

Unique Solution for a General Coupled System of Fractional Differential Equations*

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Abstract This paper discusses a new coupled system of Riemann-Liouville fractional differential equations, in which the nonlinear terms include the Riemann-Liouville fractional integrals and the boundary value problems involve three-points. We seek also for the existence and uniqueness of solutions for this new system. We first get some useful properties of the Green's function generated by the system, and then we apply a fixed point theorem of increasing φ -(h, e)-concave operators to this new coupled system. Finally, we gain the existence and uniqueness results of the solution for this problem. In the end, a concrete example is structured to illustrate the main result.

Keywords Existence and uniqueness, coupled system of fractional differential equations, fixed point theorem, φ -(h, e)-concave operators

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1. Introduction

This paper considers the existence and uniqueness of solutions for a new coupled system of fractional differential equations:

$$\begin{cases} D^\alpha u(t) + f(t, v(t), I^\gamma v(t)) = a, t \in (0, 1), \\ D^\beta v(t) + g(t, u(t), I^\delta u(t)) = b, t \in (0, 1), \\ u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \\ v(0) = v'(0) = 0, v'(1) = \mu v'(\xi), \end{cases} \quad (1.1)$$

where $2 < \alpha, \beta \leq 3$, $1 < \gamma, \delta \leq 2$, and $\alpha - \delta \geq 1$, $\beta - \gamma \geq 1$, $0 < \xi < 1$, $0 < \mu \xi^{\alpha-2} < 1$, $0 < \mu \xi^{\beta-2} < 1$, a, b are nonnegative constants. D^α and D^β are the standard Riemann-Liouville fractional derivatives, and $f, g : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions. We employ a fixed point theorem of increasing φ -(h, e)-concave operators by Zhai and Wang to study the system (1.1).

In recent decades, fractional calculus has been widely used as a tool to study many problems in different research fields, such as engineering, biology, physics,

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and so on. Because the extensive development and application of fractional calculus equation theory, the theme of fractional differential equation system has developed into a crucial research field. Based on it, the theoretical research on the existence of solutions for fractional problems has attracted widespread attention. For some gained studies on fractional differential equations, we can refer to [3, 7–13, 19–23, 27–39] and others. Many problems of coupled systems involving fractional differential equations have been investigated extensively, see [9, 15, 22, 24] and others. From literature, we know that the mathematical definition of the differential operator of fractional order has been the subject of different approaches, and the most used are the Riemann-Liouville (RL), the Grünwald-Letnikov (GL), and the Caputo's (C) definitions. Although they are different in differential equations, some fixed methods can be used to study these problems and some similar results under different conditions are also obtained. Based upon this reason, here we consider (1.1) only involving Riemann-Liouville fractional derivative rather than other derivatives.

Nowadays, many researchers devoted themselves to determining the solvability of system of fractional differential equations with different boundary conditions, specifically to the study of existence of solutions to some systems of fractional differential equations, see [4–6, 9–14, 16–21, 23] for example and the references therein. The usual methods used are Schauder's fixed point theorem, Banach's fixed point theorem, Guo-Krasnoselskii's fixed point theorem on cone, monotone iterative technique, the method of lower and upper solutions, nonlinear differentiation of Leray-Schauder type and others. In [10], the authors considered a system of fractional differential equations:

$$\begin{cases} D^\alpha u(t) + f(t, v(t), D^\gamma v(t)) = 0, 0 < t < 1, \\ D^\beta v(t) + g(t, u(t), D^\delta u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \\ v(0) = v'(0) = 0, v'(1) = \mu v'(\xi), \end{cases}$$

where, $2 < \alpha, \beta \leq 3$, $1 < \gamma, \delta \leq 2$, and $\alpha - \delta \geq 1$, $\beta - \gamma \geq 1$, $0 < \xi < 1$, $0 < \mu \xi^{\alpha-2} < 1$, $0 < \mu \xi^{\beta-2} < 1$. D^α and D^β are the Riemann-Liouville fractional derivatives, and $f, g : [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \rightarrow [0, +\infty)$ are continuous functions and depend on the unknown functions as well as their lower order fractional derivatives. The authors got the existence and uniqueness of positive solutions for the system by utilizing fixed point theorems due to Schauder and Banach.

This paper considers the existence and uniqueness of solutions for system (1.1). As is well known, no papers have considered the solutions of system (1.1) from literatures. So it is worthwhile to investigate (1.1).

2. Preliminaries and previous results

Definition 2.1. [1, 2] The Riemann-Liouville fractional integral of order $\alpha > 0$, for a function $f : (0, +\infty) \rightarrow (-\infty, +\infty)$, is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, t > 0,$$

provided the integral exists.

Definition 2.2. [1, 2] For a function $f : (0, +\infty) \rightarrow (-\infty, +\infty)$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$D^\alpha f(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, t > 0,$$

where $n = [\alpha] + 1$, in which $[\alpha]$ denotes the integer part of the number α .

Lemma 2.1. [1] If $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ belongs to $C(0, 1) \cap L(0, 1)$, then

$$I^\alpha D^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-N},$$

for some $C_i \in (-\infty, +\infty)$, $i = 1, 2, \dots, N$, where $N = [\alpha] + 1$.

Lemma 2.2. [10] Let $y_1 \in C[0, 1]$ and $\alpha \in (2, 3]$, then the problem

$$\begin{cases} D^\alpha u(t) + y_1(t) = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \end{cases}$$

has a unique solution $u(t) = \int_0^t G_1(t, s) y_1(s) ds$, where

$$G_1(t, s) = \begin{cases} \frac{(1-s)^{\alpha-2} t^{\alpha-1} - \mu(\xi-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1} (1-\mu\xi^{\alpha-2})}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1} (1-\mu\xi^{\alpha-2})}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 < \xi \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-1} - \mu(\xi-s)^{\alpha-2} t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 \leq t \leq s \leq \xi < 1, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases} \quad (2.1)$$

Lemma 2.3. The above Green's function $G_1(t, s)$ defined by (2.1) has some properties:

(i) $G_1(t, s) > 0$, for $t, s \in (0, 1)$;

(ii) $t^{\alpha-1} G_1(1, s) \leq G_1(t, s) \leq \frac{t^{\alpha-1} (1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})}$, $t, s \in (0, 1)$.

Proof. From Lemma 4 in Reference [10], we know that the condition (i) is true. In addition, when $0 < s \leq t < 1$, $s \leq \xi$, there is

$$\begin{aligned} G_1(t, s) &= \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)} [(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-\frac{s}{t})^{\alpha-1} (1-\mu\xi^{\alpha-2})] \\ &\geq \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)} [(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-s)^{\alpha-1} (1-\mu\xi^{\alpha-2})] \\ &= t^{\alpha-1} G_1(1, s). \end{aligned}$$

When $0 < \xi \leq s \leq t < 1$, we have

$$\begin{aligned} G_1(t, s) &= \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)} [(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-\frac{s}{t})^{\alpha-1} (1-\mu\xi^{\alpha-2})] \\ &\geq \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)} [(1-s)^{\alpha-2} - (1-s)^{\alpha-1} (1-\mu\xi^{\alpha-2})] \\ &= t^{\alpha-1} G_1(1, s). \end{aligned}$$

Therefore, $G_1(t, s) \geq t^{\alpha-1}G_1(1, s)$, where

$$G_1(1, s) = \frac{1}{(1 - \mu\xi^{\alpha-2})\Gamma(\alpha)} \begin{cases} (1 - s)^{\alpha-2} - \mu(\xi - s)^{\alpha-2} - (1 - s)^{\alpha-1}(1 - \mu\xi^{\alpha-2}) \\ \text{for } 0 \leq s \leq \xi, \\ (1 - s)^{\alpha-2} - (1 - s)^{\alpha-1}(1 - \mu\xi^{\alpha-2}) \text{ for } \xi \leq s \leq 1. \end{cases}$$

Obviously, by **le2.2**, there is $G_1(t, s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})}, t, s \in (0, 1)$.

That is, the condition (ii) is true. □

Remark 2.1. [10] The Green’s function $G_2(t, s)$ has the same properties as $G_1(t, s)$, where

$$G_2(t, s) = \begin{cases} \frac{(1-s)^{\beta-2}t^{\beta-1} - \mu(\xi-s)^{\beta-2}t^{\beta-1} - (t-s)^{\beta-1}(1-\mu\xi^{\beta-2})}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{(1-s)^{\beta-2}t^{\beta-1} - (t-s)^{\beta-1}(1-\mu\xi^{\beta-2})}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 < \xi \leq s \leq t \leq 1, \\ \frac{(1-s)^{\beta-2}t^{\beta-1} - \mu(\xi-s)^{\beta-2}t^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 \leq t \leq s \leq \xi < 1, \\ \frac{(1-s)^{\beta-2}t^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases} \tag{2.2}$$

Let $(E, \|\cdot\|_E)$ be a real Banach space, and it is partially ordered by a cone $P \subset E$. By θ it denotes the zero element of E . For $\forall x, y \in E, \theta \leq x \leq y$ and $\exists N > 0$, such that $\|x\|_E \leq N\|y\|_E$, then P is called normal. For $h > \theta$, we define a set $P_h = \{x \in E \mid \lambda h \leq x \leq \mu h, \lambda, \mu > 0\}$. Obviously, $P_h \subset P$. Take $e \in P$ with $\theta \leq e \leq h$, we define $P_{h,e} = \{x \in E \mid x + e \in P_h\}$. Let a vector $h = (h_1, h_2)$, and $h_1, h_2 \in P$ with $h_1, h_2 \neq \theta$, then $h \in \tilde{P} := P \times P$. Obviously, if P is normal, then \tilde{P} is also normal. Take $\theta \leq e_1 \leq h_1, \theta \leq e_2 \leq h_2$, and let $\bar{\theta} = (\theta, \theta), e = (e_1, e_2)$, then $\bar{\theta} = (\theta, \theta) \leq (e_1, e_2) \leq (h_1, h_2) = h$. That is, $\bar{\theta} \leq e \leq h$.

Definition 2.3. [26] Suppose that $T : P_{h,e} \rightarrow E$ is an operator and it satisfies: for $x \in P_{h,e}$ and $0 < \tau < 1$, there exists $\varphi(\tau) > \tau$, such that $T(\tau x + (\tau - 1)e) \geq \varphi(\tau)Tx + (\varphi(\tau) - 1)e$. Then T is called a φ -(h, e)-concave operator.

Lemma 2.4. [26] Let P be normal and T be an increasing φ - (h, e)-concave operator satisfying $Th \in P_{h,e}$, then T has a unique fixed point $x^* \in P_{h,e}$. For $\omega_0 \in P_{h,e}$, defining a sequence $\omega_n = T\omega_{n-1}, n = 1, 2, \dots$, then $\|\omega_n - x^*\| \rightarrow 0 (n \rightarrow \infty)$.

Lemma 2.5. [25] $\tilde{P}_h = \{(x, y) : x \in P_{h_1}, y \in P_{h_2}\}, \tilde{P}_h = P_{h_1} \times P_{h_2}$.

Lemma 2.6. [3] $\tilde{P}_{h,e} = P_{h_1,e_1} \times P_{h_2,e_2}$.

3. Main results

We consider a Banach space $E = C[0, 1]$, equipped with the norm

$$\|u\| = \max\{|u(t)| : t \in [0, 1]\}.$$

Let $\|(u, v)\|_E = \max\{\|u\|, \|v\|\}$, for $(u, v) \in E \times E$, so $(E \times E, \|(\cdot, \cdot)\|_E)$ is a Banach space. Let P be the cone in E given by

$$P = \{u \in E \mid u(t) \geq 0, t \in [0, 1]\},$$

then

$$\tilde{P} = \{(u, v) \in E \times E \mid u(t), v(t) \geq 0, t \in [0, 1]\},$$

is also a cone. Obviously, $\tilde{P} = P \times P \subset E \times E$ is normal, and we get

$$(u_1, v_1) \leq (u_2, v_2) \iff u_1(t) \leq u_2(t) \text{ and } v_1(t) \leq v_2(t), t \in [0, 1].$$

By Lemma 2.2 and the reference [10], we can obtain the following results.

Lemma 3.1. *Assume that f, g are continuous, then $(u, v) \in E \times E$ is a solution of system (1.1) if and only if $(u, v) \in E \times E$ is a solution of the following equations:*

$$\begin{cases} u(t) = \int_0^1 G_1(t, s) f(s, v(s), I^\gamma v(s)) ds - a \int_0^1 G_1(t, s) ds, \\ v(t) = \int_0^1 G_2(t, s) g(s, u(s), I^\delta u(s)) ds - b \int_0^1 G_2(t, s) ds. \end{cases}$$

For $(u, v) \in E \times E$, define operators $T_1 : E \rightarrow E, T_2 : E \rightarrow E$ and $T : E \times E \rightarrow E \times E$ as follows:

$$\begin{aligned} T_1 u(t) &= \int_0^1 G_1(t, s) f(s, v(s), I^\gamma v(s)) ds - a \int_0^1 G_1(t, s) ds, \\ T_2 v(t) &= \int_0^1 G_2(t, s) g(s, u(s), I^\delta u(s)) ds - b \int_0^1 G_2(t, s) ds, \\ T(u, v)(t) &= (T_1 u(t), T_2 v(t)). \end{aligned}$$

Thus, from **le3.1**, the solution of (1.1) is a fixed point of the operator T . Let

$$\begin{aligned} e_1(t) &= a \int_0^1 G_1(t, s) ds, \quad e_2(t) = b \int_0^1 G_2(t, s) ds, \\ h_1(t) &= L_1 t^{\alpha-1}, \quad h_2(t) = L_2 t^{\beta-1}, \end{aligned} \quad (3.1)$$

with $L_1 \geq \frac{a}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)(\alpha-1)}$, $L_2 \geq \frac{b}{(1-\mu\xi^{\beta-2})\Gamma(\beta)(\beta-1)}$.

Theorem 3.1. *Let $2 < \alpha, \beta \leq 3$, $a, b > 0$ and e_1, e_2, h_1, h_2 be given as in (3.1). Suppose that the following assumptions holds:*

(H₁) $f : [0, 1] \times [-e_2^*, +\infty) \times [-e_2^*, +\infty) \rightarrow [0, +\infty)$, $g : [0, 1] \times [-e_1^*, +\infty) \times [-e_1^*, +\infty) \rightarrow [0, +\infty)$ are increasing with respect to the second and third variables and continuous, where $e_2^* = \max\{e_2(t) : t \in [0, 1]\}$, $e_1^* = \max\{e_1(t) : t \in [0, 1]\}$;

(H₂) for $0 < \tau < 1$, there exists $\varphi(\tau) > \tau$ such that

$$f(t, \tau x_1 + (\tau - 1)y_1, \tau x_2 + (\tau - 1)y_1) \geq \varphi(\tau) f(t, x_1, x_2),$$

$$g(t, \tau x_1 + (\tau - 1)y_2, \tau x_2 + (\tau - 1)y_2) \geq \varphi(\tau) g(t, x_1, x_2),$$

$t \in [0, 1], x_1, x_2 \in (-\infty, +\infty), y_1 \in [0, e_2^*], y_2 \in [0, e_1^*]$;

(H₃) for $t \in [0, 1]$, $f(t, 0, 0) \geq 0, g(t, 0, 0) \geq 0$ with $f(t, 0, 0) \not\equiv 0, g(t, 0, 0) \not\equiv 0$.

Then

(1) system (1.1) has a unique solution $(u^*, v^*) \in \tilde{P}_{h,e}$, where

$$e(t) = (e_1(t), e_2(t)), h(t) = (h_1(t), h_2(t)), t \in [0, 1];$$

(2) for any point $(u_0, v_0) \in \tilde{P}_{h,e}$, construct the following sequences:

$$u_{n+1}(t) = \int_0^1 G_1(t, s)f(s, v_n(s), I^\gamma v_n(s))ds - a \int_0^1 G_1(t, s)ds, n = 0, 1, 2 \dots ,$$

$$v_{n+1}(t) = \int_0^1 G_2(t, s)g(s, u_n(s), I^\delta u_n(s))ds - b \int_0^1 G_2(t, s)ds, n = 0, 1, 2 \dots ,$$

we have

$$\|u_{n+1} - u^*\| \rightarrow 0, \|v_{n+1} - v^*\| \rightarrow 0, n \rightarrow \infty.$$

Proof. By **le2.3**, for $t \in [0, 1]$, we have

$$e_1(t) = a \int_0^1 G_1(t, s)ds \geq 0, e_2(t) = b \int_0^1 G_2(t, s)ds \geq 0,$$

and

$$\begin{aligned} e_1(t) &= a \int_0^1 G_1(t, s)ds \leq a \int_0^1 \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}ds \\ &= \frac{at^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2}ds = \frac{at^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)(\alpha-1)} \\ &\leq L_1t^{\alpha-1} = h_1(t); \end{aligned}$$

$$\begin{aligned} e_2(t) &= b \int_0^1 G_2(t, s)ds \leq b \int_0^1 \frac{(1-s)^{\beta-2}t^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}ds \\ &= \frac{bt^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)} \int_0^1 (1-s)^{\beta-2}ds = \frac{bt^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)(\beta-1)} \\ &\leq L_2t^{\beta-1} = h_2(t); \end{aligned}$$

So, $0 \leq e_1 \leq h_1, 0 \leq e_2 \leq h_2$.

Firstly, we prove that $T : \tilde{P}_{h,e} \rightarrow E \times E$ is a φ - (h, e) -concave operator.

For $(u, v) \in \tilde{P}_{h,e}, 0 < \tau < 1$, we have

$$\begin{aligned} T(\tau(u, v) + (\tau - 1)e)(t) &= T(\tau(u, v) + (\tau - 1)(e_1, e_2))(t) \\ &= T(\tau u + (\tau - 1)e_1, \tau v + (\tau - 1)e_2)(t) \\ &= (T_1(\tau u + (\tau - 1)e_1), T_2(\tau v + (\tau - 1)e_2))(t). \end{aligned}$$

By (H_2) and the continuity of Riemann-Liouville fractional integral I^γ and I^δ ,

$$\begin{aligned} T_1(\tau u + (\tau - 1)e_1)(t) &= \int_0^1 G_1(t, s)f(s, \tau v(s) + (\tau - 1)e_2, I^\gamma(\tau v(s) \\ &\quad + (\tau - 1)e_2))ds - e_1(t) \\ &\geq \varphi(\tau) \int_0^1 G_1(t, s)f(s, v(s), I^\gamma v(s))ds - e_1(t) \\ &= \varphi(\tau) \left[\int_0^1 G_1(t, s)f(s, v(s), I^\gamma v(s))ds - e_1(t) \right] \\ &\quad + [\varphi(\tau) - 1]e_1(t) \\ &= \varphi(\tau)T_1u(t) + [\varphi(\tau) - 1]e_1(t), \end{aligned}$$

$$\begin{aligned}
T_2(\tau v + (\tau - 1)e_2)(t) &= \int_0^1 G_2(t, s)g(s, \tau u(s) + (\tau - 1)e_1, I^\delta(\tau u(s) \\
&\quad + (\tau - 1)e_1))ds - e_2(t) \\
&\geq \varphi(\tau) \int_0^1 G_2(t, s)g(s, u(s), I^\delta u(s))ds - e_2(t) \\
&= \varphi(\tau) \left[\int_0^1 G_2(t, s)g(s, u(s), I^\delta u(s))ds - e_2(t) \right] \\
&\quad + [\varphi(\tau) - 1]e_2(t) \\
&= \varphi(\tau)T_2v(t) + [\varphi(\tau) - 1]e_2(t).
\end{aligned}$$

So,

$$\begin{aligned}
T(\tau(u, v) + (\tau - 1)e)(t) &\geq (\varphi(\tau)T_1u(t) + [\varphi(\tau) - 1]e_1(t), \varphi(\tau)T_2v(t) \\
&\quad + [\varphi(\tau) - 1]e_2(t)) \\
&= (\varphi(\tau)T_1u(t), \varphi(\tau)T_2v(t)) \\
&\quad + ([\varphi(\tau) - 1]e_1(t), [\varphi(\tau) - 1]e_2(t)) \\
&= \varphi(\tau)(T_1u(t), T_2v(t)) + (\varphi(\tau) - 1)(e_1(t), e_2(t)) \\
&= \varphi(\tau)T(u, v)(t) + (\varphi(\tau) - 1)e(t).
\end{aligned}$$

That is,

$$T(\tau(u, v) + (\tau - 1)e) \geq \varphi(\tau)T(u, v) + (\varphi(\tau) - 1)e, (u, v) \in \tilde{P}_{h,e}, \tau \in (0, 1).$$

Therefore, $T : \tilde{P}_{h,e} \rightarrow E \times E$ is a φ - (h, e) -concave operator.

Secondly, we prove that $T : \tilde{P}_{h,e} \rightarrow E \times E$ is increasing.

For $(u, v) \in \tilde{P}_{h,e}$, we have $(u, v) + e \in \tilde{P}_h$. From **le2.5**, $(u + e_1, u + e_2) \in P_{h_1} \times P_{h_2}$.

So, there exists $\tau_1, \tau_2 > 0$ such that

$$u(t) + e_1(t) \geq \tau_1 h_1(t), v(t) + e_2(t) \geq \tau_2 h_2(t), t \in [0, 1].$$

Thus, $u(t) \geq \tau_1 h_1(t) - e_1(t) \geq -e_1(t) \geq -e_1^*$, $v(t) \geq \tau_2 h_2(t) - e_2(t) \geq -e_2(t) \geq -e_2^*$.

By (H_1) and the monotonicities of I^γ and I^δ , operator T is increasing.

Next, we show that $Th \in \tilde{P}_{h,e}$, that is $Th + e \in \tilde{P}_h$. For $t \in [0, 1]$,

$$\begin{aligned}
Th(t) + e(t) &= T(h_1, h_2)(t) + e(t) \\
&= (T_1h_1(t), T_2h_2(t)) + (e_1(t), e_2(t)) \\
&= (T_1h_1(t) + e_1(t), T_2h_2(t) + e_2(t)).
\end{aligned}$$

By **le2.3** and (H_1) , (H_3) ,

$$\begin{aligned}
T_1h_1(t) + e_1(t) &= \int_0^1 G_1(t, s)f(s, h_2(s), I^\gamma h_2(s))ds \\
&\geq \int_0^1 G_1(1, s)t^{\alpha-1}f(s, h_2(s), I^\gamma h_2(s))ds \\
&\geq t^{\alpha-1} \int_0^1 G_1(1, s)f(s, L_2t^{\beta-1}, I^\gamma L_2t^{\beta-1})ds \\
&\geq t^{\alpha-1} \int_0^1 G_1(1, s)f(s, 0, 0)ds \\
&= \frac{1}{L_1}h_1(t) \int_0^1 G_1(1, s)f(s, 0, 0)ds,
\end{aligned}$$

$$\begin{aligned}
 T_1 h_1(t) + e_1(t) &= \int_0^1 G_1(t, s) f(s, h_2(s), I^\gamma h_2(s)) ds \\
 &\leq \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s, L_2 t^{\beta-1}, I^\gamma L_2 t^{\beta-1}) ds \\
 &\leq \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s, L_2, I^\gamma L_2) ds \\
 &\leq \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s, L_2, L_2) ds \\
 &= \frac{1}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})L_1} h_1(t) \int_0^1 (1-s)^{\alpha-2} f(s, L_2, L_2) ds.
 \end{aligned}$$

By (H_1) , (H_3) and **le2.3**,

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s, L_2, L_2) ds \geq \int_0^1 G_1(1, s) f(s, 0, 0) ds > 0.$$

Let

$$\begin{aligned}
 l_1 &:= \frac{1}{L_1} \int_0^1 G_1(1, s) f(s, 0, 0) ds, \\
 l_2 &:= \frac{1}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})L_1} \int_0^1 (1-s)^{\alpha-2} f(s, L_2, L_2) ds,
 \end{aligned}$$

and from **le2.3** and (H_3) , $0 < l_1 \leq l_2$. That is, $l_1 h_1(t) \leq T_1 h_1(t) + e_1(t) \leq l_2 h_1(t)$. So, $T_1 h_1 + e_1 \in P_{h_1}$. Similarly, we have $T_2 h_2 + e_2 \in P_{h_2}$. Therefore, by **le2.6**,

$$Th + e = (T_1 h_1 + e_1, T_2 h_2 + e_2) \in \tilde{P}_h.$$

Finally, by **le2.4**, T has a unique fixed point $(u^*, v^*) \in \tilde{P}_{h,e}$. That is, (1.1) has a unique solution $(u^*, v^*) \in \tilde{P}_{h,e}$. In addition, taking any point $(u_0, v_0) \in \tilde{P}_{h,e}$, define the sequences:

$$\begin{aligned}
 u_{n+1}(t) &= \int_0^1 G_1(t, s) f(s, v_n(s), I^\gamma v_n(s)) ds - a \int_0^1 G_1(t, s) ds, n = 0, 1, 2, \dots, \\
 v_{n+1}(t) &= \int_0^1 G_2(t, s) g(s, u_n(s), I^\delta u_n(s)) ds - b \int_0^1 G_2(t, s) ds, n = 0, 1, 2, \dots,
 \end{aligned}$$

we must have

$$\|u_{n+1} - u^*\| \rightarrow 0, \|v_{n+1} - v^*\| \rightarrow 0, n \rightarrow \infty.$$

□

Example 3.1. We consider the following system:

$$\begin{cases}
 D^{\frac{5}{2}} u(t) + f(t, v(t), I^{\frac{7}{6}} v(t)) = 1, t \in (0, 1), \\
 D^{\frac{7}{3}} v(t) + g(t, u(t), I^{\frac{5}{4}} u(t)) = 1, t \in (0, 1), \\
 u(0) = u'(0) = 0, u'(1) = \frac{1}{3} u'(\frac{1}{2}) dt, \\
 v(0) = v'(0) = 0, v'(1) = \frac{1}{3} v'(\frac{1}{2}) dt,
 \end{cases} \tag{3.2}$$

where, $\alpha = \frac{5}{2}$, $\beta = \frac{7}{3}$, $\gamma = \frac{7}{6}$, $\delta = \frac{5}{4}$, $a = b = 1$, $\mu = \frac{1}{3}$, $\xi = \frac{1}{2}$ and

$$\begin{aligned} f(t, x_1, x_2) &= \left\{ \frac{1}{\Gamma(\frac{10}{3})} (-Ax_1 - 1)t^{\frac{7}{3}} + \frac{18\sqrt[3]{2} - 3}{(24\sqrt[3]{2} - 8)\Gamma(\frac{7}{3})} (Ax_1 + 1)t^{\frac{4}{3}} \right\}^{\frac{1}{3}} \\ &\quad + \left\{ \frac{1}{\Gamma(\frac{10}{3})} (-Ax_2 - 1)t^{\frac{7}{3}} + \frac{1}{\Gamma(\frac{7}{3})} \frac{18\sqrt[3]{2} - 3}{24\sqrt[3]{2} - 8} (Ax_2 + 1)t^{\frac{4}{3}} \right\}^{\frac{1}{3}}, \\ g(t, x_1, x_2) &= \left\{ \frac{1}{\Gamma(\frac{7}{2})} (-A_1x_1 - 1)t^{\frac{5}{2}} + \frac{6\sqrt{2} - 1}{(9\sqrt{2} - 3)\Gamma(\frac{5}{2})} (A_1x_1 + 1)t^{\frac{3}{2}} \right\}^{\frac{1}{3}} \\ &\quad + \left\{ \frac{1}{\Gamma(\frac{7}{2})} (-A_1x_2 - 1)t^{\frac{5}{2}} + \frac{6\sqrt{2} - 1}{(9\sqrt{2} - 3)\Gamma(\frac{5}{2})} (A_1x_2 + 1)t^{\frac{3}{2}} \right\}^{\frac{1}{3}}, \end{aligned}$$

$$A = \frac{1}{\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)} - \frac{1}{\Gamma(\frac{10}{3})}}, \quad A_1 = \frac{1}{\frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)} - \frac{1}{\Gamma(\frac{7}{2})}}.$$

$$\begin{aligned} e_1(t) &= \int_0^1 G_1(t, s) ds \\ &= -\frac{1}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} + \frac{6\sqrt{2} - 1}{\Gamma(\frac{5}{2})(9\sqrt{2} - 3)} t^{\frac{3}{2}} > 0, \end{aligned}$$

$$\begin{aligned} e_2(t) &= \int_0^1 G_2(t, s) ds \\ &= -\frac{1}{\Gamma(\frac{10}{3})} t^{\frac{7}{3}} + \frac{18\sqrt[3]{2} - 3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2} - 8)} t^{\frac{4}{3}} > 0, \end{aligned}$$

$$e_1^* = \max\{e_1(t), t \in [0, 1]\} = -\frac{1}{\Gamma(\frac{7}{2})} + \frac{6\sqrt{2} - 1}{\Gamma(\frac{5}{2})(9\sqrt{2} - 3)},$$

$$e_2^* = \max\{e_2(t), t \in [0, 1]\} = -\frac{1}{\Gamma(\frac{10}{3})} + \frac{18\sqrt[3]{2} - 3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2} - 8)}.$$

Set $h_1(t) = L_1 t^{\frac{3}{2}}$, $h_2(t) = L_2 t^{\frac{4}{3}}$, where

$$L_1 \geq \frac{2\sqrt{2}}{\Gamma(\frac{5}{2})(3\sqrt{2} - 1)}, \quad L_2 \geq \frac{9\sqrt[3]{2}}{\Gamma(\frac{7}{3})(12\sqrt[3]{2} - 4)}.$$

So,

$$\begin{aligned} e_1(t) &= \frac{6\sqrt{2} - 1}{\Gamma(\frac{5}{2})(9\sqrt{2} - 3)} t^{\frac{3}{2}} - \frac{1}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} < \frac{6\sqrt{2} - 1}{\Gamma(\frac{5}{2})(9\sqrt{2} - 3)} t^{\frac{3}{2}} < \frac{2\sqrt{2}}{\Gamma(\frac{5}{2})(3\sqrt{2} - 1)} \\ &= L_1 t^{\frac{3}{2}} = h_1(t), \end{aligned}$$

$$\begin{aligned} e_2(t) &= \frac{18\sqrt[3]{2} - 3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2} - 8)} t^{\frac{4}{3}} - \frac{1}{\Gamma(\frac{10}{3})} t^{\frac{7}{3}} < \frac{18\sqrt[3]{2} - 3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2} - 8)} t^{\frac{4}{3}} < \frac{9\sqrt{2} - 3}{\Gamma(\frac{7}{3})(12\sqrt[3]{2} - 4)} t^{\frac{4}{3}} \\ &= L_2 t^{\frac{4}{3}} = h_2(t), \end{aligned}$$

and

$$\begin{aligned} f(t, 0, 0) &= \left\{ -\frac{1}{\Gamma(\frac{10}{3})} t^{\frac{7}{3}} + \frac{18\sqrt[3]{2} - 3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2} - 8)} t^{\frac{4}{3}} \right\}^{\frac{1}{3}} \\ &\quad + \left\{ -\frac{1}{\Gamma(\frac{10}{3})} t^{\frac{7}{3}} + \frac{18\sqrt[3]{2} - 3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2} - 8)} t^{\frac{4}{3}} \right\}^{\frac{1}{3}} = 2(e_2(t))^{\frac{1}{3}} \geq 0, \end{aligned}$$

$$g(t, 0, 0) = \left\{ -\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}} \right\}^{\frac{1}{3}} \\ + \left\{ -\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}} \right\}^{\frac{1}{3}} = 2(e_1(t))^{\frac{1}{3}} \geq 0,$$

with $f(t, 0, 0) \neq 0$, $g(t, 0, 0) \neq 0$. Obviously,

$$f(t, x_1, x_2) = \{Ae_2(t)x_1 + e_2(t)\}^{\frac{1}{3}} + \{Ae_2(t)x_2 + e_2(t)\}^{\frac{1}{3}}, \\ g(t, x_1, x_2) = \{A_1e_1(t)x_1 + e_1(t)\}^{\frac{1}{3}} + \{A_1e_1(t)x_2 + e_1(t)\}^{\frac{1}{3}}.$$

For $\tau \in (0, 1)$, $x_1, x_2 \in (-\infty, +\infty)$, $y_1 \in [0, e_2^*]$, $y_2 \in [0, e_1^*]$, taking $\varphi(\tau) = \tau^{\frac{1}{3}}$, we have

$$f(t, \tau x_1 + (\tau - 1)y_1, \tau x_2 + (\tau - 1)y_2) \\ = \{Ae_2(t)(\tau x_1 + (\tau - 1)y_1) + e_2(t)\}^{\frac{1}{3}} + \{Ae_2(t)(\tau x_2 + (\tau - 1)y_2) + e_2(t)\}^{\frac{1}{3}} \\ = \tau^{\frac{1}{3}}\{Ae_2(t)(x_1 + (1 - \frac{1}{\tau})y_1) + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} + \tau^{\frac{1}{3}}\{Ae_2(t)(x_2 + (1 - \frac{1}{\tau})y_2) + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} \\ = \tau^{\frac{1}{3}}\{Ae_2(t)x_1 + Ae_2(t)(\tau - 1)y_1 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} + \tau^{\frac{1}{3}}\{Ae_2(t)x_2 \\ + Ae_2(t)(\tau - 1)y_2 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} \\ \geq \tau^{\frac{1}{3}}\{Ae_2(t)x_1 + (1 - \frac{1}{\tau})y_1 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} + \tau^{\frac{1}{3}}\{Ae_2(t)x_2 + (1 - \frac{1}{\tau})y_2 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} \\ = \tau^{\frac{1}{3}}f(t, x_1, x_2) = \varphi(\tau)f(t, x_1, x_2),$$

Similarly, $g(t, \tau x_1 + (\tau - 1)y_2, \tau x_2 + (\tau - 1)y_1) \geq \varphi(\tau)g(t, x_1, x_2)$.

By **th3.1**, problem (3.2) has a unique solution $(u^*, v^*) \in \tilde{P}_{h,e}$, where

$$e(t) = (e_1(t), e_2(t)) = \left(-\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}, -\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}} + \frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}} \right),$$

$$h(t) = (h_1(t), h_2(t)) = (L_1t^{\frac{3}{2}}, L_2t^{\frac{4}{3}}), t \in [0, 1].$$

For $\forall (u_0, v_0) \in \tilde{P}_{h,e}$, construct the following two sequences:

$$u_{n+1}(t) = \int_0^1 G_1(t, s)f(s, v_n(s), I^{\frac{7}{6}}v_n(s))ds + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}, \\ v_{n+1}(t) = \int_0^1 G_2(t, s)f(s, u_n(s), I^{\frac{5}{4}}u_n(s))ds + \frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}} - \frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}},$$

$n = 0, 1, 2, \dots$. Then,

$$\|u_{n+1} - u^*\| \rightarrow 0, \|v_{n+1} - v^*\| \rightarrow 0, n \rightarrow \infty.$$

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