\left(-\frac{2^{2/3}\sqrt[3]{A_0}}{\alpha_5^3\beta_9^2} - \frac{18\sqrt[3]{2}\alpha_5^3\left(\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3(\beta_2 - \beta_4 + \beta_7)\beta_9\right)}{\sqrt[3]{A_0}} \right. \\ \qquad \left. + \frac{6\left(3\sqrt{A_1} - 2\beta_5 + 2\beta_8\right)}{\beta_9}\right), \\ \text{either } (y_3, y_4) = (y_{3_1}, y_{4_1}), \text{ or } (y_3, y_4) = (y_{3_2}, y_{4_2}), \end{cases}$$

if A_0 and A_1 are positive, where A_1 is equal to

$$\begin{aligned} &\left(\beta_5 - \beta_8 + \beta_9\left(\frac{\sqrt[3]{A_0}}{9\sqrt[3]{2}\alpha_5^3\beta_9^2} - \frac{\alpha_5^3\beta_5\beta_9 - \alpha_5^3\beta_8\beta_9}{3\alpha_5^3\beta_9^2} - \Theta\right)\right)^2 \\ &-4\beta_9\left(\beta_9\left(\frac{\sqrt[3]{A_0}}{9\sqrt[3]{2}\alpha_5^3\beta_9^2} - \frac{\alpha_5^3\beta_5\beta_9 - \alpha_5^3\beta_8\beta_9}{3\alpha_5^3\beta_9^2} - \Theta\right)^2\right. \\ &+\beta_5\left(\frac{\sqrt[3]{A_0}}{9\sqrt[3]{2}\alpha_5^3\beta_9^2} - \frac{\alpha_5^3\beta_5\beta_9 - \alpha_5^3\beta_8\beta_9}{3\alpha_5^3\beta_9^2} - \Theta\right) \\ &\left.-\beta_8\left(\frac{\sqrt[3]{A_0}}{9\sqrt[3]{2}\alpha_5^3\beta_9^2} - \frac{\alpha_5^3\beta_5\beta_9 - \alpha_5^3\beta_8\beta_9}{3\alpha_5^3\beta_9^2} - \Theta\right) + \beta_2 - \beta_4 + \beta_7\right), \end{aligned}$$

where

$$\Theta = \frac{\sqrt[3]{2}\left(27\alpha_5^3\beta_9^2\left(\beta_2\beta_9\alpha_5^3 - \beta_4\beta_9\alpha_5^3 + \beta_7\beta_9\alpha_5^3\right) - 9\left(\alpha_5^3\beta_5\beta_9 - \alpha_5^3\beta_8\beta_9\right)^2\right)}{9\sqrt[3]{A_0}\alpha_5^3\beta_9^2}.$$

Since the expressions of y_1 in the four solutions of s_k for $k = 1, 2, 3, 4$ are the same, these four solutions can produce at most one limit cycle.

Case 2: $\alpha_5 = 0$ and $\beta_9 \neq 0$. From system (3.1) we obtain

$$\begin{aligned} H_2(-1,y_1) - H_2(-1,y_2) &= \beta_2 - \beta_4 + \beta_7 + (\beta_5 - \beta_8)(y_1 + y_2) + \beta_9(y_1^2 + y_2 y_1 \\ &\quad + y_2^2) = 0, \\ H_3(-1,y_1) - H_3(1,y_3) &= -2\beta_1 + 4\beta_3 - 6\beta_6 - 4(\gamma_1 + \gamma_3) - 3y_3(\beta_2 - \beta_4 + \beta_7 \\ &\quad + 2\gamma_4 + y_3(\beta_5 + \beta_8 + \beta_9 y_3)) + 3\beta_9 y_1^3 + 3(\beta_5 - \beta_8)y_1^2 \\ &\quad + 3(\beta_2 - \beta_4 + \beta_7)y_1 = 0, \\ H_3(-1,y_2) - H_3(1,y_4) &= -2\beta_1 + 4\beta_3 - 6\beta_6 - 4(\gamma_1 + \gamma_3) - 3y_4(\beta_2 - \beta_4 + \beta_7 \\ &\quad + 2\gamma_4 + y_4(\beta_5 + \beta_8 + \beta_9 y_4)) + 3\beta_9 y_2^3 + 3(\beta_5 - \beta_8)y_2^2 \\ &\quad + 3(\beta_2 - \beta_4 + \beta_7)y_2 = 0, \\ H_1(1,y_3) - H_1(1,y_4) &= \alpha_2 + \alpha_4 = 0. \end{aligned}$$

From the fourth equation we obtain $\alpha_2 + \alpha_4 = 0$. Then Taking $\alpha_2 = -\alpha_4$ it remains a polynomial system with three equations and four unknowns, y_k for $k = 1, \ldots, 4$, so if this system has a solution it has a continuum of solutions, which can produce a continuum of periodic orbits and then no limit cycles.

Case 3: $\beta_9 = 0$ and $\alpha_5 \neq 0$. Therefore system (3.1) gives

$$\begin{aligned} H_2(-1,y_1) - H_2(-1,y_2) &= \beta_2 - \beta_4 + \beta_7 + (\beta_5 - \beta_8)(y_1 + y_2) = 0, \\ H_1(-1,y_1) - H_1(1,y_3) &= -2\beta_1 + 4\beta_3 - 6\beta_6 - 4(\gamma_1 + \gamma_3) - 3y_3(\beta_2 - \beta_4 + \beta_7 + 2\gamma_4 \\ &\quad + (\beta_5 + \beta_8)y_3) + 3(\beta_5 - \beta_8)y_1^2 + 3(\beta_2 - \beta_4 + \beta_7)y_1 = 0, \\ H_1(-1,y_2) - H_1(1,y_4) &= -2\beta_1 + 4\beta_3 - 6\beta_6 - 4(\gamma_1 + \gamma_3) - 3y_4(\beta_2 - \beta_4 + \beta_7 + 2\gamma_4 \\ &\quad + (\beta_5 + \beta_8)y_4) + 3(\beta_5 - \beta_8)y_2^2 + 3(\beta_2 - \beta_4 + \beta_7)y_2 = 0, \\ H_3(1,y_3) - H_3(1,y_4) &= \alpha_2 + \alpha_4 + \alpha_5(y_3 + y_4) = 0. \end{aligned}$$

All solutions of this polynomial system give $y_3 = y_4 = (-\alpha_2 - \alpha_4)/(2\alpha_5)$. Then no limit cycles can be produced.

Case 4: $\alpha_5 = 0$ and $\beta_9 = 0$. From system (3.1) we obtain

$$\begin{aligned} H_2(-1,y_1) - H_2(-1,y_2) &= \beta_2 - \beta_4 + \beta_7 + (\beta_5 - \beta_8)(y_1 + y_2) = 0, \\ H_3(-1,y_1) - H_3(1,y_3) &= -2\beta_1 + 4\beta_3 - 6\beta_6 - 4(\gamma_1 + \gamma_3) - 3y_3(\beta_2 - \beta_4 + \beta_7 \\ &\quad + 2\gamma_4 + y_3(\beta_5 + \beta_8)) + 3(\beta_5 - \beta_8)y_1^2 + 3(\beta_2 - \beta_4 \\ &\quad + \beta_7)y_1 = 0, \\ H_3(-1,y_2) - H_3(1,y_4) &= -2\beta_1 + 4\beta_3 - 6\beta_6 - 4(\gamma_1 + \gamma_3) - 3y_4(\beta_2 - \beta_4 + \beta_7 \\ &\quad + 2\gamma_4 + y_4(\beta_5 + \beta_8)) + 3(\beta_5 - \beta_8)y_2^2 + 3(\beta_2 - \beta_4 \\ &\quad + \beta_7)y_2 = 0, \\ H_1(1,y_3) - H_1(1,y_4) &= \alpha_2 + \alpha_4 = 0. \end{aligned}$$

The fourth equation yields the result $\alpha_2 + \alpha_4 = 0$. After substituting $\alpha_2 = -\alpha_4$, the polynomial system has three equations and four unknowns, y_k for $k = 1, \ldots, 4$,

therefore if it has a solution, it has an infinite number of solutions that might result in an infinite number of periodic orbits and no limit cycles.
Case 5: $\beta_7 = (\beta_5^2 - 2\beta_8\beta_5 + \beta_8^2 - 3\beta_2\beta_9 + 3\beta_4\beta_9)/(3\beta_9)$, $\beta_9 \neq 0$ and $\alpha_5 \neq 0$. We solve system (3.1) with respect to y_1, y_2, y_3, y_4, k_1, k_2, k_3 and k_4, and we obtain that all the solutions have the pair (y_3, y_4) equal to either

$$(y_{3_1}, y_{4_1}) = \left(-\frac{3\,(\alpha_2+\alpha_4)\,\alpha_5\beta_9^2 + \sqrt{3}\sqrt{A_0}}{6\alpha_5^2\beta_9^2}, \frac{\sqrt{3}\sqrt{A_0} - 3\,(\alpha_2+\alpha_4)\,\alpha_5\beta_9^2}{6\alpha_5^2\beta_9^2}\right),$$

or

$$(y_{3_2}, y_{4_2}) = \left(\frac{\sqrt{3}\sqrt{A_0} - 3\,(\alpha_2+\alpha_4)\,\alpha_5\beta_9^2}{6\alpha_5^2\beta_9^2}, -\frac{3\,(\alpha_2+\alpha_4)\,\alpha_5\beta_9^2 + \sqrt{3}\sqrt{A_0}}{6\alpha_5^2\beta_9^2}\right),$$

if $A_0 = -\alpha_5^2\beta_9^2(4((\beta_5-\beta_8)^2+6\beta_9\gamma_4)\alpha_5^2 - 12(\alpha_2+\alpha_4)(\beta_5+\beta_8)\beta_9\alpha_5 + 9(\alpha_2+\alpha_4)^2\beta_9^2)$ is positive. Since $(y_{3_1}, y_{4_1}) = (y_{4_2}, y_{3_2})$, all the solutions of system (3.1) can produce at most one limit cycle. Then for this class of piecewise differential systems, the calculations show that there is only one allowed solution, which implies that there is only one limit cycle.

As the final step under the assumptions of Theorem 1.2 we provide the following piecewise differential system with exactly one crossing limit cycle

$$\begin{aligned}
&\dot{x} = x + 2y - 1, && \dot{y} = -2x - y + 1, && \text{in } x > 1,\\
&\dot{x} = x^2 - 2xy - 3y^2 + 200y, && \dot{y} = 30x^2 - 2xy + 2x + y^2 - 1, && \text{in } x < -1,\\
&\dot{x} = x^2 - 2xy - 3y^2 + 200y, && \dot{y} = 30x^2 - 2xy + 2x + y^2 - 1, && \text{in } -1 < x < 1,
\end{aligned}$$

with first integrals

$$\begin{aligned}
H_1(x,y) &= -x^2 - xy + x - y^2 + y,\\
H_2(x,y) &= 10x^3 - x^2y + x^2 + xy^2 - x + y^3 - 100y^2,\\
H_3(x,y) &= 10x^3 - x^2y + x^2 + xy^2 - x + y^3 - 100y^2,
\end{aligned}$$

respectively. Figure 2 shows the crossing limit cycle in this case. Here the points of intersection with the straight line $x = -1$ are

$$\begin{aligned}
y_1 = \tfrac{1}{3}\Big(&101 - 2\sqrt{7653}\sin\left(\tfrac{1}{3}\tan^{-1}\left(\tfrac{3\sqrt{62587095}}{514831}\right)\right)\\
&- 2\sqrt{2551}\cos\left(\tfrac{1}{3}\tan^{-1}\left(\tfrac{3\sqrt{62587095}}{514831}\right)\right)\Big) \approx 0.896506,
\end{aligned}$$

and

$$\begin{aligned}
y_2 = \tfrac{1}{3}\Big(&101 + 2\sqrt{7653}\sin\left(\tfrac{1}{3}\tan^{-1}\left(\tfrac{3\sqrt{62587095}}{514831}\right)\right)\\
&- 2\sqrt{2551}\cos\left(\tfrac{1}{3}\tan^{-1}\left(\tfrac{3\sqrt{62587095}}{514831}\right)\right)\Big) \approx -0.896506,
\end{aligned}$$

while the points of intersection with the straight line $x = 1$ are $y_3 = 1$ and $y_4 = -1$.
□

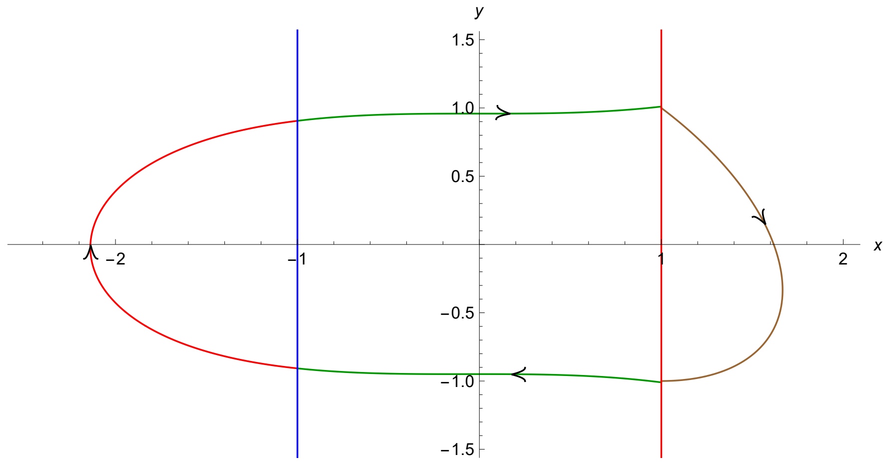

Figure 2. A limit cycle under the assumptions of Theorem 1.2.

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