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An Example of Piecewise Linear Systems with Infinitely Many Limit Cycles Separated by a Piecewise Linear Curve and Its Perturbations*

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Abstract In this paper, we present an example of piecewise linear systems with infinitely many crossing limit cycles defined in two zones separated by a piecewise linear curve with countable corners. Then we prove that under piecewise linear perturbations, the perturbed system can have infinitely many limit cycles, or exactly ℓ limit cycles for any given nonnegative integer ℓ.

Keywords Piecewise linear system, crossing limit cycle, piecewise linear perturbation, piecewise smooth switching curve, stability

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1. Introduction

A classical problem in the theory of dynamical systems is to determine the number and relative configuration of limit cycles. For example, the second part of the Hilbert's 16th problem is to find a uniform upper bound $\mathcal{H}(n)$ for the maximum number of limit cycles of planar polynomial systems of degree n such that $\mathcal{H}(n)$ depends only on n. This problem is extremely difficult and is still open. In fact, it remains unsolved whether $\mathcal{H}(n)$ is finite even for $n = 2$.

In the past decades, stimulated by real world applications from applied science such as mechanics, electronic engineering and control theory, it is natural to consider the same problem for planar piecewise smooth (PWS) systems. In particular, there is considerable interest in finding a uniform upper bound $\mathcal{H}_p^c(n)$ depending only on n of the maximum number of crossing limit cycles in the planar piecewise polynomial systems of degree n defined in two zones Σ_L^+ and Σ_L^- separated by exactly one switching line Σ_L given by the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} (P_n^+(x,y),\, Q_n^+(x,y))^T, \text{ if } (x,y) \in \Sigma_L^+, \\ (P_n^-(x,y),\, Q_n^-(x,y))^T, \text{ if } (x,y) \in \Sigma_L^-, \end{cases} \tag{1.1}$$

where $P_n^{\pm}(x,y)$ and $Q_n^{\pm}(x,y)$ are real polynomials of degree n, Σ_L^+ and Σ_L^- are disjoint open sets of $\mathbb{R}^2$, $\Sigma_L \subset \mathbb{R}^2$ is a straight line, $\Sigma_L^+ \cup \Sigma_L \cup \Sigma_L^- = \mathbb{R}^2$. Here the Fillipov's convention is assumed for the solutions of system (1.1) on Σ_L. A

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crossing limit cycle of system (1.1) refers to an isolated periodic orbit of (1.1) which intersects Σ_L transversally. Thus no sliding or grazing occurs. All of the limit cycles mentioned in the sequel are crossing limit cycles.

To date, most attentions have been paid to the cases when $n = 1$ and $n = 2$. The simplest case of (1.1) is piecewise linear (PWL) systems, i.e. when $n = 1$. In 1991, Lum and Chua conjectured that system (1.1) has at most one limit cycle under the continuity hypothesis when $n = 1$ [28]. This conjecture was proved in 1998 by Freire et al. in [13]. In [19], Han and Zhang constructed examples of (1.1) without the continuity hypothesis that have two limit cycles when $n = 1$ and conjectured that the upper bound for the maximum number of limit cycles of such a kind of systems is two. This conjecture was disproved by Huan and Yang in [21] by presenting concrete examples of such a kind of systems with three limit cycles. Since then, it had remained an open question whether there exists a unform upper bound for the maximum number of limit cycles of PWL systems defined in two zones separated by a straight line. Progress has been made recently by Carmona et al. in [3]. They provided a positive answer to this question by obtaining a unform upper bound $L^* \leq 8$ for the maximum number of limit cycles such a type of PWL systems has. Moreover, it was proved in [33] that two limit cycles can be bifurcated from the perturbations of PWL Hamiltonian systems with two saddles.

Like for smooth systems, the case of system (1.1) is intractable when $n = 2$ for the present time. Thus many researchers paid their attention to the center-focus and cyclicity problems under the assumption that $(0, 0)$ is a nondegenerate *pseudo-focus* of (1.1). That is to determine whether $(0, 0)$ is a center or a focus of (1.1) and to find the maximum number of small amplitude limit cycles bifurcating from $(0, 0)$. Those problems were first studied by Coll et al. in [7]. They pointed out that a pseudo-focus of system (1.1) can be classified into four types. They are *focus-focus* (FF), *focus-parabolic* (FP), *parabolic-focus* (PF) and *parabolic-parabolic* (PP). In [7, 16], planar PWS quadratic systems having four and five limit cycles were presented respectively. The center-focus and cyclicity problems for switching Bautin systems were further investigated in [4, 32] and it was proved in [32] that at least ten small amplitude limit cycles can bifurcate from a center in planar PWS quadratic systems of FF type. In [25], Llibre and Mereu proved that five limit cycles can bifurcate from the isochronous centers of planar PWS quadratic systems by using the generalized averaging theory of first order. Gouveia and Torregrosa proved in [17] that at least thirteen small amplitude limit cycles can bifurcate from an equilibrium of planar PWS quadratic systems with one switching line. In [8] da Cruz et al. proved that sixteen limit cycles can bifurcate from the period annulus of some isochronous quadratic centers in a class of planar PWS quadratic systems with one switching line. Thus $\mathcal{H}_p^c(2) \geq 16$. As far as we know, this is the best lower bound of the cyclicity of system (1.1) for $n = 2$.

In recent years, the attention has been paid to planar PWS quadratic systems with FP (or PF) and PP type critical points. Concrete examples of planar PWS quadratic systems with a FP type critical point that have at least four limit cycles and with a PP type critical point that have at least one limit cycle were constructed in [7]. It was proved in [31] that at least six limit cycles can bifurcate from a FP type critical point in a planar PWS quadratic system. Small amplitude limit cycles in planar PWS Hamiltonian systems with invisible fold-fold (i.e. PP type) critical points were discussed in [10]. In the work of Novaes and Silva [29], an example of planar PWS quadratic systems with a PP type critical point $(0, 0)$ that has five limit cycles bifurcated from $(0,0)$ was presented. This result was improved to seven in [12]. Moreover, system (1.1) for $n=2$ with a nilpotent equilibrium point $(0,0)$ was investigated in [5,6]. It was proved in [5] that at least seven small amplitude limit cycles can bifurcate from $(0,0)$ under small quadratic perturbations.

In real applications, discontinuities may occur on multiple lines or nonlinear curves and surfaces. The work of Braga and Mello in [9] showed that the shape and smoothness of the discontinuity sets have great impact on the maximum number of crossing limit cycles of a planar piecewise polynomial system. In [15], Gasull proposed to improve the lower bounds of the number $\mathcal{L}(n)$, which is defined as the maximum number of crossing limit cycles of planar PWL systems with two zones separated by a branch of an algebraic curve of degree n. Andrade et al. proved that $\mathcal{L}(2) \geq 4$, $\mathcal{L}(3) \geq 8$, $\mathcal{L}(n) \geq 7$ for $n \geq 4$ even, and $\mathcal{L}(n) \geq 9$ for $n \geq 5$ odd [2]. In [30], Novaes improved those results and proved that $\mathcal{L}(n)$ grows as fast as n^2. Limit cycles in small perturbations of a PWL Hamiltonian system with a non-regular separation line were investigated in [24]. Limit cycles of planar PWS systems formed by linear centers and separated by two circles were considered by Anacleto et al. in [1]. In [22], Ke et al. studied limit cycles of planar PWS systems arising from the perturbation of a quadratic isochronous system with two switching lines. Li and Llibre obtained an upper bound for the maximum number of limit cycles for a class of planar piecewise polynomial Hamiltonian systems of degree n separated by the curve $y=x^m$, where m and n are positive integers in [23]. The existence and number of limit cycles in a class of planar piecewise C^k systems defined in two zones separated by a C^k curve were studied in [18]. The number of limit cycles bifurcated from a period annulus in a class of planar piecewise Hamiltonian systems with a non-regular separation line was studied in [20].

Despite the widespread works on the study of the number of limit cycles of planar PWS systems, the literature on planar PWS systems with nonsmooth discontinuity sets, particularly, those with discontinuity sets given by piecewise smooth curves with countable corners, is very limited. In this paper we aim to make some efforts on this. More specifically, we present an example of planar PWL systems with infinitely many crossing limit cycles defined in two zones separated by a PWL switching curve. One branch of the switching curve consists of countable vertical and slant line segments. Thus it has countable corners. Then we prove that under PWL perturbations, the perturbed system can have infinitely many limit cycles, or exactly ℓ limit cycles for any given nonnegative integer ℓ.

It is worth mentioning that, PWL systems with infinitely many crossing limit cycles were constructed in [26,27]. The systems presented in those works are defined in infinitely many zones separated by the straight lines $|x|=2n-1$ for $n=1,2,\cdots$. An example of PWL systems defined in two zones separated by an analytical curve which has exactly n limit cycles for any given positive integer n was presented in [34]. Recently, the work [14] presented an example of PWL systems defined in two zones separated by an analytical curve which has infinitely many crossing limit cycles. To the best of our knowledge, the example given in this paper is the first example of PWL systems with infinitely many crossing limit cycles defined in two zones separated by a PWS curve having countable corners.

Our presentation is organized as follows. In Section 2, we present an example of planar PWL systems with infinitely many crossing limit cycles defined in two zones separated by a PWL switching curve. In Section 3, we investigate that under PWL perturbations, the number of limit cycles of the perturbed system can have. Some concluding remarks are given in Section 4.

2. An example with infinitely many limit cycles

Let m be a positive integer. Define

$$\begin{aligned}
\mathcal{L}_{-1} &= \left\{(x,y)\in\mathbb{R}^2:\ x\ge 0, y=0\right\},\\
\mathcal{L}_{0} &= \left\{(x,y)\in\mathbb{R}^2:\ x=0, 0\le y\le 2\right\},\\
\mathcal{L}_{m}^{p} &= \left\{(x,y)\in\mathbb{R}^2:\ x=2(m-1), 2\le y\le \frac{5}{2}\right\},\\
\mathcal{L}_{m}^{s} &= \left\{(x,y)\in\mathbb{R}^2:\ y=-\frac{1}{4}x+\left(2+\frac{1}{2}m\right), 2(m-1)\le x\le 2m\right\}.
\end{aligned}$$

Let $\mathcal{L}\subset\mathbb{R}^2$ be the discontinuity set given by

$$\mathcal{L}=\mathcal{L}_{-1}\bigcup\mathcal{L}_0\bigcup_{m=1}^{+\infty}\left(\mathcal{L}_m^p\bigcup\mathcal{L}_m^s\right).$$

Then $\mathbb{R}^2$ is split into two disjoint open regions Ω_1 and Ω_2 by $\mathcal{L}$, where $\Omega_1\subset\mathbb{R}^2$ is the narrow open belt region bounded by $\mathcal{L}$ in the first quadrant and $\Omega_2=\mathbb{R}^2-(\Omega_1\bigcup\mathcal{L})$. Consider the following PWL system:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}-2x-8y+3\\ x+2y+\frac{1}{2}\end{pmatrix}, \text{ if } (x,y)\in\Omega_1,\\[2ex] \begin{pmatrix}-2x-5y\\ x+2y+\frac{1}{2}\end{pmatrix}, \quad \text{ if } (x,y)\in\Omega_2.\end{cases} \tag{2.1}$$

The subsystem of (2.1) in the region Ω_1 is called the inner system of (2.1) and the one in Ω_2 is called the outer system of (2.1). The crossing region $\mathcal{L}^{\text{cross}}$ and the sliding region $\mathcal{L}^{\text{slid}}$ of system (2.1) are respectively given by (see, for example [11]):

$$\begin{aligned}
\mathcal{L}^{\text{cross}} &= \left\{(x,0)\in\mathbb{R}^2: x>0\right\}\bigcup\left\{(0,y)\in\mathbb{R}^2: \frac{3}{8}<y\le 2\right\}\bigcup\mathcal{L}_m^p\bigcup\mathcal{L}_m^s,\\
\mathcal{L}^{\text{slid}} &= \left\{(0,y)\in\mathbb{R}^2: 0<y<\frac{3}{8}\right\}.
\end{aligned}$$

A point in the sliding (resp. crossing) regions is called a *sliding* (resp. *crossing*) *point* of system (2.1). The Filippov's convention is assumed for the solutions of system (2.1) on $\mathcal{L}$. In this paper we are interested in the crossing limit cycles of system (2.1). A crossing limit cycle is a limit cycle that must intersect $\mathcal{L}$ at some points and these points are all crossing points. Again, all of the limit cycles mentioned in the sequel are crossing limit cycles.

We have the following result.

Theorem 2.1. *System* (2.1) *has infinitely many limit cycles.*

Proof of Theorem 2.1. Ignoring the discontinuity set $\mathcal{L}$ and extending the two subsystems of (2.1) to the whole plane $\mathbb{R}^2$, both of the inner and the outer systems have only one equilibrium given by $P : \left(-\frac{5}{2}, 1\right)$, which is a center for both subsystems. Furthermore, the flows of both subsystems turn counterclockwise around the center P. Clearly, the inner (resp. outer) system of (2.1) has a first integral $H_1(x, y)$ (resp. $H_2(x, y)$), where

$$H_1(x, y) = (x + 2y)^2 - 6y + x + 4y^2, \quad H_2(x, y) = (x + 2y)^2 + x + y^2.$$

For each of $k = 0, 1, \cdots$, the orbit Γ_k of (2.1) given by $\Gamma_k = \Gamma_k^1 \bigcup \Gamma_k^2$ is a periodic orbit of (2.1) that intersects the discontinuity set $\mathcal{L}$ transversally exactly twice at $P_k : (2(k + 2), 0) \in \mathcal{L}_{-1}$ and $Q_k : (2k, 2) \in \mathcal{L}_{k+1}^p$ respectively, where for $j = 1, 2$,

$$\Gamma_k^j = \left\{(x, y) \in \Omega_j : \ H_j(x, y) = 2(k + 2)(2k + 5)\right\}.$$

It is clear that, for any $k = 0, 1, 2, \cdots$, Γ_k is inside Γ_{k+1}. Γ_0, Γ_1 and Γ_2 are plotted in Fig. 1.

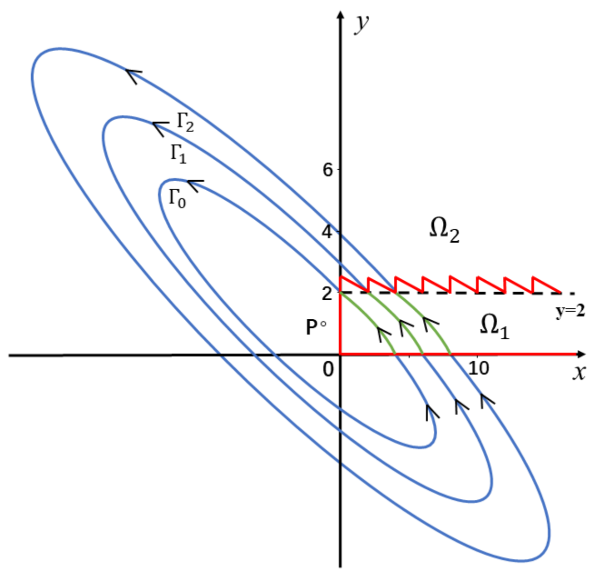

Figure 1. Three limit cycles Γ_0, Γ_1 and Γ_2 among the infinitely many limit cycles Γ_k for $k = 0, 1, 2, \cdots$ of system (2.1).

For each of $k = 0, 1, \cdots$, consider the orbits of (2.1) between Γ_k and Γ_{k+1}. Let $(x_k, 0) \in \mathcal{L}_{-1}$ be an arbitrary point between Γ_k and Γ_{k+1} on $\mathcal{L}_{-1}$, where $2(k + 2) < x_k < 2(k + 3)$ and consider the orbit γ_k of (2.1) starting from $(x_k, 0)$. γ_k first enters Ω_1 and is given by $H_1(x, y) = x_k^2 + x_k$. For $2(k + 2) < x_k \le \nu_k$, γ_k intersects $\mathcal{L}_{k+1}^p$ transversally at $(2k, y_k^p)$, from which γ_k enters Ω_2 and is given by $H_2(x, y) = h_k^p$, and returns to $\mathcal{L}_{-1}$ for the first time at $(x_k^p, 0)$ where

$$\begin{aligned}
\nu_k &= \frac{\sqrt{16k^2 + 88k + 141} - 1}{2}, \\
h_k^p &= \frac{(12k + 15)\sqrt{8x_k^2 + 8x_k - 16k^2 - 40k + 9} + 20x_k^2 + 20x_k + 45}{32},
\end{aligned}$$

$$x_k^p = \frac{-1 + \sqrt{4h_k^p + 1}}{2},$$
$$y_k^p = \frac{3 - 4k + \sqrt{8x_k^2 + 8x_k - 16k^2 - 40k + 9}}{8}.$$

We have $(x_k^p)^2 + x_k^p = (x_k)^2 + x_k + 6y_k^p - 3(y_k^p)^2$ and $y_k^p > 2$, from which we obtain $(x_k^p - x_k)(x_k^p + x_k + 1) = 3y_k^p(2 - y_k^p) < 0$, implying that $2(k+2) < x_k^p < x_k$. For $\nu_k < x_k < 2(k+3)$, γ_k intersects $\mathcal{L}_{k+1}^s$ transversally at $(-4y_k^s + 10 + 2k, y_k^s)$, from which γ_k enters Ω_2 and is given by $H_2(x,y) = h_k^s$, and returns to $\mathcal{L}_{-1}$ for the first time at $(x_k^s, 0)$ where

$$h_k^s = \frac{(12k+51)\sqrt{8x_k^2 + 8x_k - 16k^2 - 136k - 255} + 20x_k^2 + 20x_k + 45}{32},$$
$$x_k^s = \frac{-1 + \sqrt{4h_k^s + 1}}{2},$$
$$y_k^s = \frac{25 + 4k - \sqrt{8x_k^2 + 8x_k - 16k^2 - 136k - 255}}{8}.$$

We have $(x_k^s)^2 + x_k^s = (x_k)^2 + x_k + 6y_k^s - 3(y_k^s)^2$ and $y_k^s > 2$, from which we get $(x_k^s - x_k)(x_k^s + x_k + 1) = 3y_k^s(2 - y_k^s) < 0$, implying that $2(k+2) < x_k^s < x_k$. Thus in either case, γ_k spirals towards Γ_k. Hence, $\Gamma_1, \Gamma_2, \cdots$ are all semi-stable limit cycles of (2.1).

Now we prove that Γ_0 is also a limit cycle of (2.1). In the above we have proved that all of the orbits between Γ_0 and Γ_1 are all spiral towards Γ_0. Thus in the following we only need to consider the orbits inside Γ_0.

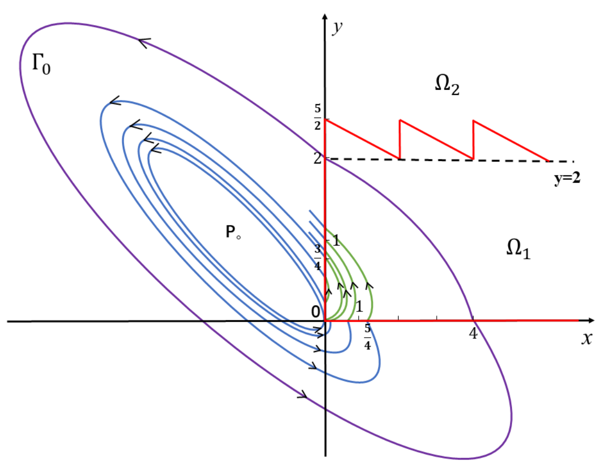

Figure 2. The limit cycle Γ_0 and the orbits inside Γ_0.

For the outer system of (2.1), there is a periodic orbit $\mathcal{C}_0$ given by $H_2(x,y) = 0$, which is tangent to the y-axis at $(0,0)$. System (2.1) has an annulus of periodic orbits $\mathcal{C}_h$ given by $H_2(x,y) = h$ for $h \in \left(-\frac{5}{4}, 0\right]$ around the center $\left(-\frac{5}{2}, 1\right)$ and bounded by $\mathcal{C}_0$. Furthermore, if we ignore the discontinuity set $\mathcal{L}$ and extend the inner system to the whole plane $\mathbb{R}^2$, it has a periodic orbit around the center $\left(-\frac{5}{2}, 1\right)$ and is given by $H_1(x,y) = -\frac{9}{8}$, which is tangent to the y-axis at $\left(0, \frac{3}{8}\right)$.

In the following we consider the orbit starting from $(0, y_*)$, where $y_* \in [0, 2)$. Our discussion is divided into three cases as follows.

(1) If $y_* = 0$, then besides the aforementioned periodic orbit $\mathcal{C}_0$, there is another orbit $\bar{\gamma}_1$ of (2.1) starting from $(0, 0)$. $\bar{\gamma}_1$ first enters Ω_1 and is given by $H_1(x, y) = 0$. Then it intersects the y-axis transversally at $\left(0, \frac{3}{4}\right)$, from which $\bar{\gamma}_1$ enters Ω_2 and is given by $H_2(x, y) = \frac{45}{16}$. Then it intersects the x-axis transversally at $\left(\frac{5}{4}, 0\right)$ and enters Ω_1 again and is given by $H_1(x, y) = \frac{45}{16}$. Finally it returns to the y-axis for the first time at $(0, \bar{y}_{*0})$, where $\bar{y}_{*0} = \frac{3}{8} + \frac{3}{16}\sqrt{14} \in \left(\frac{3}{4}, 2\right)$.

(2) If $y_* \in \left(0, \frac{3}{8}\right)$, then there is an orbit $\bar{\gamma}_2$ of (2.1) starting from $(0, y_*)$ at some initial time $t = t_*$. For $t > t_*$ and t increases, $\bar{\gamma}_2$ first enters Ω_1 and is given by $H_1(x, y) = 8y_*^2 - 6y_*$. Then it intersects the y-axis transversally at $\left(0, \frac{3}{4} - y_*\right)$ with $\frac{3}{8} < \frac{3}{4} - y_* < \frac{3}{4}$, from which $\bar{\gamma}_2$ enters Ω_2 and is given by $H_2(x, y) = 5\left(\frac{3}{4} - y_*\right)^2$. Then it intersects the x-axis transversally at $\left(\bar{x}_{*1}^{+}, 0\right)$ and enters Ω_1 again and is given by $H_1(x, y) = 5(\frac{3}{4} - y_*)^2$, where

$$\bar{x}_{*1}^{+} = -\frac{1}{2} + \frac{\sqrt{80y_*^2 - 120y_* + 49}}{4} \in \left(0, \frac{5}{4}\right).$$

Finally it returns to the y-axis for the first time at $\left(0, \bar{y}_{*1}^{+}\right)$, where

$$\bar{y}_{*1}^{+} = \frac{3}{8} + \frac{\sqrt{160y_*^2 - 240y_* + 126}}{16} \in \left(\frac{3}{4}, 2\right).$$

For $t < t_*$ and t decreases, $\bar{\gamma}_2$ first enters Ω_2 and is given by $H_2(x, y) = 5y_*^2$. Then it intersects the x-axis transversally at $\left(\bar{x}_{*1}^{-}, 0\right)$ and enters Ω_1 again and is given by $H_1(x, y) = 5y_*^2$, where

$$\bar{x}_{*1}^{-} = -\frac{1}{2} + \frac{\sqrt{20y_*^2 + 1}}{2} \in \left(0, \bar{x}_{*1}^{+}\right).$$

Then it returns to the y-axis at $\left(0, \bar{y}_{*1}^{-}\right)$, where

$$\bar{y}_{*1}^{-} = \frac{3}{8} + \frac{\sqrt{40y_*^2 + 9}}{8} \in \left(\frac{3}{4}, \bar{y}_{*1}^{+}\right).$$

(3) If $y_* \in \left(\frac{3}{8}, 2\right)$, then there is an orbit $\bar{\gamma}_3$ of (2.1) starting from $(0, y_*)$. $\bar{\gamma}_3$ first enters Ω_2 and is given by $H_2(x, y) = 5y_*^2$. Then it intersects the x-axis transversally at $(\bar{x}_{*2}, 0)$ and enters Ω_1 and is given by $H_1(x, y) = 5y_*^2$, where

$$\bar{x}_{*2} = -\frac{1}{2} + \frac{\sqrt{20y_*^2 + 1}}{2} \in \left(\frac{5}{4}, 4\right).$$

Finally it returns to the y-axis for the first time at $(0, \bar{y}_{*2})$, where

$$\bar{y}_{*2} = \frac{3}{8} + \frac{\sqrt{40y_*^2 + 9}}{8} \in (y_*, 2).$$

Thus the orbits of (2.1) near Γ_0 are all spiral towards Γ_0. Please see Fig. 2. Consequently Γ_0 is a stable limit cycle of (2.1).

In summary, we have proved that system (2.1) has infinitely many nested limit cycles given by Γ_k for $k = 0, 1, 2, \cdots$. Furthermore, Γ_0 is stable and for any $k \geq 1$, Γ_k is semi-stable.

The proof is complete. $\square$

3. Piecewise linear perturbations

In this section we investigate the number of limit cycles bifurcated from the PWL perturbations of system (2.1) given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -2x - 8y + 3 + \varepsilon(a_1 x + a_2 y) \\ x + 2y + \frac{1}{2} + \varepsilon(a_3 x - a_1 y) \end{pmatrix}, \text{ if } (x,y) \in \Omega_1, \\ \\ \begin{pmatrix} -2x - 5y + \varepsilon(b_1 x + b_2 y) \\ x + 2y + \frac{1}{2} + \varepsilon(b_3 x - b_1 y) \end{pmatrix}, \quad \text{if } (x,y) \in \Omega_2, \end{cases} \tag{3.1}$$

where $\varepsilon, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ are parameters, and $\varepsilon > 0$ is sufficiently small. The crossing region $\mathcal{L}_\varepsilon^{\text{cross}}$ and the sliding region $\mathcal{L}_\varepsilon^{\text{slid}}$ of system (3.1) are respectively given by:

$$\mathcal{L}_\varepsilon^{\text{cross}} = \big\{(x,0) \in \mathbb{R}^2 : x > 0\big\} \bigcup \left\{(0,y) \in \mathbb{R}^2 : \frac{3}{8 - \varepsilon a_2} < y \le 2\right\} \bigcup \mathcal{L}_m^p \bigcup \mathcal{L}_m^s,$$
$$\mathcal{L}_\varepsilon^{\text{slid}} = \left\{(0,y) \in \mathbb{R}^2 : 0 < y < \frac{3}{8 - \varepsilon a_2}\right\}.$$

Let $\mu_{1\varepsilon} = \varepsilon a_1 - 2$, $\mu_{2\varepsilon} = \varepsilon a_2 - 8$, $\mu_{3\varepsilon} = \varepsilon a_3 + 1$, $\nu_{1\varepsilon} = \varepsilon b_1 - 2$, $\nu_{2\varepsilon} = \varepsilon b_2 - 5$, $\nu_{3\varepsilon} = \varepsilon b_3 + 1$. Then the inner (resp. outer) system of (3.1) has a first integral $H_{1\varepsilon}(x,y)$ (resp. $H_{2\varepsilon}(x,y)$), given by

$$H_{1\varepsilon}(x,y) = \mu_{3\varepsilon} x^2 - \mu_{2\varepsilon} y^2 - 2\mu_{1\varepsilon} xy + x - 6y,$$
$$H_{2\varepsilon}(x,y) = \nu_{3\varepsilon} x^2 - \nu_{2\varepsilon} y^2 - 2\nu_{1\varepsilon} xy + x.$$

Let m be a positive integer, $\alpha_{i\varepsilon} = \varepsilon(b_i - a_i)$ for $i = 1,2$. Define

$$\xi_{\varepsilon 0} = \frac{6}{3 + \alpha_{2\varepsilon}},$$
$$\xi_{m\varepsilon,1} = \frac{6 - 4(m-1)\alpha_{1\varepsilon}}{3},$$
$$\xi_{m\varepsilon,2} = \frac{6 - 4(m-1)\alpha_{1\varepsilon}}{3 + \alpha_{2\varepsilon}},$$
$$\xi_{m\varepsilon,3} = \frac{6 - 4(m+4)\alpha_{1\varepsilon}}{3 - 8\alpha_{1\varepsilon}},$$
$$\xi_{m\varepsilon,4} = \frac{6 - 4(m+4)\alpha_{1\varepsilon}}{3 - 8\alpha_{1\varepsilon} + \alpha_{2\varepsilon}}.$$

Clearly, for $\varepsilon > 0$ sufficiently small, $3 + \alpha_{2\varepsilon}$, $3 - 8\alpha_{1\varepsilon}$ and $3 - 8\alpha_{1\varepsilon} + \alpha_{2\varepsilon}$ are all nonzero. Thus $\xi_{\varepsilon 0}$ and $\xi_{m\varepsilon,j}$ for $j = 1, \cdots, 4$ are well-defined. For any $\lambda \in \mathbb{R}$, let $[\lambda]$ be the integer part of λ. Let

$$k_0 = \left[-\frac{3}{8\alpha_{1\varepsilon}}\right], \quad k_1 = \left[-\frac{3 + 5\alpha_{2\varepsilon}}{8\alpha_{1\varepsilon}}\right], \quad k_2 = \left[-\frac{\alpha_{2\varepsilon}}{2\alpha_{1\varepsilon}}\right],$$
$$K_0 = -\frac{3}{8\alpha_{1\varepsilon}} - k_0, \quad K_1 = -\frac{3 + 5\alpha_{2\varepsilon}}{8\alpha_{1\varepsilon}} - k_1, \quad K_2 = -\frac{\alpha_{2\varepsilon}}{2\alpha_{1\varepsilon}} - k_2.$$

We remark that in the following we only need the values of k_j and K_j for $j = 0, 1, 2$ when $\alpha_{1\varepsilon} \neq 0$. Clearly, $K_j \geq 0$ for $j = 0, 1, 2$.

We have the following result showing that, for any given nonnegative integer ℓ, system (3.1) have exactly ℓ limit cycles under certain conditions. Furthermore, system (3.1) can have infinitely many limit cycles under certain conditions.

Theorem 3.1. *Suppose that $a_3 = b_3$ and $\varepsilon > 0$ is sufficiently small. Then we have the following results:*

(1) *If $a_1 = b_1$ and $a_2 = b_2$, then system (3.1) has one limit cycle which is stable and crosses $(0, 2)$, as well as infinitely many limit cycles which are all semi-stable and cross $\mathcal{L}$ at $(2m, 2)$ for $m = 1, 2, 3, \cdots$.*

(2) *If $a_1 \leq b_1$ and $a_2 < b_2$, then system (3.1) has exactly one limit cycle which is stable and crosses $\mathcal{L}$ at $(0, \xi_{\varepsilon 0})$. If $a_1 < b_1$ and $a_2 = b_2$, then system (3.1) has exactly one limit cycle which is stable and crosses $\mathcal{L}$ at $(0, 2)$.*

(3) *If $a_1 = b_1$ and $a_2 > b_2$, then system (3.1) has infinitely many limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{\varepsilon 0})$ and infinitely many limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{\varepsilon 0} + 8 + 2m, \xi_{\varepsilon 0})$ for $m = 1, 2, 3, \cdots$.*

(4) *If $a_1 > b_1$ and $a_2 = b_2$, then when $K_0 > 0$, system (3.1) has k_0+1 limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,1})$ and $k_0 + 1$ limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,3} + 8 + 2m, \xi_{m\varepsilon,3})$ for $m = 1, 2, \cdots, k_0 + 1$; when $K_0 = 0$, system (3.1) has k_0 limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,1})$, k_0 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,3} + 8 + 2m, \xi_{m\varepsilon,3})$ for $m = 1, 2, \cdots, k_0$ and one limit cycle which is semi-stable and crosses $\mathcal{L}$ at $(2k_0, \frac{5}{2})$.*

(5) *If $a_1 > b_1$ and $a_2 > b_2$, then when $K_1 > 0$, system (3.1) has k_1+1 limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ and $k_1 + 1$ limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4} + 8 + 2m, \xi_{m\varepsilon,4})$ for $m = 1, 2, \cdots, k_1 + 1$; when $K_1 = 0$, system (3.1) has k_1 limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$, k_1 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4} + 8 + 2m, \xi_{m\varepsilon,4})$ for $m = 1, 2, \cdots, k_1$ and one limit cycle which is semi-stable and crosses $\mathcal{L}$ at $(2k_1, \frac{5}{2})$.*

(6) *If $a_1 > b_1$ and $a_2 < b_2$, then system (3.1) has a stable limit cycle which crosses $\mathcal{L}$ at $(0, \xi_{\varepsilon 0})$. In addition to this one, (i) if $K_1 > 0$ and $K_2 > 0$, then system (3.1) has $k_1 - k_2$ limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ for $k_2 + 2 \leq m \leq k_1 + 1$ and $k_1 - k_2 + 1$ limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4} + 8 + 2m, \xi_{m\varepsilon,4})$ for $k_2 + 1 \leq m \leq k_1 + 1$; (ii) if $K_1 = 0$ and $K_2 > 0$, then system (3.1) has $k_1 - k_2 - 1$ limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ for $k_2 + 2 \leq m \leq k_1$, $k_1 - k_2$ limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4} + 8 + 2m, \xi_{m\varepsilon,4})$ for $k_2 + 1 \leq m \leq k_1$ and one limit cycle which is semi-stable and crosses $\mathcal{L}$ at $(2k_1, \frac{5}{2})$; (iii) if $K_1 > 0$ and $K_2 = 0$, then system (3.1) has $k_1 - k_2$ limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ for $k_2 + 2 \leq m \leq k_1 + 1$, $k_1 - k_2 + 1$ limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4} + 8 + 2m, \xi_{m\varepsilon,4})$ for $k_2 + 1 \leq m \leq k_1 + 1$ and one limit cycle which is semi-stable and crosses $\mathcal{L}$ at $(2k_2, 2)$; (iv) if $K_1 = K_2 = 0$, then system (3.1) has $k_1 - k_2 - 1$ limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ for $k_2 + 2 \leq m \leq k_1$, $k_1 - k_2$ limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4} + 8 + 2m, \xi_{m\varepsilon,4})$ for $k_2 + 1 \leq m \leq k_1$ and two limit cycles which are semi-stable and cross $\mathcal{L}$ at $(2k_2, 2)$ and $(2k_1, \frac{5}{2})$ respectively.*

(7) *If $a_1 < b_1$ and $a_2 > b_2$, then when $K_2 > 0$, system (3.1) has $k_2 + 1$ limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ for $m = 1, 2, \cdots, k_2 + 1$*

and k_2 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m, \xi_{m\varepsilon,4})$ for $m=1,2,\cdots,k_2$; when $K_2=0$, system (3.1) *has k_2 limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1), \xi_{m\varepsilon,2})$ for $m=1,2,\cdots,k_2$, k_2-1 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m, \xi_{m\varepsilon,4})$ for $m=1,2,\cdots,k_2-1$ and one limit cycle which is semi-stable and crosses $\mathcal{L}$ at $(2k_2, 2)$.*

Proof of Theorem 3.1. Suppose that $\varepsilon>0$ is sufficiently small. For any nonnegative integer j and $i=1,2$, let

$$\begin{aligned}
\Delta_1(j) &= 8\mu_{3\varepsilon}\left(2j^2\mu_{3\varepsilon}-4j\mu_{1\varepsilon}-2\mu_{2\varepsilon}+j-6\right),\\
\Delta_2(j) &= \mu_{3\varepsilon}\left(16j^2\mu_{3\varepsilon}-40j\mu_{1\varepsilon}-25\mu_{2\varepsilon}+8j-60\right),\\
\Lambda_i(j) &= \frac{1}{2\mu_{3\varepsilon}}\left(-1+\sqrt{1+\Delta_i(j)}\right).
\end{aligned}$$

Let m be a positive integer. Consider the orbit Γ starting from $(x_1,0)\in\mathcal{L}_{-1}$. Under the flow of system (3.1), Γ first enters Ω_1. Then it intersects $\mathcal{L}_0$ when $0\le x_1\le\Lambda_1(0)$, intersects $\mathcal{L}_m^p$ when $\Lambda_1(m-1)\le x_1\le\Lambda_2(m-1)$ and intersects $\mathcal{L}_m^s$ when $\Lambda_2(m-1)\le x_1\le\Lambda_1(m)$. Let (x_0,y_0) be the corresponding intersection point. Finally, Γ enters Ω_2 and intersects $\mathcal{L}_{-1}$ at a point denoted by $(x_2,0)$. All of those intersections are transversal when $\varepsilon>0$ is sufficiently small. From $H_{1\varepsilon}(x_1,0)=H_{1\varepsilon}(x_0,y_0)$ and $H_{2\varepsilon}(x_2,0)=H_{2\varepsilon}(x_0,y_0)$ we obtain:

$$\begin{cases}
\mu_{3\varepsilon}x_1^2+x_1=\mu_{3\varepsilon}x_0^2-\mu_{2\varepsilon}y_0^2-2\mu_{1\varepsilon}x_0y_0+x_0-6y_0,\\[2ex]
\nu_{3\varepsilon}x_2^2+x_2=\nu_{3\varepsilon}x_0^2-\nu_{2\varepsilon}y_0^2-2\nu_{1\varepsilon}x_0y_0+x_0.
\end{cases}\tag{3.2}$$

Clearly Γ is a closed orbit if and only if $x_1=x_2$. Our discussions are divided into three cases as follows.

Case 1: $0\le x_1\le\Lambda_1(0)$**.**

In this case, Γ intersects $\mathcal{L}_0$ at (x_0,y_0) with $x_0=0$ and $y_0\in[0,2]$. From (3.2) we have:

$$\left[\mu_{3\varepsilon}(x_1+x_2)+1\right](x_1-x_2)=(3+\alpha_{2\varepsilon})\,y_0^2-6y_0.$$

For a crossing limit cycle, we require that $x_1>0$ and $y_0>0$. If $a_2=b_2$, then $x_1=x_2$ if and only if $y_0=2$. Moreover, if $0<y_0<2$, then $x_1<x_2$. Thus system (3.1) has a periodic orbit passing through $(0,2)$ and the orbits inside it spiral toward it when $a_2=b_2$. If $a_2>b_2$, then for any $y_0\in(0,2]$, we have $x_1<x_2$, implying that system (3.1) has no periodic orbit in this case. If $a_2<b_2$, then $x_1=x_2$ if and only if $y_0=\xi_{\varepsilon 0}$. If $0<y_0<\xi_{\varepsilon 0}$, then $x_1<x_2$. If $\xi_{\varepsilon 0}<y_0\le 2$, then $x_1>x_2$. Thus system (3.1) has a limit cycle which is stable and crosses $\mathcal{L}$ at $(0,\xi_{\varepsilon 0})$ when $a_2<b_2$.

Case 2: $\Lambda_1(m-1)\le x_1\le\Lambda_2(m-1)$**.**

In this case, Γ intersects $\mathcal{L}_m^p$ at (x_0,y_0) with $x_0=2(m-1)$ and $y_0\in\left[2,\frac{5}{2}\right]$. From (3.2) we have:

$$\left[\mu_{3\varepsilon}(x_1+x_2)+1\right](x_1-x_2)=(3+\alpha_{2\varepsilon})\,y_0^2-\left[6-4\alpha_{1\varepsilon}(m-1)\right]y_0.\tag{3.3}$$

We divide our discussions further into the following nine subcases:

(2.1) If $a_1=b_1$ and $a_2=b_2$, then from (3.3), we have $\operatorname{sgn}(x_1-x_2)=\operatorname{sgn}(y_0-2)y_0$ for sufficiently small $\varepsilon>0$. Thus $x_1=x_2$ if and only if $y_0=2$. If $y_0>2$, then

$x_1 > x_2$. Consequently, system (3.1) has infinitely many periodic orbits which are all outer stable and cross $\mathcal{L}$ at $(2(m-1),2)$ for $m=1,2,3,\cdots$.

(2.2) If $a_1=b_1$ and $a_2<b_2$, then from (3.3), we have $x_1>x_2$ when $y_0\in[2,\frac{5}{2}]$. Thus Γ is not a periodic orbit of system (3.1).

(2.3) If $a_1=b_1$ and $a_2>b_2$, then from (3.3), we have $x_1=x_2$ if and only if $y_0=\xi_{\varepsilon 0}$. If $2\le y_0<\xi_{\varepsilon 0}$, then $x_1<x_2$. If $\xi_{\varepsilon 0}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus system (3.1) has infinitely many limit cycles which are all stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{\varepsilon 0})$ for $m=1,2,3,\cdots$.

(2.4) If $a_1<b_1$ and $a_2=b_2$, then from (3.3), we have $\operatorname{sgn}(x_1-x_2)=\operatorname{sgn}\{3y_0+[4\alpha_{1\varepsilon}(m-1)-6]\}y_0$. If $m=1$, then $x_1=x_2$ if and only if $y_0=2$. If $y_0>2$, then $x_1>x_2$. If $m>1$, then for any of $y_0\in[2,\frac{5}{2}]$, we have $x_1>x_2$. Thus Γ is a periodic orbit of (3.1) if and only if $m=1$. In this case, Γ crosses $\mathcal{L}$ at $(0,2)$ and all orbits outside Γ spiral towards it.

(2.5) If $a_1>b_1$ and $a_2=b_2$, then when $m=1$, Γ is a periodic orbit of (3.1) which crosses $\mathcal{L}$ at $(0,2)$ and all orbits outside Γ spiral towards it. When $m>1$, if $K_0>0$, then $x_1=x_2$ if and only if $y_0=\xi_{m\varepsilon,1}\in(2,\frac{5}{2})$. If $2\le y_0<\xi_{m\varepsilon,1}$, then $x_1<x_2$. If $\xi_{m\varepsilon,1}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus there are k_0 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,1})$ for $1<m\le k_0+1$. Similarly, if $K_0=0$, there are k_0-1 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,1})$ with $1<m\le k_0$. Moreover, there is a periodic orbit which is inner stable and crosses $\mathcal{L}$ at $(2k_0,\frac{5}{2})$.

(2.6) If $a_1<b_1$ and $a_2<b_2$, then from (3.3), we have $x_1-x_2>0$ for $2\le y_0\le\frac{5}{2}$. Then Γ is not a periodic orbit.

(2.7) If $a_1>b_1$ and $a_2>b_2$, then when $K_1>0$, $x_1=x_2$ if and only if $y_0=\xi_{m\varepsilon,2}\in(2,\frac{5}{2})$. If $2\le y_0<\xi_{m\varepsilon,2}$, then $x_1<x_2$. If $\xi_{m\varepsilon,2}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus there are k_1+1 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $1\le m\le k_1+1$. Similarly, if $K_1=0$, then there are k_1 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $1\le m\le k_1$. Moreover, there is a periodic orbit which is inner stable and crosses $\mathcal{L}$ at $(2k_1,\frac{5}{2})$.

(2.8) If $a_1>b_1$ and $a_2<b_2$, then when $K_1>0$ and $K_2>0$, $x_1=x_2$ if and only if $y_0=\xi_{m\varepsilon,2}\in(2,\frac{5}{2})$. If $2\le y_0<\xi_{m\varepsilon,2}$, then $x_1<x_2$. If $\xi_{m\varepsilon,2}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus there are k_1-k_2 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $k_2+2\le m\le k_1+1$. Similarly, if $K_1=0$ and $K_2>0$, then there are k_1-k_2-1 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $k_2+2\le m\le k_1$. Moreover, there is a periodic orbit which is inner stable and crosses $\mathcal{L}$ at $(2k_1,\frac{5}{2})$. If $K_1>0$ and $K_2=0$, then there are k_1-k_2 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $k_2+2\le m\le k_1+1$. Moreover, there is a periodic orbit which is outer stable and crosses $\mathcal{L}$ at $(2k_2,2)$. If $K_1=K_2=0$, then there are k_1-k_2-1 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $k_2+2\le m\le k_1$. Moreover, there is a periodic orbit which is outer stable and crosses $\mathcal{L}$ at $(2k_2,2)$ and a periodic orbit which is inner stable and crosses $\mathcal{L}$ at $(2k_1,\frac{5}{2})$.

(2.9) If $a_1<b_1$ and $a_2>b_2$, then when $K_2>0$, we can similarly prove that there are k_2+1 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $1\le m\le k_2+1$. If $K_2=0$, then there are k_2 limit cycles which are stable and cross $\mathcal{L}$ at $(2(m-1),\xi_{m\varepsilon,2})$ for $1\le m\le k_2$. Moreover, there is a periodic orbit which is outer stable and crosses $\mathcal{L}$ at $(2k_2,2)$.

Case 3: $\Lambda_2(m-1)\le x_1\le\Lambda_1(m)$**.**

In this case, Γ intersects $\mathcal{L}_m^s$ at (x_0,y_0) with $x_0=-4y_0+8+2m$ and $y_0\in\left[2,\frac{5}{2}\right]$.

From (3.2) we have:

$$[\mu_{3\varepsilon}(x_1+x_2)+1]\,(x_1-x_2)=(3+\alpha_{2\varepsilon})\,y_0^2-6y_0+2\alpha_{1\varepsilon}(-4y_0+8+2m)y_0. \tag{3.4}$$

Similar to the discussions for Case 2 given above, we divide our discussions further into the following eight subcases:

(3.1) If $a_1=b_1$ and $a_2=b_2$, then from (3.4), we have $\operatorname{sgn}(x_1-x_2)=\operatorname{sgn}(y_0-2)y_0$ for sufficiently small $\varepsilon>0$. Thus system (3.1) has infinitely many periodic orbits which are all inner unstable and cross $\mathcal{L}$ at $(2m,2)$ for $m=1,2,3,\cdots$. If $a_1=b_1$ and $a_2<b_2$, then Γ is not a periodic orbit of system (3.1).

(3.2) If $a_1=b_1$ and $a_2>b_2$, then $x_1=x_2$ if and only if $y_0=\xi_{\varepsilon 0}$. If $2\le y_0<\xi_{\varepsilon 0}$, then $x_1<x_2$. If $\xi_{\varepsilon 0}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus system (3.1) has infinitely many limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{\varepsilon 0}+8+2m,\xi_{\varepsilon 0})$ for $m=1,2,3,\cdots$.

(3.3) If $a_1<b_1$ and $a_2=b_2$, then $x_1>x_2$ for any $y_0\in[2,\frac{5}{2}]$. Thus Γ is not a periodic orbit of system (3.1).

(3.4) If $a_1>b_1$ and $a_2=b_2$, then when $K_0>0$, $x_1=x_2$ if and only if $y_0=\xi_{m\varepsilon,3}\in(2,\frac{5}{2})$. If $2\le y_0<\xi_{m\varepsilon,3}$, then $x_1<x_2$. If $\xi_{m\varepsilon,3}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus there are k_0+1 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,3}+8+2m,\xi_{m\varepsilon,3})$ for $1\le m\le k_0+1$. Similarly, if $K_0=0$, then there are k_0 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,3}+8+2m,\xi_{m\varepsilon,3})$ for $1\le m\le k_0$. Moreover, there is a periodic orbit which is outer unstable and crosses $\mathcal{L}$ at $(2k_0,\frac{5}{2})$.

(3.5) If $a_1<b_1$ and $a_2<b_2$, then $x_1-x_2>0$ for any $y_0\in[2,\frac{5}{2}]$. Thus Γ is not a periodic orbit of system (3.1).

(3.6) If $a_1>b_1$ and $a_2>b_2$, then when $K_1>0$, $x_1=x_2$ if and only if $y_0=\xi_{m\varepsilon,4}\in(2,\frac{5}{2})$. If $2\le y_0<\xi_{m\varepsilon,4}$, then $x_1<x_2$. If $\xi_{m\varepsilon,4}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus there are k_1+1 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $1\le m\le k_1+1$. Similarly, if $K_1=0$, then there are k_1 limit cycles which are all unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $1\le m\le k_1$. Moreover, there is a periodic orbit which is outer unstable and crosses $\mathcal{L}$ at $(2k_1,\frac{5}{2})$.

(3.7) If $a_1>b_1$ and $a_2<b_2$, then when $K_1>0$ and $K_2>0$, $x_1=x_2$ if and only if $y_0=\xi_{m\varepsilon,4}$. If $2\le y_0<\xi_{m\varepsilon,4}$, then $x_1<x_2$. If $\xi_{m\varepsilon,4}<y_0\le\frac{5}{2}$, then $x_1>x_2$. Thus there are k_1-k_2+1 limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $k_2+1\le m\le k_1+1$. Similarly, if $K_1=0$ and $K_2>0$, then there are k_1-k_2 limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $k_2+1\le m\le k_1$. Moreover, there is a periodic orbit which is outer unstable and crosses $\mathcal{L}$ at $(2k_1,\frac{5}{2})$. If $K_1>0$ and $K_2=0$, then there are k_1-k_2+1 limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $k_2+1\le m\le k_1+1$. Moreover, there is a periodic orbit which is inner unstable and crosses $\mathcal{L}$ at $(2k_2,2)$. If $K_1=K_2=0$, then there are k_1-k_2 limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $k_2+1\le m\le k_1$. Moreover, there is a periodic orbit which is inner unstable and crosses $\mathcal{L}$ at $(2k_2,2)$ and a periodic orbit which is outer unstable and crosses $\mathcal{L}$ at $(2k_1,\frac{5}{2})$.

(3.8) If $a_1<b_1$ and $a_2>b_2$, then when $K_2>0$, we can similarly prove that there are k_2 limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $1\le m\le k_2$. If $K_2=0$, then there are k_2-1 limit cycles which are unstable and cross $\mathcal{L}$ at $(-4\xi_{m\varepsilon,4}+8+2m,\xi_{m\varepsilon,4})$ for $1\le m\le k_2-1$. Moreover, there is a periodic orbit which is inner unstable and crosses $\mathcal{L}$ at $(2k_2,2)$.

From the above analysis, it is clear that the results for $a_2 = b_2$ obtained in Case 1 along with those in subcases (2.1) and (3.1) imply that the statement (1) of Theorem 3.1 is true. The results for $a_2 < b_2$ obtained in Case 1 along with those in subcases (2.2), (2.6) and (3.1), (3.5) imply that the first part of statement (2) of Theorem 3.1 is true; the second part of statement (2) of Theorem 3.1 can be derived from the results for $a_2 = b_2$ in Case 1 along with those in subcases (2.4) and (3.3). The results given in the statement (3) of Theorem 3.1 can be derived from the results for $a_2 > b_2$ of Case 1 along with those in subcases (2.3) and (3.2). The results given in the statement (4) of Theorem 3.1 can be derived from the results for $a_2 = b_2$ of Case 1 along with those in subcases (2.5) and (3.4). The results given in the statement (5) of Theorem 3.1 can be derived from the results for $a_2 > b_2$ of Case 1 along with those in subcases (2.7) and (3.6). The results given in the statement (6) of Theorem 3.1 can be derived from the results for $a_2 < b_2$ of Case 1 along with those in subcases (2.8) and (3.7). The results given in the statement (7) of Theorem 3.1 can be derived from the results for $a_2 > b_2$ of Case 1 along with those in subcases (2.9) and (3.8).

The proof of Theorem 3.1 is complete. $\square$

4. Concluding remarks

In this paper, we construct a concrete example of planar PWL systems defined in two zones separated by a PWL switching curve, namely system (2.1). The PWL nature of the system as well as the switching curve enables us to obtain analytical expressions of solutions for each of the subsystems of (2.1). By "gluing" those solutions along the switching curve, we prove that system (2.1) has infinitely many crossing limit cycles. Using the same method, we prove that under PWL perturbations, for sufficiently small $\varepsilon > 0$ the perturbed system (3.1) can have infinitely many limit cycles, or exactly ℓ limit cycles for any given nonnegative integer ℓ.

We would like to point out that the method used and the results obtained in this paper can be modified to construct more general PWS systems that have infinitely many limit cycles. First, the "saw-tooth" shape part, i.e. $\mathcal{S}_{\mathcal{W}} := \cup_{m=1}^{+\infty} (\mathcal{L}_m^p \cup \mathcal{L}_m^s)$, of the switching curve $\mathcal{L}$ can be replaced by a properly chosen smooth or piecewise smooth curve with countable minimum points at $(2k, 2)$ for $k = 0, 1, \cdots$. For example, one can choose the function $2 + \mu|\sin(\pi x)|$ $(x \geq 0)$ with $0 < \mu < \frac{1}{2}$ to replace $\mathcal{S}_{\mathcal{W}}$ and Theorem 2.1 is still true. But in this case, more computational difficulties arise when considering the stabilities of the limit cycles because it is hard to compute the contact points of the orbits between Γ_k and Γ_{k+1}. Second, the two subsystems given in (2.1) can also be replaced by nonlinear differential equations, such as nonlinear Hamiltonian systems. Again, more complicated computations will be involved.

In real applications, discontinuities may occur on multiple lines or nonlinear curves and surfaces, even on piecewise smooth curves with countably infinitely many corners such as the one presented in systems (2.1) and (3.1). Thus it is very important to investigate the bifurcation phenomena of such a kind of piecewise smooth systems. Due to computational difficulties, we did not investigate the dynamical behaviors of system (2.1) under nonlinear autonomous or periodic perturbations. In those cases, the closed forms of the solutions of the subsystems cannot be obtained. Thus perturbation methods have to be used and much richer discontinuity-induced bifurcation phenomena, such as grazing, sliding and corner bifurcations, may occur. In our future work, we plan to focus on those problems, which are more challenging.

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