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Limit Cycle Bifurcations of a Cubic Polynomial System via Melnikov Analysis*

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Abstract In this paper, a linear perturbation up to any order in ϵ for a cubic center with a multiple line of critical points is considered. By the algorithm of any order Melnikov function, the sharp upper bound of the number of limit cycles is 2.

Keywords Melnikov functions, bifurcations, limit cycles

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1. Introduction

One of the important problems in the qualitative theory of differential systems is the Hilbert's 16th problem, which aims to find the distributions and numbers of limit cycles of planar polynomials differential systems. The perturbation of the integrable non-Hamiltonian system as follows:

$$\begin{cases} \dot{x} = yh(x,\, y) - \epsilon p(x,\, y,\, \epsilon), \\ \dot{y} = -xh(x,\, y) + \epsilon q(x,\, y,\, \epsilon), \end{cases} \tag{1.1}$$

is closely related to the weak Hilbert's 16th problem of determining an upper bound of the number of zeros for the following integral

$$I(h) = \oint_{\frac{1}{2}(x^2+y^2)=h} \frac{p(x,\, y,\, 0)dy + q(x,\, y,\, 0)dx}{h(x,\, y)},$$

where $p(x,\, y,\, \epsilon)$ and $q(x,\, y,\, \epsilon)$ are polynomials in x, y depending analytically on ϵ, and here $h(x,\, y)$ is a polynomial in x, y with $h(0, 0) \neq 0$.

Many researchers focus on system (1.1) with different $h(x,\, y)$ and the difficulty reflects on how to deal with the Abel integral with a denominator of $h(x,\, y)$, as discussed in [1, 3, 5, 8–12] and references therein. For $h(x,\, y) = ax^2 + bx + 1$, the authors in [10–12] studied system (1.1) with different ranges of a and b by the first

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order Melnikov function, respectively. Buică et al. [2] considered system (1.1) with $h(x,\,y) = 1 + x$ by the first three order Melnikov functions. The authors in [13] considered system (1.1) with $h(x) = (1+x)^2$ under the perturbation up to the first order in ϵ by any order Melnikov functions.

Inspired by the works mentioned above, we would like to see the influence of the linear perturbation up to any order in ϵ on the number of limit cycles by higher order Melnikov functions.

More precisely, in this paper, we consider the following system:

$$\begin{cases} \dot{x} = y(1+x)^2 - \displaystyle\sum_{i=0}^{N} \epsilon^{i+1} P_i(x,\,y), \\ \dot{y} = -x(1+x)^2 + \displaystyle\sum_{i=0}^{N} \epsilon^{i+1} Q_i(x,\,y), \end{cases} \tag{1.2}$$

where $P_i(x,\,y) = a_{i0} + a_{i1}x + a_{i2}y$ and $Q_i(x,\,y) = b_{i0} + b_{i1}x + b_{i2}y$ for $0 \le i \le N$. Here $N \ge 1$ is an integer. System (1.2) can be rewritten as follows,

$$dH = \epsilon(\omega_0 + \epsilon\omega_1 + \cdots + \epsilon^N \omega_N),$$

where

$$H(x,\,y) = \frac{1}{2}(x^2+y^2), \quad \omega_i = \frac{P_i(x,\,y)dy + Q_i(x,\,y)dx}{(1+x)^2}.$$

When $\epsilon = 0$, there exist a family of periodic orbits $\Gamma_h : \{\frac{1}{2}(x^2+y^2) = h,\, h \in (0,\, \frac{1}{2})\}$. And we denote $M_k(h)$ as the $k-$th order Melnikov function of system (1.2) by the displacement function

$$d(h,\,\epsilon) = \epsilon M_1(h) + \epsilon^2 M_2(h) + \cdots + \epsilon^k M_k(h) + \cdots,\, h \in (0, \frac{1}{2}).$$

Then we give our main result in the following theorem.

Theorem 1.1. *For system (1.2), the following statements hold.*

(i) If the first order Melnikov function $M_1(h)$ is not zero identically, then $M_1(h)$ has at most one isolated zero, multiplicity taken into account.

(ii) If $M_j(h) \equiv 0$ for $1 \le j \le k-1$ and $M_k(h) \not\equiv 0$ with $k \le N+1$, then the $k-$th order Melnikov function $M_k(h)$ has at most two isolated zeros, multiplicity taken into account.

(iii) If $M_j(h) \equiv 0$ for $1 \le j \le N+1$ and $M_{N+2}(h) \not\equiv 0$, then the $N+2-$th order Melnikov function $M_{N+2}(h)$ has no isolated zero.

(iv) If $M_j(h) \equiv 0$ for $1 \le j \le N+2$, system (1.2) is integrable.

In short, the maximum number of limit cycles bifurcated from the cubic center is 2 by any order Melnikov function, taking into account their multiplicities. All the upper bounds mentioned above can be reached with proper parameters.

2. The calculation of $M_k(h)$

In this section we shall give the calculation of any order Melnikov function according to the algorithm of higher order Melnikov functions proposed in [4, 7, 13].

Firstly, denoting

$$\omega_{ij}^k=\frac{x^iy^j}{(1+x)^k}dx,\qquad \delta_{ij}^k=\frac{x^iy^j}{(1+x)^k}dy,\qquad J_k=\oint_{\Gamma_h}\delta_{00}^k,\quad i,\,j,\,k\in\mathbb{N},$$

then we make a decomposition with every ω_i for $0\le i\le N$, $i\in\mathbb{N}$ which is presented in the next lemma.

Lemma 2.1. *The* $1-$*forms* ω_i *for* $0\le i\le N$ *can be decomposed into the following forms:*

$$\omega_i=\bar{q}_idH+d\bar{Q}_i+N_i, \tag{2.1}$$

where

$$\begin{aligned}\bar{q}_i&=a_{i2}\frac{1}{(1+x)^2},\\ d\bar{Q}_i&=\Omega_{i1}(x)+\Omega_{i2}(x,\,y),\\ N_i&=(b_{i2}+a_{i1})\delta_{00}^1+(a_{i0}-a_{i1})\delta_{00}^2,\end{aligned}$$

and

$$\begin{aligned}\Omega_{i1}(x)&=\Big(b_{i0}\frac{1}{(1+x)^2}+(b_{i1}-a_{i2})\frac{x}{(1+x)^2}\Big)dx,\\ \Omega_{i2}(x,\,y)&=b_{i2}\Big(\frac{y}{(1+x)^2}dx-\frac{1}{1+x}dy\Big).\end{aligned}$$

Proof. A routine calculating gives to

$$\begin{aligned}\omega_{00}^2&=\frac{1}{(1+x)^2}dx=d\Big(-\frac{1}{1+x}\Big),\\ \omega_{10}^2&=\frac{x}{(1+x)^2}dx=d\Big(-\frac{x}{1+x}+\ln(1+x)\Big),\\ \omega_{01}^2&=\frac{y}{(1+x)^2}dx=d\Big(-\frac{y}{1+x}\Big)+\delta_{00}^1,\\ \delta_{10}^2&=\frac{x}{(1+x)^2}dy=\delta_{00}^1-\delta_{00}^2,\\ \delta_{01}^2&=\frac{y}{(1+x)^2}dy=\frac{1}{(1+x)^2}dH+d\Big(\frac{x}{1+x}-\ln(1+x)\Big).\end{aligned} \tag{2.2}$$

Substituting (2.2) into ω_i gives

$$\begin{aligned}\omega_i&=b_{i0}\omega_{00}^2+b_{i1}\omega_{10}^2+b_{i2}\omega_{01}^2+a_{i0}\delta_{00}^2+a_{i1}\delta_{10}^2+a_{i2}\delta_{01}^2\\ &=\bar{q}_idH+d\bar{Q}_i+N_i,\end{aligned} \tag{2.3}$$

where $\bar{q}_i$, $d\bar{Q}_i$ and N_i are given in (2.1). This completes the proof. □

Proposition 2.1. *If the first order Melnikov function* $M_1(h)$ *for system (1.2) is not zero identically, then it has at most one isolated zero, multiplicity taken into account, and this upper bound can be reached.*

Proof. The first order Melnikov function has the form

$$M_1(h)=(b_{02}+a_{01})J_1+(a_{00}-a_{01})J_2.$$

Denoting $z = \sqrt{1-2h}$, then we have

$$M_1(h) = \frac{2\pi}{z^3}(A_1 z^3 + (A_2 - A_1)z^2 - A_2),$$

where $A_1 = a_{01} + b_{02}$ and $A_2 = a_{00} - a_{01}$. It is easy to prove that $(1,\, z^2,\, z^3)$ is an ECT-system for $z \in (0,\, +\infty)$ and $z = 1$ is a zero, which implies that $M_1(z)$ has at most one zero for $z \in (0,\, 1)$. This leads to a conclusion of Proposition 2.1. $\square$

When $M_1(h) \equiv 0$, we have $a_{00} = a_{01}$ and $b_{02} = -a_{01}$. This displays,

$$\omega_0 = q_0 dH + dQ_0 = \bar{q_0} dH + \Omega_{01}(x) + \Omega_{02}(x,\, y),$$

where $\bar{q_0}$, $\Omega_{01}(x)$ and $\Omega_{02}(x,\, y)$ are given in (2.1).

Proposition 2.2. *If $M_1(h) \equiv 0$ and $M_2(h) \not\equiv 0$, then the second order Melnikov function $M_2(h)$ for system (1.2) has at most two isolated zeros, multiplicity taken into account, and two can be reached.*

Proof. The second order Melnikov function $M_2(h)$ can be expressed as

$$M_2(h) = \oint_{\Gamma_h} q_0\omega_0 + \omega_1 = \oint_{\Gamma_h} (q_0^2 + \bar{q_1})dH + (q_0\Omega_{01}(x) + d\bar{Q}_1) + q_0\Omega_{02}(x,\, y) + N_1.$$

It is straightforward to obtain that

$$q_0\Omega_{02}(x,\, y) = a_{02}b_{02}\frac{1}{(1+x)^2}\Big(\frac{y}{(1+x)^2}dx - \frac{1}{1+x}dy\Big) = a_{02}b_{02}(\omega_{01}^4 - \delta_{00}^3).$$

It is easy to decompose ω_{01}^4 into $d(-\frac{1}{3}\frac{y}{(1+x)^3}) + \frac{1}{3}\frac{1}{(1+x)^3}dy$. Hence we have

$$M_2(h) = -\frac{2}{3}a_{02}b_{02}J_3 + (a_{10} - a_{11})J_2 + (b_{12} + a_{11})J_1.$$

Let $z = \sqrt{1-2h}$. Then after a routine calculation, we have

$$W_1(J_1) = \frac{2\pi(z-1)}{z},\;\; W_2(J_1,\, J_2) = \frac{-4\pi^2(z^3 - 3z + 2)}{z^5},$$

$$W_3(J_1,\, J_2,\, J_3) = \frac{-24\pi^3(3z^5 - 10z^3 + 15z - 8)}{z^{12}}.$$

It is easy to obtain that $(J_1,\, J_2,\, J_3)$ is an ECT-system, which means that $M_2(h)$ has at most 2 zeros, and two zeros can appear with proper parameters. $\square$

If $M_1(h) = M_2(h) \equiv 0$, then $a_{00} = a_{01}$, $b_{02} = -a_{01}$, $a_{02}b_{02} = 0$, $a_{10} = a_{11}$, $b_{12} = -a_{11}$. It can be split into three subcases as follows:
Case 21: $a_{00} = a_{01} = b_{02} = 0$, $a_{10} = a_{11}$, $b_{12} = -a_{11}$, $a_{02} \neq 0$;
Case 22: $a_{00} = a_{01}$, $b_{02} = -a_{01}$, $a_{02} = 0$, $a_{10} = a_{11}$, $b_{12} = -a_{11}$, $b_{02} \neq 0$;
Case 23: $a_{00} = a_{01} = a_{02} = b_{02} = 0$, $a_{10} = a_{11}$, $b_{12} = -a_{11}$.

Proposition 2.3. *If $M_1(h) = M_2(h) \equiv 0$ and $M_3(h) \not\equiv 0$, then the third order Melnikov function $M_3(h)$ for system (1.2) has at most two isolated zeros, multiplicity taken into account, and two can be reached.*

Proof. For Case 21, the conditions $b_{02} = 0$ and $a_{02} \neq 0$ lead to a conclusion that $\Omega_{02}(x, y) \equiv 0$. This implies that $dQ_0 = \Omega_{01}(x)$. Under Case 21, we also get $q_1 = q_0^2 + \bar{q}_1 = \frac{a_{02}^2}{(1+x)^4} + \frac{a_{12}}{(1+x)^2}$ and $dQ_1 = q_0\Omega_{01}(x) + \Omega_{11}(x) + \Omega_{12}(x, y)$. Then

$$
\begin{aligned}
M_3(h) &= \oint_{\Gamma_h} q_1\omega_0 + q_0\omega_1 + \omega_2 \\
&= \oint_{\Gamma_h} (q_1q_0 + q_0\bar{q}_1 + \bar{q}_2)dH + q_1\Omega_{01}(x) + q_0\Omega_{11}(x) + \Omega_{21}(x) + \Omega_{22}(x, y) \\
&\quad + q_0\Omega_{12}(x, y) + N_2 \\
&= \oint_{\Gamma_h} q_0\Omega_{12}(x, y) + N_2 \\
&= -\frac{2}{3}a_{02}b_{12}J_3 + (a_{20} - a_{21})J_2 + (b_{22} + a_{21})J_1.
\end{aligned}
$$

For Case 22, the conditions $a_{02} = 0$ and $b_{02} \neq 0$ lead to a conclusion that $q_0 \equiv 0$. This implies that $q_1 = \bar{q}_1 = \frac{a_{12}}{(1+x)^2}$ and $dQ_1 = d\bar{Q}_1 = \Omega_{11}(x) + \Omega_{12}(x, y)$. Then

$$
\begin{aligned}
M_3(h) &= \oint_{\Gamma_h} q_1\omega_0 + q_0\omega_1 + \omega_2 \\
&= \oint_{\Gamma_h} \bar{q}_1\Omega_{01}(x) + \Omega_{21}(x) + \Omega_{22}(x, y) + \bar{q}_1\Omega_{02}(x, y) + N_2 \\
&= \oint_{\Gamma_h} \bar{q}_1\Omega_{02}(x, y) + N_2 \\
&= -\frac{2}{3}a_{12}b_{02}J_3 + (a_{20} - a_{21})J_2 + (b_{22} + a_{21})J_1.
\end{aligned}
$$

For Case 23, the conditions $a_{02} = 0$ and $b_{02} = 0$ lead to a conclusion that $q_0 \equiv 0$ and $\Omega_{02}(x, y) \equiv 0$. This implies that $q_1 = \bar{q}_1 = \frac{a_{12}}{(1+x)^2}$ and $dQ_1 = d\bar{Q}_1 = \Omega_{11}(x) + \Omega_{12}(x, y)$. Then

$$
\begin{aligned}
M_3(h) &= \oint_{\Gamma_h} q_1\omega_0 + q_0\omega_1 + \omega_2 \\
&= \oint_{\Gamma_h} \bar{q}_1\Omega_{01}(x) + \Omega_{21}(x) + \Omega_{22}(x, y) + N_2 \\
&= \oint_{\Gamma_h} N_2 \\
&= (a_{20} - a_{21})J_2 + (b_{22} + a_{21})J_1.
\end{aligned}
$$

It is easy to prove that $M_3(h)$ has at most two isolated zeros for Case 21 and Case 22, and at most one isolated zero for Case 23, multiplicity taken into account. These upper bounds can be all reached with proper parameters. □

If $M_1(h) = M_2(h) = M_3(h) \equiv 0$, then

Case 31: $a_{00} = a_{01} = b_{02} = b_{12} = a_{10} = a_{11} = 0$, $a_{20} = a_{21}$, $b_{22} = -a_{21}$, $a_{02} \neq 0$;

Case 32: $a_{02} = a_{12} = 0$, $a_{00} = a_{01}$, $b_{02} = -a_{01}$, $a_{10} = a_{11}$, $b_{12} = -a_{11}$, $a_{20} = a_{21}$, $b_{22} = -a_{21}$, $b_{02} \neq 0$;

Case 33: $a_{00} = a_{01} = a_{02} = b_{02} = 0$, $a_{10} = a_{11}$, $b_{12} = -a_{11}$, $a_{20} = a_{21}$, $b_{22} = -a_{21}$.

By induction, we have the following lemma.

Lemma 2.2. *For system (1.2),* $M_1(h) = M_2(h) = \cdots = M_k(h) \equiv 0$, $1 < k \le N+1$, *if and only if one of the following conditions holds.*

(i)

$$a_{i2} = b_{i2} = 0,\ 0 \le i \le \left[\frac{k}{2}\right] - 1,$$
$$a_{i0} = a_{i1} = -b_{i2},\ 0 \le i \le k-1;$$

(ii) For any $l \in \{0, 1, \cdots, [\frac{k}{2}] - 1\}$,

$$b_{i2} = 0,\ 0 \le i \le l-1,\ b_{l2} \ne 0,$$
$$a_{i2} = 0,\ 0 \le i \le k-2-l,$$
$$a_{i0} = a_{i1} = -b_{i2},\ 0 \le i \le k-1;$$

(iii) For any $l \in \{0, 1, \cdots, [\frac{k}{2}] - 1\}$,

$$a_{i2} = 0,\ 0 \le i \le l-1,\ a_{l2} \ne 0,$$
$$b_{i2} = 0,\ 0 \le i \le k-2-l,$$
$$a_{i0} = a_{i1} = -b_{i2},\ 0 \le i \le k-1.$$

Proof. We shall prove this lemma according that k is even and odd, respectively.
Case 1: k is even.

We would like to prove $M_j(h) \equiv 0$ for $0 \le j \le k$ when k is even, if and only if one of the conditions (i)-(iii) holds by induction.

For $k = 2$, it can be easily obtained by Case 21-Case 23.

Suppose that for $k = n = 2s$, one of the conditions (i)-(iii) holds. Next we shall prove that it also holds for $k = n+1 = 2s+1$.

Firstly, assume that (i) holds for $k = 2s$. Then we claim that (i) also holds for $k = 2s+1$.

This assumption implies that $a_{i2} = b_{i2} = 0$, $0 \le i \le s-1$. This displays that $\bar{q}_i = 0$ and $d\bar{Q}_i = \Omega_{i1}(x)$ for $0 \le i \le s-1$. The conditions $a_{i0} = a_{i1} = -b_{i2}$, $0 \le i \le 2s-1$ give to $N_i \equiv 0$, $0 \le i \le 2s-1$. Then

$$\begin{aligned} M_{2s+1}(h) &= \oint_{\Gamma_h} \sum_{j=0}^{2s-1} q_j \omega_{2s-1-j} + \omega_{2s} \\ &= \oint_{\Gamma_h} \sum_{j=s}^{2s-1} q_j \omega_{2s-1-j} + \omega_{2s} \\ &= \oint_{\Gamma_h} \sum_{j=s}^{2s-1} q_j \Omega_{2s-1-j,\,2}(x,\,y) + \omega_{2s} \\ &= \oint_{\Gamma_h} q_s \Omega_{s-1,\,2}(x,\,y) + \cdots + q_{2s-1}\Omega_{0,\,2}(x,\,y) + \omega_{2s} \\ &= (a_{2s,\,0} - a_{2s,\,1})J_2 + (b_{2s,\,2} + a_{2s,\,1})J_1. \end{aligned} \tag{2.4}$$

If $M_{2s+1}(h) \equiv 0$, then we have $a_{2s,\,0} = a_{2s,\,1} = -b_{2s,\,2}$. This displays that for $k = 2s+1$, (i) also holds.

Secondly, suppose that (ii) holds for $k = 2s$. Then we claim that (ii) also holds for $k = 2s+1$.

This assumption implies that for any given $l \in \{0, 1, \cdots, s-1\}$, $a_{i2} = 0$, $0 \leq i \leq 2s-2-l$. It can be obtained directly $\bar{q}_i = 0$ for $0 \leq i \leq 2s-2-l$. On the other hand, $b_{i2} = 0$, $0 \leq i \leq l-1$, $b_{l2} \neq 0$ give to $d\bar{Q}_i = \Omega_{i1}(x)$ for $0 \leq i \leq l-1$. The conditions $a_{i0} = a_{i1} = -b_{i2}$, $0 \leq i \leq 2s-1$ give to $N_i \equiv 0$, $0 \leq i \leq 2s-1$. Then

$$
\begin{aligned}
M_{2s+1}(h) &= \oint_{\Gamma_h} \sum_{j=0}^{2s-1} q_j \omega_{2s-1-j} + \omega_{2s} \\
&= \oint_{\Gamma_h} \sum_{j=2s-1-l}^{2s-1} q_j \omega_{2s-1-j} + \omega_{2s} \\
&= \oint_{\Gamma_h} \sum_{j=2s-1-l}^{2s-1} q_j \Omega_{2s-1-j,\,2}(x,\, y) + \omega_{2s} \\
&= -\frac{2}{3} a_{2s-1-l,\,2} b_{l2} J_3 + (a_{2s,\,0} - a_{2s,\,1}) J_2 + (b_{2s,\,2} + a_{2s,\,1}) J_1.
\end{aligned} \tag{2.5}
$$

If $M_{2s+1}(h) \equiv 0$, then we have $a_{2s-1-l,\,2} b_{l2} = 0$, $a_{2s,\,0} - a_{2s,\,1} = 0$, $b_{2s,\,2} + a_{2s,\,1} = 0$. Noting that $b_{l2} \neq 0$, we have $a_{2s-1-l,\,2} = 0$ and $a_{2s,\,0} = a_{2s,\,1} = -b_{2s,\,2}$. It means that for $k = 2s+1$, (ii) also holds.

Finally, assume that for $k = 2s$, (iii) holds. Then we claim that for $k = 2s+1$, (iii) also holds.

This assumption implies that for any given $l \in \{0, 1, \cdots, s-1\}$, $a_{i2} = 0$, $0 \leq i \leq l-1$, $a_{l2} \neq 0$. Then we have $\bar{q}_i = 0$ for $0 \leq i \leq l-1$, $\bar{q}_l \neq 0$. On the other hand, $b_{i2} = 0$, $0 \leq i \leq 2s-2-l$ give to $d\bar{Q}_i = \Omega_{i1}(x)$ for $0 \leq i \leq 2s-2-l$. The conditions $a_{i0} = a_{i1} = -b_{i2}$, $0 \leq i \leq 2s-1$ give to $N_i \equiv 0$, $0 \leq i \leq 2s-1$. Then

$$
\begin{aligned}
M_{2s+1}(h) &= \oint_{\Gamma_h} \sum_{j=0}^{2s-1} q_{2s-1-j} \omega_j + \omega_{2s} \\
&= \oint_{\Gamma_h} \sum_{j=2s-1-l}^{2s-1} q_{2s-1-j} \omega_j + \omega_{2s} \\
&= \oint_{\Gamma_h} \bar{q}_l \Omega_{2s-1-l,\,2}(x,\, y) + \omega_{2s} \\
&= -\frac{2}{3} a_{l2} b_{2s-1-l,\,2} J_3 + (a_{2s,\,0} - a_{2s,\,1}) J_2 + (b_{2s,\,2} + a_{2s,\,1}) J_1.
\end{aligned} \tag{2.6}
$$

If $M_{2s+1}(h) \equiv 0$, then we have $a_{l2} b_{2s-1-l,\,2} = 0$, $a_{2s,\,0} - a_{2s,\,1} = 0$, $b_{2s,\,2} + a_{2s,\,1} = 0$. Noting that $a_{l2} \neq 0$, we have $b_{2s-1-l,\,2} = 0$ and $a_{2s,\,0} = a_{2s,\,1} = -b_{2s,\,2}$. Hence we get that (iii) also holds for $k = 2s+1$.

This completes the proof for Case 1.

Case 2: k is odd.

We shall prove this case by induction.

When $k = 3$, it can be easily obtained by Case 31-Case 33.

Then we assume that for $k = n = 2s+1$, one of the conditions (i)-(iii) holds. We need to prove that it also holds for $k = n+1 = 2s+2$.

Firstly, assume that for $k = 2s+1$, (i) holds. Then we claim that for $k = 2s+2$, the condition (i) holds.

The assumption means that $a_{i2} = 0$, $0 \leq i \leq s-1$. We have $\bar{q}_i = 0$ for $0 \leq i \leq s-1$. On the other hand, $b_{i2} = 0$, $0 \leq i \leq s-1$ give to $d\bar{Q}_i = \Omega_{i1}(x)$ for $0 \leq i \leq s-1$.

The conditions $a_{i0} = a_{i1} = -b_{i2}$, $0 \le i \le 2s$ give to $N_i \equiv 0$, $0 \le i \le 2s$. Then

$$
\begin{aligned}
M_{2s+2}(h) &= \oint_{\Gamma_h} \sum_{j=0}^{2s} q_j \omega_{2s-j} + \omega_{2s+1} \\
&= \oint_{\Gamma_h} \sum_{j=s}^{2s} q_j \omega_{2s-j} + \omega_{2s+1} \\
&= \oint_{\Gamma_h} \sum_{j=s}^{2s} q_j \Omega_{2s-j,\,2}(x,\,y) + \omega_{2s+1} \\
&= \oint_{\Gamma_h} q_s \Omega_{s,\,2}(x,\,y) + q_{s+1}\Omega_{s-1,\,2}(x,\,y) + \cdots + q_{2s}\Omega_{0,\,2}(x,\,y) + \omega_{2s+1} \\
&= -\frac{2}{3} a_{s2} b_{s2} J_3 + (a_{2s+1,\,0} - a_{2s+1,\,1}) J_2 + (b_{2s+1,\,2} + a_{2s+1,\,1}) J_1.
\end{aligned} \tag{2.7}
$$

If $M_{2s+2}(h) \equiv 0$, then $a_{s2}b_{s2} = 0$, $a_{2s+1,\,0} = a_{2s+1,\,1} = -b_{2s+1,\,2}$. If $a_{s2} = b_{s2} = 0$, then it means that (i) holds for $k = 2s+2$.

In addition, when $a_{s2} = 0$, $b_{s2} \neq 0$, it belongs to (ii) with $l = [\frac{2s+2}{2}] - 1$. When $a_{s2} \neq 0$, $b_{s2} = 0$, it belongs to (iii) with $l = [\frac{2s+2}{2}] - 1$.

Secondly, suppose that when $k = 2s+1$, (ii) or (iii) holds. We want to prove that for $k = 2s+2$, the condition (ii) or (iii) also holds, respectively.

Following the same process of proof for Case 1, it is easy to prove the claim for any given $l \in \{0,\,1,\,\cdots,\,s-1\}$. Combining with the analysis of $M_{2s+2}(h) \equiv 0$ in formula (2.7), it is straightforward to obtain that for $l = [\frac{k+1}{2}] - 1 = s$, it also holds. This finishes the proof for Case 2.

By induction, the conclusion holds for each k. This leads to a complete proof of this lemma. $\square$

Proposition 2.4. *For system (1.2), $M_1(h) = M_2(h) = \cdots = M_k(h) \equiv 0$, $1 \le k \le N$ and $M_{k+1}(h) \not\equiv 0$, then $M_{k+1}(h)$ has at most two isolated zeros, multiplicity taken into account, and two can be reached.*

Proof. From Lemma 2.2, when the condition (i) holds, we have to consider this condition by $k = 2s$ and $k = 2s+1$, respectively. When $k = 2s$, $M_{2s+1}(h)$ can be expressed as

$$M_{2s+1}(h) = (a_{2s,\,0} - a_{2s,\,1}) J_2 + (b_{2s,\,2} + a_{2s,\,1}) J_1.$$

When $k = 2s+1$, we get

$$M_{2s+2}(h) = -\frac{2}{3} a_{s2} b_{s2} J_3 + (a_{2s+1,\,0} - a_{2s+1,\,1}) J_2 + (b_{2s+1,\,2} + a_{2s+1,\,1}) J_1.$$

If (ii) holds, then we have for any given l,

$$M_{k+1}(h) = -\frac{2}{3} a_{k-1-l,\,2} b_{l,\,2} J_3 + (a_{k0} - a_{k1}) J_2 + (b_{k2} + a_{k1}) J_1.$$

For the condition (iii), for any given l, it can be obtained from (2.6) that

$$M_{k+1}(h) = -\frac{2}{3} a_{l,\,2} b_{k-1-l,\,2} J_3 + (a_{k0} - a_{k1}) J_2 + (b_{k2} + a_{k1}) J_1.$$

In short, according to the fact that (J_1, J_2, J_3) is an ECT-system, $M_{k+1}(h)$ has at most two isolated zeros, multiplicity taken into account, and two zeros can appear with some proper parameters. □

Proposition 2.5. *For system (1.2), if* $M_1(h) = M_2(h) = \cdots = M_{N+1}(h) \equiv 0$ *and* $M_{N+2}(h) \not\equiv 0$, *then* M_{N+2} *has no isolated zero. If* $M_1(h) = M_2(h) = \cdots = M_{N+2}(h) \equiv 0$, *then system (1.2) is integrable.*

Proof. For $k = N + 2$, according to the proof of Lemma 2.2 and Proposition 2.4, we have $M_{N+2}(h) = m_{n+2}J_3$ or $M_{N+2}(h) \equiv 0$, where m_{n+2} is a constant formed by some parameters $a_{i,2}$ and $b_{j,2}$ with $0 \leq i, j \leq N$. Therefore M_{N+2} has no isolated zero. And if $M_1(h) = M_2(h) = \cdots = M_{N+2}(h) \equiv 0$, then $M_k(h) \equiv 0$ for any $k \geq N + 2$, which means that system (1.2) is integrable. Hence we complete the proof. □

Finally, we give the proof of Theorem 1.1.

Proof. [The proof of Theorem 1.1]

According to the relationship between the number of limit cycles bifurcated from the periodic orbits and the number of zeros of Melnikov functions mentioned in Theorem 3.4 in [6], and combining Proposition 2.1-2.5, we can get Theorem 1.1 proved. □

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