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Discontinuous Fractional Sturm-Liouville Problems with Hilfer Derivatives*

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Abstract In this paper, we study discontinuous Sturm-Liouville problem with fractional Hilfer derivatives. By defining an operator A in the Hilbert space $L_2[-1,1]$, this research shows that the eigenvalues and corresponding eigenfunctions of the main problem coincide with the eigenvalues and corresponding eigenfunctions of the constructed operator. Moreover, the characteristic function is also constructed such that the eigenvalues of the problem are coincide with the zeros of this function.

Keywords Sturm-Liouville problem, Hilfer derivatives, eigenparameter, fractional boundary conditions, fractional transmission conditions

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1. Introduction

Named after mathematicians Jacques Charles Francois Sturm and Joseph Liouville, Sturm-Liouville(S-L) problem is a mathematical concept that deals with the eigenvalue problem of a differential equation. In related fields such as quantum mechanics, heat transfer, and vibration analysis, it has attracted much attention and plays an important role in mathematical physics. Although first proposed more than 170 years ago, Sturm-Liouville theory has produced numerous research papers and monographs, it remains one of the most thriving areas of research [1–3], and many references with respect to physics and mechanics problems are contained in [4–6].

In general, as a class of boundary value problems, the Sturm-Liouville problem involves finding the eigenfunctions and eigenvalues of a second-order linear differential equation of the form:

$$-\frac{\mathrm{d}}{\mathrm{d}x}[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}]+q(x)y=\lambda w(x)y,$$

where $p(x)$, $q(x)$, and $w(x)$ are specified functions defined on an interval $[a,b]$. λ is the spectral parameter with certain boundary value conditions, while y is

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the eigenfunction. The differential equation mentioned above, combined with the boundary conditions in forms

$$c_1y(a) + c_2y'(a) = 0, (c_1^2 + c_2^2 > 0),$$
$$d_1y(b) + d_2y'(b) = 0, (d_1^2 + d_2^2 > 0),$$

is referred to as regular Sturm-Liouville problems (SLPs) if $p(x)$,$w(x) > 0$ and $p'(x)$, $q(x)$ and $w(x)$ are continuous functions over the finite interval $[a, b]$. To solve this problem, it is necessary to apply several techniques such as separation of variables, Fourier series, and Green's functions in order to obtain the eigenvalues and eigenfunctions that meet the boundary conditions.

While a significant amount of research has been conducted on the general theory and methodologies for boundary value problems with continuous coefficients, little is currently known about similar problems that involve discontinuities. Several studies, such as those detailed in [7–10], have explored the discontinuous boundary value problems with transmission conditions and its potential application to boundary value problems in parabolic equations. Additionally, some research has also examined transmission condition problems in mechanics. A particular focus has been placed on studying the discontinuous Sturm-Liouville problem, especially when the eigenparameter is presented in both the differential equation and the boundary and transmission conditions, as discussed in [11–14]. One common method for solving discontinuous Sturm-Liouville problems is to divide the domain into multiple sub-domains, each with its own set of continuous coefficients. The solutions for each sub-domain can then be matched at the points of discontinuity using boundary conditions. Another approach is to use a weighted inner product to define the space of functions over which the problem is defined. This allows for a wider range of functions to be used in the solution, including those that are not continuous. This problem can provide valuable insights into a wide range of physical phenomena.

Fractional Sturm-Liouville problem is a type of differential equation problem that involves fractional derivatives in the Sturm-Liouville operator. The history of fractional calculus can be found in [15, 16]. One important application of fractional calculus is in modeling problems involving anomalous diffusion, viscoelasticity, and other phenomena that exhibit fractal behavior. Researches [17–21] have demonstrated that fractional derivative models typically provide more accurate solutions for real processes of anomalous systems compared to models based on integer-order derivatives. The traditional methods used to solve ordinary Sturm-Liouville problems may not be applicable to fractional Sturm-Liouville problems. Specialized techniques, such as the fractional calculus and spectral methods, are often employed to solve these problems.

There exist various definitions for fractional derivatives and integrals in fractional calculus, such as Riemann-Liouville, Caputo, Grunwald-Letnikov, and others. Each of these has its own advantages and disadvantages. The selection of the appropriate definition to use relies on the specific problem being investigated.The Riemann-Liouville and Caputo derivative definitions are among the most commonly utilized tools in fractional calculus, particularly for the purpose of modeling physical systems. Additionally, a new generalized definition of fractional derivatives has been proposed by Hilfer, which has garnered significant attention in recent years. The Hilfer derivative is defined by two parameters α and β, with Riemann-Liouville and Caputo fractional derivatives being specific cases of $\beta = 0$ and $\beta = 1$ ([22–24]), respectively. Although the Hilfer derivative has only recently been introduced, it has already found applications in diverse fields ([25, 26]), such as engineering, physics, biology, and finance. Nonetheless, additional research is necessary to gain a comprehensive understanding of its characteristics and determine its effectiveness in various contexts.

This paper aims to investigate the applicability of utilizing the Hilfer fractional derivative in discontinuous Sturm-Liouville problems. Specifically, we will examine the properties of the discontinuous fractional Sturm-Liouville problem with transmission conditions for the Hilfer operator. The structure of this paper is as follows: Section 2 provides an overview of fundamental properties of fractional derivatives, including the Riemann-Liouville, Caputo, and Hilfer definitions. Additionally, some lemmas that will be used to establish more intricate results later in the paper are presented. In Section 3, we focus on the construction of a novel type of fractional Sturm-Liouville problem involving discontinuous functions with Hilfer derivatives. Section 4 analyzes the operator's characteristics, such as its symmetry, the orthogonality of its eigenfunctions, the reality of its eigenvalues, and the properties of its eigenfunctions.

2. Some auxiliary definitions and results

This section will review some fundamental concepts and principles of fractional calculus that are essential for the paper's progression (see also [15,16]). Additionally, several lemmas will be presented and proven as necessary.

Definition 2.1 (c.f. [15])**.** (Left and right Riemann-Liouville (R-L) fractional integrals)

Let $[a, b] \subset \mathbb{R}, Re(\alpha) > 0$ and $f \in L^1[a, b]$. Then the left and right Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order $\alpha \in \mathbb{C}$ are given by

$$
\begin{aligned}
I^{\alpha}_{a^+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\mathrm{d}t}{(x-t)^{1-\alpha}}, \qquad x \in (a, b],\\
I^{\alpha}_{b^-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\mathrm{d}t}{(t-x)^{1-\alpha}}, \qquad x \in [a, b),
\end{aligned}
$$

respectively.

Definition 2.2 (c.f. [16])**.** (Left and right Riemann-Liouville (R-L) fractional derivatives) Let $[a, b] \subset \mathbb{R}, Re(\alpha) \in (0, 1)$ and $f \in L^1[a, b]$. Then the left and right Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ of function f are defined as

$$
\begin{aligned}
D^{\alpha}_{a^+} f(x) &:= D I^{1-\alpha}_{a^+} f(x), \qquad x \in (a, b],\\
D^{\alpha}_{b^-} f(x) &:= -D I^{1-\alpha}_{b^-} f(x), \qquad x \in [a, b),
\end{aligned}
$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator.

Definition 2.3 (c.f. [16])**.** (Left and right Caputo fractional derivatives) Let $[a, b] \subset \mathbb{R}, Re(\alpha) \in (0, 1)$ and $f \in L^1[a, b]$. Then the left and right Caputo fractional derivatives of order $\alpha \in \mathbb{C}$ are

$$
\begin{aligned}
{}^c D^{\alpha}_{a^+} f(x) &:= I^{1-\alpha}_{a^+} D f(x), \qquad x \in (a, b],\\
{}^c D^{\alpha}_{b^-} f(x) &:= -I^{1-\alpha}_{b^-} D f(x), \qquad x \in [a, b),
\end{aligned}
$$

respectively, where $D = \frac{d}{dx}$ is the usual differential operator.

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Definition 2.4 (c.f. [27]). (Left and right Hilfer fractional derivatives) Let $[a,b] \subset \mathbb{R}, \alpha \in (0,1), \beta \in [0,1]$ and $f \in L^1[a,b]$. Then the left and right Hilfer fractional derivatives of order are defined as

$$D_{a^+}^{\alpha,\beta} f(x) := I_{a^+}^{\beta(1-\alpha)} D(I_{a+}^{(1-\beta)(1-\alpha)} f(x),$$
$$D_{b^-}^{\alpha,\beta} f(x) := I_{b^-}^{(1-\alpha)(1-\beta)} D I_{b-}^{\beta(1-\alpha)} f(x),$$

respectively, where $D = \frac{\mathrm{d}}{\mathrm{d}x}$ is the usual differential operator.

Definition 2.5. (New integral-type fractional operator) Let $[a,b]\subset\mathbb{R}, \alpha\in(0,1), \beta\in(0,1)$ and $f \in L^1[a,b]$.

To unify the fractional derivative form with Hilfer's, a combination of different versions of fractional integral and derivatives is defined as a new integral-type fractional operator

$$l_{a^+}^{\alpha,\beta} f(x) := D_{a^+}^{(1-\beta)(1-\alpha)} I_{a^+}^1 D_{a+}^{\beta(1-\alpha)} f(x),$$
$$l_{b^-}^{\alpha,\beta} f(x) := D_{b^-}^{\beta(1-\alpha)} I_{b^-}^1 D_{b-}^{(1-\beta)(1-\alpha)} f(x).$$

Using property 2.2 in [16], the direct calculation gives

$$\begin{aligned} l_{a^+}^{\alpha,\beta} f(x) &= D_{a^+}^{(1-\beta)(1-\alpha)} I_{a^+}^1 D_{a+}^{\beta(1-\alpha)} f(x) \\ &= I_{a^+}^{1-(1-\alpha)(1-\beta)} D_{a+}^{\beta(1-\alpha)} f(x) \\ &= I_{a^+}^{\alpha} I_{a^+}^{\beta-\alpha\beta} D_{a+}^{\beta-\alpha\beta} f(x) \\ &= I_{a^+}^{\alpha} f(x), \end{aligned}$$

Property 2.1 (c.f. [15]).

$$D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(x) = f(x),$$
$$D_{b^-}^{\alpha} I_{b^-}^{\alpha} f(x) = f(x),$$

and

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} I_{a^+}^{1-\alpha} f(a),$$
$$I_{b^-}^{\alpha} D_{b^-}^{\alpha} f(x) = f(x) - \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} I_{a^+}^{1-\alpha} f(b),$$

where $\alpha \in (0,1)$.

According to the above equations, we can see that the R-L derivative is the left inverse of the R-L integral, but not the right inverse.

Property 2.2 (c.f. [15]).

$${}^c D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(x) = f(x),$$
$${}^c D_{b^-}^{\alpha} I_{b^-}^{\alpha} f(x) = f(x),$$

and

$$I_{a^+}^{\alpha}\ {}^c D_{a^+}^{\alpha} f(x) = f(x) - f(a),$$
$$I_{b^-}^{\alpha}\ {}^c D_{b^-}^{\alpha} f(x) = f(x) - f(b),$$

where $\alpha \in (0,1)$.

Now, we state and prove the following lemmas which is used in the next section.

Lemma 2.1 (c.f. [16]). *Let $f \in L_2(a,b)$ and $\alpha \in (0,1)$. Then*

$$I_{a^+}^{\alpha}\, {}^cD_{b^-}^{\alpha} = M_g(x) - (f(x) - f(b)),$$
$$I_{a^+}^{\alpha}\, {}^cD_{b^-}^{\alpha} = (f(x) - f(a)) - I_{a^+}^{\alpha} N_f^1(x),$$

where

$$M_g(x) = \frac{1}{\Gamma(\alpha)} \int_a^b |x-t|^{\alpha-1} g(t)\,dt,$$
$$N_f^1(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^b |x-t|^{-\alpha} f'(t)\,dt,$$

and

$$g(x) = {}^cD_{b^-}^{\alpha} f(x).$$

Proof. In view of Definition 2.1, we have

$$\begin{aligned} M_g(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x |x-t|^{\alpha-1} g(t)\mathrm{d}t + \frac{1}{\Gamma(\alpha)} \int_x^b |x-t|^{\alpha-1} g(t)\mathrm{d}t \\ &= I_{a^+}^{\alpha} g(x) + I_{b^-}^{\alpha} g(x). \end{aligned}$$

Then it leads to

$$I_{a^+}^{\alpha} g(x) = M_g(x) - I_{b^-}^{\alpha} g(x).$$

To prove (2), by Definition 2.3, we obtain

$$\begin{aligned} N_f^1(x) &= \frac{1}{\Gamma(1-\alpha)} \int_a^b |x-t|^{-\alpha} f'(t)\mathrm{d}t \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^x |x-t|^{-\alpha} f'(t)\mathrm{d}t + \frac{1}{\Gamma(1-\alpha)} \int_x^b |x-t|^{-\alpha} f'(t)\mathrm{d}t \\ &= {}^cD_{a^+}^{\alpha} f(x) - \frac{1}{\Gamma(1-\alpha)} \int_x^b |t-x|^{-\alpha} (-f')(t)\mathrm{d}t \\ &= {}^cD_{a^+}^{\alpha} f(x) - {}^cD_{b^-}^{\alpha} f(x), \end{aligned}$$

which gives

$${}^cD_{b^-}^{\alpha} f(x) = {}^cD_{a^+}^{\alpha} f(x) - N_f^1(x).$$

By applying the fractional operator$I_{a^+}^{\alpha}$ to both sides, we get

$$\begin{aligned} I_{a^+}^{\alpha}\, {}^cD_{b^-}^{\alpha} f(x) &= I_{a^+}^{\alpha}\, {}^cD_{a^+}^{\alpha} f(x) - I_{a^+}^{\alpha} N_f^1(x) \\ &= (f(x) - f(a)) - I_{a^+}^{\alpha} N_f^1(x). \end{aligned}$$

□

Property 2.3 (c.f. [16]). If $[a,b] \subset \mathbb{R}, \alpha \in (0,1), \beta \in (0,1)$ $Re(\alpha) > 0$ and $Re(\beta) > 0$, for$f \in L^1[a,b]$, the following relations hold almost everywhere on $[a,b]$.

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(x) = I_{a^+}^{\alpha+\beta} f(x),$$
$$I_{b^-}^{\alpha} I_{b^-}^{\beta} f(x) = I_{b^-}^{\alpha+\beta} f(x),$$

and

$$D_{a^+}^{\beta} I_{a^+}^{\alpha} f(x) = I_{a^+}^{\alpha-\beta} f(x),$$
$$D_{b^-}^{\beta} I_{b^-}^{\beta} f(x) = I_{b^-}^{\alpha-\beta} f(x).$$

Theorem 2.1 (c.f. [28]). *Let $[a,b]\subset\mathbb{R},\alpha\in(0,1),\beta\in(0,1)$, $f(x)\in L^1[a,b]$, and $g(x)\in L^1[a,b]$, the integration by parts formula for the new operator defined by definition 2.4 is given*

$$\int_a^b f(x)D_{a^+}^{\alpha,\beta}g(x)dx=-\int_a^b g(x)D_{b^-}^{\alpha,\beta}f(x)dx+I_{b^-}^{\beta(1-\alpha)}f(x)I_{a^+}^{(1-\beta)(1-\alpha)}g(x)|_{x=a}^{x=b}.$$

Theorem 2.2. *Let $\alpha\in(0,1)$ and $\beta\in(0,1)$. If $f(x)\in L^1[a,b]$ is generalized differentiable and integrable, then*

$$\begin{aligned}
l_{a^+}^{\alpha,\beta}D_{a^+}^{\alpha,\beta}f(x)&=f(x)-\frac{(x-a)^{-(1-\beta)(1-\alpha)}}{\Gamma(1-(1-\beta)(1-\alpha))}I_{a^+}^{(1-\beta)(1-\alpha)}f(a),\\
l_{b^-}^{\alpha,\beta}D_{b^-}^{\alpha,\beta}f(x)&=-f(x)+\frac{(b-x)^{-\beta(1-\alpha)}}{\Gamma(1+\alpha\beta-\beta)}I_{b^-}^{\beta(1-\alpha)}f(b).
\end{aligned}$$

The fractional derivatives are defined in Riemann-Liouville sense.

Proof.

$$\begin{aligned}
l_{a^+}^{\alpha,\beta}D_{a^+}^{\alpha,\beta}f(x)&=(D_{a^+}^{(1-\beta)(1-\alpha)}I_{a^+}^1D_{a^+}^{\beta(1-\alpha)}I_{a^+}^{\beta(1-\alpha)}DI_{a^+}^{(1-\beta)(1-\alpha)})f(x)\\
&=(D_{a^+}^{(1-\beta)(1-\alpha)}I_{a^+}^1DI_{a^+}^{(1-\beta)(1-\alpha)})f(x)\\
&=(D_{a^+}^{(1-\beta)(1-\alpha)}(I_{a^+}^{(1-\beta)(1-\alpha)})f(x)-I_{a^+}^{(1-\beta)(1-\alpha)})f(a))\\
&=f(x)-\frac{(x-a)^{-(1-\beta)(1-\alpha)}}{\Gamma(1-(1-\beta)(1-\alpha))}I_{a^+}^{(1-\beta)(1-\alpha)}f(a).
\end{aligned}$$

$$\begin{aligned}
l_{b^-}^{\alpha,\beta}D_{b^-}^{\alpha,\beta}f(x)&=(D_{b^-}^{\beta(1-\alpha)}I_{b^-}^1D_{b^-}^{(1-\beta)(1-\alpha)}I_{b^-}^{(1-\beta)(1-\alpha)}DI_{b^-}^{\beta(1-\alpha)})f(x)\\
&=(D_{b^-}^{\beta(1-\alpha)}I_{b^-}^1DI_{b^-}^{\beta(1-\alpha)})f(x)\\
&=D_{b^-}^{\beta(1-\alpha)}(I_{b^-}^{\beta(1-\alpha)}f(b)-I_{b^-}^{\beta(1-\alpha)}f(x))\\
&=-f(x)+\frac{(b-x)^{-\beta(1-\alpha)}}{\Gamma(1+\alpha\beta-\beta)}I_{b^-}^{\beta(1-\alpha)}f(b).
\end{aligned}$$

□

Theorem 2.3. *Let $\alpha\in(0,1)$ and $\beta\in(0,1)$. If $f(x)\in L^1[a,b]$ is generalized differentiable and integrable as theorem 2.2, then*

$$I_{a^+}^{\alpha+\beta-\alpha\beta}D_{b^-}^{\alpha,\beta}f(x)=-I_{b^-}^{\beta(1-\alpha)}f(x)+I_{b^-}^{\beta(1-\alpha)}f(a)+I_{a^+}^{\alpha+\beta-\alpha\beta}N_f,$$

where

$$N_f=\frac{1}{\Gamma[(1-\alpha)(a-\beta)]}\int_a^b|x-t|^{\alpha\beta-\alpha-\beta}I_{b^-}^{\beta-\alpha\beta-1}f(t)dt.$$

Proof.

$$\begin{aligned}
I_{a^+}^{\alpha+\beta-\alpha\beta}D_{b^-}^{\alpha,\beta}f(x)&=I_{a^+}^{\alpha+\beta-\alpha\beta}(I_{b^-}^{(1-\alpha)(1-\beta)}DI_{b-}^{\beta(1-\alpha)})f(x)\\
&=I_{a^+}^{\alpha+\beta-\alpha\beta}[-I_{b^-}^{1-(\alpha+\beta-\alpha\beta)}D(-I_{b-}^{\beta(1-\alpha)})]\\
&=I_{a^+}^{\alpha+\beta-\alpha\beta c}D_{b^-}^{\alpha+\beta-\alpha\beta}(-I_{b-}^{\beta(1-\alpha)})f(x)\\
&=-I_{b-}^{\beta(1-\alpha)}f(x)+I_{b-}^{\beta(1-\alpha)}f(a)+I_{a^+}^{\alpha+\beta-\alpha\beta}N_f,
\end{aligned}$$

where

$$\begin{aligned}
N_f &= \frac{1}{\Gamma[(1-\alpha)(1-\beta)]}\int_a^b |x-t|^{\alpha\beta-\alpha-\beta}(I_{b^-}^{\beta(1-\alpha)}f(t))'\mathrm{d}t\\
&= \frac{1}{\Gamma[(1-\alpha)(1-\beta)]}\int_a^x |x-t|^{\alpha\beta-\alpha-\beta}(I_{b^-}^{\beta(1-\alpha)}f(t))'\mathrm{d}t\\
&\quad + \frac{1}{\Gamma[(1-\alpha)(1-\beta)]}\int_x^b |x-t|^{\alpha\beta-\alpha-\beta}(I_{b^-}^{\beta(1-\alpha)}f(t))'\mathrm{d}t\\
&= {}^cD_{a^+}^{\alpha+\beta-\alpha\beta}(I_{b^-}^{\beta(1-\alpha)}f(x)) - {}^cD_{b^-}^{\alpha+\beta-\alpha\beta}(I_{b^-}^{\beta(1-\alpha)}f(x)).
\end{aligned}$$

□

Theorem 2.4. *Let* $\alpha\in(0,1)$, $\beta\in[0,1]$ *and* $f\in L^1[a,b]$. *Then*

$$\|D_{a^+}^{\alpha,\beta}f(x)\| \le \frac{1}{(1+\beta-\alpha\beta)\Gamma(\beta-\alpha\beta)\Gamma[1+(1-\beta)(1-\alpha)]}\|f(x)\|.$$

Proof. Using Lemma 2.1 and Corollary 2.3 in [16] we obtain

$$\begin{aligned}
\|D_{a^+}^{\alpha,\beta}f(x)\| &= \|I_{a^+}^{\beta(1-\alpha)}D(I_{a+}^{(1-\beta)(1-\alpha)}f(x)\|\\
&= \|I_{a^+}^{1-[1-\beta(1-\alpha)]}D(I_{a+}^{(1-\beta)(1-\alpha)}f(x)\|\\
&= \|{}^cD_{a^+}^{1-\beta+\alpha\beta}(I_{a+}^{(1-\beta)(1-\alpha)}f(x)\|\\
&\le \frac{1}{(1+\beta-\alpha\beta)\Gamma(\beta-\alpha\beta)\Gamma[1+(1-\beta)(1-\alpha)]}\|f(x)\|.
\end{aligned}$$

□

3. Discontinuous Hilfer fractional Sturm-Liouville problems with transmission conditions

In this section, we consider the following Hilfer fractional S-L differential expression $\pounds_\alpha$ defined as

$$\pounds_\alpha y := D_{1^-}^{\alpha,\beta}D_{-1^+}^{\alpha,\beta}y + q(x)y,\ x\in[-1,0)\cup(0,1].$$

We shall consider the following fractional S-L problem on I, where $I=[-1,0)\cup(0,1]$,

$$\pounds_\alpha u + \lambda u = 0, \tag{3.1}$$

with boundary conditions

$$L_1(u) := I_{-1^+}^{(1-\beta)(1-\alpha)}u(-1) = 0, \tag{3.2}$$

$$L_2(u) := I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(1) = 0, \tag{3.3}$$

and transmission conditions

$$L_3(u) := I_{-1^+}^{(1-\beta)(1-\alpha)}u(-0) - kI_{-1^+}^{(1-\beta)(1-\alpha)}u(+0) = 0, \tag{3.4}$$

$$L_4(u) := I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(-0) - \frac{1}{k}I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(+0) = 0, \tag{3.5}$$

where $0<\alpha<1$ in (3.1)-(3.5), $\lambda\in\mathbb{C}$ and λ is the eigenparameter in (3.1). $q(x)$ is real-valued and continuous in both $[-1,0)$ and $(0,1]$, which also has finite limits $q(\pm 0) := \lim\limits_{x\to\pm 0} q(x)$, and $k\neq 0$ is a real number.

4. The operator formulation of the problem

We define the following inner product in the Hilbert space $L^2[-1,1]$ by

$$\langle f,g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}\mathrm{d}x, \tag{4.1}$$

where $F := f(x)$, $G := g(x) \in L^2[-1,1]$. In this Hilbert space, we define the operator A with domain

$$D(A) := \left\{ \begin{array}{c} f = f(x) \text{ and } D_{-1^+}^{\alpha,\beta} f(x) \\ \text{are absolutely continuous on } [-1,0)\cup(0,1], \\ \text{and } f(\pm 0), I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(\pm 0), I_{-1^+}^{(1-\beta)(1-\alpha)} u(\pm 0) \text{ have finite limits,} \\ L_i f = 0, i = 1,2,3,4 \end{array} \right\} \tag{4.2}$$

and the action law

$$Af = D_{1^-}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta} f + q(x) f. \tag{4.3}$$

Thus problems (3.1)-(3.5) can be written in the operator form as

$$Au = \lambda u.$$

It is clear that the eigenvalues and corresponding eigenfunctions of problems (3.1)-(3.5) correspond to the eigenvalues and eigenelements of the operator A.

Theorem 4.1. *The linear operator A is symmetric.*

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Proof. For each $f, g \in D(A)$, using (4.1), we write

$$\begin{aligned}
\langle Af, g\rangle &= \int_{-1}^{1} Af(x)\overline{g(x)}\mathrm{d}x \\
&= \int_{-1}^{1} (D_{1^-}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta} f(x))\overline{g(x)}\mathrm{d}x + \int_{-1}^{1} q(x)f(x)\overline{g(x)}\mathrm{d}x \\
&= \int_{-1}^{1} (D_{-1^+}^{\alpha,\beta} f(x))(D_{-1^+}^{\alpha,\beta}\overline{g(x)})\mathrm{d}x \\
&\quad + I_{-1^+}^{(1-\alpha)(1-\beta)}\overline{g(x)} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} f(x)|_{-1}^{1} + \int_{-1}^{1} q(x)f(x)\overline{g(x)}\mathrm{d}x \\
&= \int_{-1}^{1} f(x)(D_{1^-}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta}\overline{g(x)})\mathrm{d}x \\
&\quad - I_{-1^+}^{(1-\alpha)(1-\beta)} f(x) I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta}\overline{g(x)}|_{-1}^{1} \\
&\quad + I_{-1^+}^{(1-\alpha)(1-\beta)}\overline{g(x)} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} f(x)|_{-1}^{1} + \int_{-1}^{1} q(x)f(x)\overline{g(x)}\mathrm{d}x \\
&= \langle f, Ag\rangle \\
&\quad - I_{-1^+}^{(1-\alpha)(1-\beta)} f(x) I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta}\overline{g(x)}|_{0^+}^{1} \\
&\quad - I_{-1^+}^{(1-\alpha)(1-\beta)} f(x) I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta}\overline{g(x)}|_{-1}^{0^-} \\
&\quad + I_{-1^+}^{(1-\alpha)(1-\beta)}\overline{g(x)} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} f(x)|_{0^+}^{1} \\
&\quad + I_{-1^+}^{(1-\alpha)(1-\beta)}\overline{g(x)} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} f(x)|_{-1}^{0^-}.
\end{aligned} \tag{4.4}$$

By considering the Hilfer fractional boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5), we have

$$\langle Af, g\rangle = \langle f, Ag\rangle,$$

which proves that the operator A is symmetric. □

Corollary 4.1. *All eigenvalues of problems (3.1)-(3.5) are real.*

Naturally, we can now assume that all eigenfunctions of problems (3.1)-(3.5) are real-valued.

Corollary 4.2. *The eigenfunctions corresponding to different eigenvalues of Hilfer fractional Sturm-Liouville problems (3.1)-(3.5) are orthogonal.*

Proof. Let λ_1 and λ_2 be two different eigenvalues corresponding to eigenfunctions $y(x,\lambda_1)$ and $y(x,\lambda_2)$, respectively, for problems (3.1) to (3.5).

$$\begin{aligned}
\pounds_\alpha y(x,\lambda_1) + \lambda_1 y(x,\lambda_1) = 0, \\
\pounds_\alpha y(x,\lambda_2) + \lambda_2 y(x,\lambda_2) = 0.
\end{aligned}$$

Multiplying the last two equations to $\overline{y(x,\lambda_2)}$ and $\overline{y(x,\lambda_1)}$, respectively, subtracting from each other and integrating from -1 to 1, because of the symmetry of the operator $\pounds_\alpha$, we have

$$(\lambda_1 - \lambda_2)\langle y(x,\lambda_1), y(x,\lambda_2)\rangle = 0.$$

Since $\lambda_1 \neq \lambda_2$, and the proof completes. □

Lemma 4.1. *The equivalent integral form of equation*

$$\pounds_\alpha u(x) + \lambda u(x) = 0, \qquad x \in [-1, 0), \tag{4.5}$$

with Hilfer fractional conditions

$$\begin{aligned} I_{-1^+}^{(1-\beta)(1-\alpha)} u(-1) = 0, \\ I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(-1) = 1, \end{aligned} \tag{4.6}$$

is given as

$$u(x) = l_{-1^+}^{\alpha,\beta} y_{\alpha,\beta}(x) + l_{-1^+}^{\alpha,\beta} l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)), \tag{4.7}$$

where

$$y_{\alpha,\beta}(x) = \frac{(1-x)^{-\beta(1-\alpha)}}{\Gamma(1+\alpha\beta-\beta)} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(1). \tag{4.8}$$

Proof. Let us consider (3.1),

$$D_{1^-}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta} u(x) + (\lambda + q(x))u(x) = 0.$$

Using the fractional integral operators $l_{1^-}^{\alpha,\beta}$ acting on this equation and by Theorem 2.2, we obtain

$$l_{1^-}^{\alpha,\beta}(D_{1^-}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta} u(x)) + l_{1^-}^{\alpha,\beta}(\lambda + q(x))u(x) = 0, \tag{4.9}$$

and

$$\begin{aligned} D_{-1^+}^{\alpha,\beta} u(x) &= \frac{(1-x)^{-\beta(1-\alpha)}}{\Gamma(1+\alpha\beta-\beta)} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(1) + l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)), \\ D_{-1^+}^{\alpha,\beta} u(x) &= y_{\alpha,\beta}(x) + l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)). \end{aligned} \tag{4.10}$$

Applying $l_{-1^+}^{\alpha,\beta}$ on both sides of (4.8) and using condition (4.6), we find

$$\begin{aligned} l_{-1^+}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta} u(x) &= l_{-1^+}^{\alpha,\beta} y_{\alpha,\beta}(x) + l_{-1^+}^{\alpha,\beta} l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)), \\ u(x) &= \frac{(x+1)^{-(1-\beta)(1-\alpha)}}{\Gamma(1-(1-\alpha)(1-\beta)} I_{-1^+}^{(1-\alpha)(1-\beta)} u(-1) \\ &\quad + l_{-1^+}^{\alpha,\beta} y_{\alpha,\beta}(x) + l_{-1^+}^{\alpha,\beta} l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)), \\ u(x) &= l_{-1^+}^{\alpha,\beta} y_{\alpha,\beta}(x) + l_{-1^+}^{\alpha,\beta} l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)). \end{aligned} \tag{4.11}$$

We reach

$$u(x) = l_{-1^+}^{\alpha,\beta} y_{\alpha,\beta}(x) + l_{-1^+}^{\alpha,\beta} l_{1^-}^{\alpha,\beta}((\lambda + q(x))u(x)). \tag{4.12}$$

□

Theorem 4.2. *Let $Q := \max_{x\in[-1,1]} |q(x)|$, then the following initial value problem*

$$\begin{aligned} D_{1^-}^{\alpha,\beta} D_{-1^+}^{\alpha,\beta} u(x) + (\lambda + q(x))u(x) = 0, \qquad x \in [-1, 0), \\ I_{-1^+}^{(1-\beta)(1-\alpha)} u(-1) = 0, \\ I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(-1) = 1, \end{aligned} \tag{4.13}$$

has a unique solution $\phi_1(x, \lambda)$ on $[-1, 0)$ provided that

$$\frac{|\lambda| + Q}{\Gamma^2(1+\alpha)} < 1. \tag{4.14}$$

Proof. If we use a similar way in Lemma 4.1, we get a corresponding integral equation of the problem as follows:

$$u(x) = l^{\alpha,\beta}_{-1^+} y_{\alpha,\beta}(x) + l^{\alpha,\beta}_{-1^+} l^{\alpha,\beta}_{1^-}((\lambda + q(x))u(x)), \tag{4.15}$$

where

$$\begin{aligned} y_{\alpha,\beta}(x) &= \frac{(1-x)^{-\beta(1-\alpha)}}{\Gamma(1+\alpha\beta-\beta)} I^{\beta(1-\alpha)}_{1^-} D^{\alpha,\beta}_{-1^+} u(1) \\ &= \frac{(1-x)^{-\beta(1-\alpha)}}{\Gamma(1+\alpha\beta-\beta)} I^{\beta(1-\alpha)}_{1^-}(1). \end{aligned} \tag{4.16}$$

Let us construct the integral equation by

$$\phi = T\phi, \tag{4.17}$$

where the mapping T is defined as

$$Tf = l^{\alpha,\beta}_{-1^+} y_{\alpha,\beta} + l^{\alpha,\beta}_{-1^+} l^{\alpha,\beta}_{1^-}((\lambda + q(x))f), \tag{4.18}$$

then, we have

$$\|Tf - Tg\| = \|l^{\alpha,\beta}_{-1^+} l^{\alpha,\beta}_{1^-}((\lambda + q(x))(f-g)\|. \tag{4.19}$$

By applying Lemma 2.1 in [16], we get

$$\begin{aligned} \|Tf - Tg\| &\leq \frac{1}{\Gamma(1+\alpha)\Gamma(1+\alpha)} \|(\lambda + q(x))(f-g)\|, \\ &\leq \frac{|\lambda| + Q}{\Gamma^2(1+\alpha)} \|f - g\|. \end{aligned} \tag{4.20}$$

By condition (4.13), the mapping T is a contraction on the space $\langle C[-1,0), \|\cdot\|\rangle$. Consequently, there exists a unique solution $\phi_1(x,\lambda)$ of equation (4.16) .

□

Lemma 4.2. *The equivalent integral form of equation*

$$\pounds_\alpha u + \lambda u = 0, \qquad x \in (0,1],$$

with transmission conditions (3.4), (3.5) is given by

$$u(x) = u_0(x) + l^{\alpha,\beta}_{0^+}({}^cD^{\beta(1-\alpha)}_{1^-} I^{\alpha+\beta-\alpha\beta}_{0^+})[N_u + (\lambda + q(x))u(x)], \tag{4.21}$$

where

$$N_u = {}^cD^{\alpha+\beta-\alpha\beta}_{0^+}(I^{\beta(1-\alpha)}_{1^-} D^{\alpha,\beta}_{-1^+} u(x)) - {}^cD^{\alpha+\beta-\alpha\beta}_{1^-}(I^{\beta(1-\alpha)}_{1^-} D^{\alpha,\beta}_{-1^+} u(x)).$$

Proof.

Considering the equation

$$D^{\alpha,\beta}_{1^-} D^{\alpha,\beta}_{-1^+} u(x) + (\lambda + q(x))u(x) = 0,$$

applying the integral operator $I^{\alpha+\beta-\alpha\beta}_{0^+}$ on this equation and by Theorem 2.3, we obtain

$$\begin{aligned} I^{\alpha+\beta-\alpha\beta}_{0^+} D^{\alpha,\beta}_{1^-} D^{\alpha,\beta}_{-1^+} u(x) &= -I^{\alpha+\beta-\alpha\beta}_{0^+}(\lambda + q(x))u(x), \\ I^{\beta(1-\alpha)}_{1^-} D^{\alpha,\beta}_{-1^+} u(x) &= I^{\beta(1-\alpha)}_{1^-} D^{\alpha,\beta}_{-1^+} u(+0) + I^{\alpha+\beta-\alpha\beta}_{0^+} N_u \\ &\quad + I^{\alpha+\beta-\alpha\beta}_{0^+}(\lambda + q(x))u(x), \end{aligned} \tag{4.22}$$

where

$$N_u = {}^cD_{0^+}^{\alpha+\beta-\alpha\beta}(I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(x)) - {}^cD_{1^-}^{\alpha+\beta-\alpha\beta}(I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(x)). \tag{4.23}$$

Applying ${}^cD_{1^-}^{\beta(1-\alpha)}$ on both sides of (4.21) and using conditions (3.5)-(3.6), we find

$$\begin{aligned} {}^cD_{1^-}^{\beta(1-\alpha)}I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(x) &= {}^cD_{1^-}^{\beta(1-\alpha)}I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}u(+0) \\ &\quad + {}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u(x)], \\ D_{-1^+}^{\alpha,\beta}u(x) &= kD_{-1^+}^{\alpha,\beta}\phi(-0) + {}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u(x)]. \end{aligned} \tag{4.24}$$

Since $u(x)$ vanishes on $[-1,0)$, we rewrite

$$D_{0^+}^{\alpha,\beta}u(x) = kD_{-1^+}^{\alpha,\beta}\phi(-0) + {}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u(x)]. \tag{4.25}$$

Using $l_{0^+}^{\alpha,\beta}$ acting on the both sides, based on Theorem 2.2 we find

$$\begin{aligned} u(x) &= \frac{x^{(1-\beta)(1-\alpha)}}{\Gamma(1-(1-\beta)(1-\alpha))}I_{0^+}^{(1-\beta)(1-\alpha)}u(+0) + kD_{-1^+}^{\alpha,\beta}\phi(-0) \\ &\quad + l_{0^+}^{\alpha,\beta}{}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u(x)], \\ u(x) &= \frac{1}{k}\frac{x^{(1-\beta)(1-\alpha)}}{\Gamma(1-(1-\beta)(1-\alpha))}I_{0^+}^{(1-\beta)(1-\alpha)}\phi(-0) + kD_{-1^+}^{\alpha,\beta}\phi(-0) \\ &\quad + l_{0^+}^{\alpha,\beta}{}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta})[N_u + (\lambda+q(x))u(x)]. \end{aligned} \tag{4.26}$$

Then we reach

$$u(x) = u_0(x) + l_{0^+}^{\alpha,\beta}{}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u(x)]. \tag{4.27}$$

□

We next define $u_m(x,\lambda)$ to construct the successive approximations

$$u_m(x,\lambda) = u_0(x,\lambda) + l_{0^+}^{\alpha,\beta}{}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u_{m-1}(x)].$$

Lemma 4.3. *Let* $Q := \max_{x\in[-1,1]}|q(x)|$, $P_R := \max_{|\lambda|\leq\mathbb{R}} \ P(\lambda)$ *and* $P(\lambda) := \max_{x\in(0,1]} \ u_0(x,\lambda)$,
$k_{\alpha,\beta} := \frac{1}{(1+\beta-\alpha\beta)(2-\alpha-\beta+\alpha\beta)\Gamma(1-(1-\alpha)(1-\beta))\Gamma(1+\beta-\alpha\beta)\Gamma(\beta-\alpha\beta)\Gamma(1+(1-\alpha)(1-\beta))}$,
$j_{(\alpha,\beta)} := \frac{1}{\Gamma(\alpha+1)(2-\alpha)\Gamma(1-\alpha)\Gamma(\alpha+\beta-\alpha\beta+1)}$.
Then the following estimate

$$\|u_m(x,\lambda) - u_{m-1}(x,\lambda)\| \leq P_R\{j_{\alpha,\beta}(k_{\alpha,\beta} + |\lambda| + Q)\}^m \tag{4.28}$$

holds for all m .

Proof. Let us apply the mathematical induction for m. For $m=1$, we have

$$\|u_1(x,\lambda) - u_0(x,\lambda)\| = \|l_{0^+}^{\alpha,\beta}{}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_u + (\lambda+q(x))u_0(x)]\|.$$

By using Lemma 2.1 and Corollary 2.3 in [16], we have

$$\begin{aligned} &\|u_1(x,\lambda) - u_0(x,\lambda)\| \\ &\leq \frac{1}{\Gamma(\alpha+1)}\frac{1}{(2-\alpha)\Gamma(1-\alpha)}\frac{1}{\Gamma(\alpha+\beta-\alpha\beta+1)}\|N_u + (\lambda+q(x))u_0(x)\| \\ &\leq j_{\alpha,\beta}(\|N_u\| + \|(\lambda+q(x))u_0(x)\|) \\ &\leq j_{\alpha,\beta}(k_{\alpha,\beta} + |\lambda| + Q)P_R. \end{aligned}$$

Suppose that (4.27) holds for $m-1$, i.e.,

$$\|u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda)\| \leq P_R\left\{j_{\alpha,\beta}(k_{\alpha,\beta}+|\lambda|+Q)\right\}^{m-1}.$$

Then we have

$$\begin{aligned}
&\|u_m(x,\lambda)-u_{m-1}(x,\lambda)\| \\
&= \|l_{0^+}^{\alpha,\beta\,c}D_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta}[N_{m-1}-N_{m-2}+(\lambda+q(x))(u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda))]\| \\
&\leq j_{\alpha,\beta}[\|N_{m-1}-N_{m-2}\|+\|(\lambda+q(x))(u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda))\|] \\
&\leq j_{\alpha,\beta}(k_{\alpha,\beta}+|\lambda|+Q)\|\|u_{m-1}(x,\lambda)-u_{m-2}(x,\lambda)\| \\
&\leq P_R\left\{j_{\alpha,\beta}(k_{\alpha,\beta}+|\lambda|+Q)\right\}^{m}.
\end{aligned}$$

$\square$

Next, we take into account the initial value problem of differential equation

$$\pounds_\alpha \widetilde{u}(x)+\lambda\widetilde{u}=0, \qquad x\in(0,1], \tag{4.29}$$

$$I_{-1^+}^{(1-\beta)(1-\alpha)}\widetilde{u}(+0)=\frac{1}{k}I_{-1^+}^{(1-\beta)(1-\alpha)}u(-0), \tag{4.30}$$

$$D_{-1^+}^{\alpha,\beta}\widetilde{u}(+0)=kD_{-1^+}^{\alpha,\beta}u(-0), \tag{4.31}$$

where

$$u(x):=\begin{cases} 0 & x\in[-1,0) \\ \widetilde{u}(x) & x\in(0,1]. \end{cases} \tag{4.32}$$

By Lemma 4.2, (4.27) has an equivalent integral form as follows

$$u(x)=u_0(x)+l_{0^+}^{\alpha,\beta}({}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta})[N_u+(\lambda+q(x))u(x)]. \tag{4.33}$$

We next define $u_m(x,\lambda)$ to construct the successive approximations

$$u_m(x,\lambda)=u_0(x,\lambda)+l_{0^+}^{\alpha,\beta}({}^cD_{1^-}^{\beta(1-\alpha)}I_{0^+}^{\alpha+\beta-\alpha\beta})[N_{m-1}+(\lambda+q(x))u_{m-1}(x,\lambda)]. \tag{4.34}$$

Now let us consider the series

$$u^*(x,\lambda)=\lim_{n\to\infty}(u_n(x,\lambda)-u_0(x,\lambda))=\sum_{i=0}^{\infty}(u_i(x,\lambda)-u_{i-1}(x,\lambda)). \tag{4.35}$$

According to the estimate in lemma 4.3, for $0<x\leq 1$, the absolute value of its terms is smaller than that of the corresponding terms of the convergent numeric series

$$P_R\sum_{i=0}^{\infty}\left\{j_{\alpha,\beta}(k_{\alpha,\beta}+|\lambda|+Q)\right\}^{i}.$$

Hence, series (4.34) converges uniformly. Obviously, each term $u_i(x,\lambda)-u_{i-1}(x,\lambda)$ of series (4.34) is continuous on $x\in(0,1]$. Therefore, the sum of series (4.34) is continuous on $x\in(0,1]$ and

$$\phi_2(x,\lambda)=\lim_{n\to\infty}(u_n(x,\lambda)=u_0(x,\lambda)+u^*(x,\lambda)$$

is continuous on $x \in (0,1]$.

The uniform convergence of the sequence $u_m(x,\lambda)$ implies that we can take $m \to \infty$ in (4.34), resulting in (4.33) that establishes $\phi_2(x,\lambda)$, the limit function of the process defined by (4.33), as being a solution to (4.26). Moreover, it is evident that $\phi_2(x,\lambda)$ satisfies the initial conditions (4.29)-(4.30). Finally, the function $\phi(x,\lambda)$ given by

$$\phi(x,\lambda) := \begin{cases} \phi_1(x,\lambda) & x \in [-1,0) \\ \phi_2(x,\lambda) & x \in (0,1], \end{cases} \tag{4.36}$$

satisfies the differential equation (3.1), Hilfer fractional boundary condition (3.2) and transmission conditions (3.4) and (3.5).

By a similar approach, we can prove the next theorem.

Theorem 4.3. *For any $\lambda \in \mathbb{C}$, satisfying $P_R j_{\alpha,\beta}(k_{\alpha,\beta}+|\lambda|+Q) < 1$, the differential equation has a unique solution*

$$\chi(x,\lambda) := \begin{cases} \chi_1(x,\lambda) & x \in [-1,0) \\ \chi_2(x,\lambda) & x \in (0,1] \end{cases}, \tag{4.37}$$

satisfying fractional boundary condition (3.3) and transmission conditions (3.4)-(3.5), for each $x \in [-1,0) \cup (0,1]$ where $\chi_2(x,\lambda)$ is the unique solution of the initial value problem

$$\begin{aligned} \pounds_\alpha u(x) + \lambda u(x) = 0, &\qquad x \in (0,1], \\ I_{-1^+}^{(1-\beta)(1-\alpha)} u(1) = 1, & \\ I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(1) = 0, & \end{aligned}$$

and $\chi_1(x,\lambda)$ is the unique solution of the initial value problem

$$\begin{aligned} \pounds_\alpha u(x) + \lambda u(x) &= 0, \qquad x \in [-1,0), \\ I_{-1^+}^{(1-\beta)(1-\alpha)} u(-0) &= k I_{-1^+}^{(1-\beta)(1-\alpha)} \chi_2(+0), \\ I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} u(-0) &= \frac{1}{k} I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} \chi_2(+0). \end{aligned}$$

Let us consider the Hilfer fractional Wronskians

$$\begin{aligned} \omega_i(x,\lambda) &:= W_F(\phi_i(x,\lambda), \chi_i(x,\lambda) \\ &:= I_{-1^+}^{(1-\beta)(1-\alpha)} \phi_i(x,\lambda) I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} \chi_i(x,\lambda) \\ &\quad - I_{-1^+}^{(1-\beta)(1-\alpha)} \chi_i(x,\lambda) I_{1^-}^{\beta(1-\alpha)} D_{-1^+}^{\alpha,\beta} \phi_i(x,\lambda), i = 1,2, \end{aligned} \tag{4.38}$$

which is independent of x and is an entire function. The direct calculation gives

$$\omega_1(x,\lambda) = \omega_2(x,\lambda). \tag{4.39}$$

Now we may introduce the consideration of the characteristic function

$$\omega(x,\lambda) := \omega_1(x,\lambda) = \omega_2(x,\lambda). \tag{4.40}$$

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Lemma 4.4. *For every* $\lambda \in \mathbb{C}$,

$$W_F(x,\lambda) = -\omega^3(x,\lambda),$$

where

$$W_F(x,\lambda) = \begin{vmatrix} L_1(\phi_1) & L_1(\chi_1) & L_1(\phi_2) & L_1(\chi_2) \\ L_2(\phi_1) & L_2(\chi_1) & L_2(\phi_2) & L_2(\chi_2) \\ L_3(\phi_1) & L_3(\chi_1) & L_3(\phi_2) & L_3(\chi_2) \\ L_4(\phi_1) & L_4(\chi_1) & L_4(\phi_2) & L_4(\chi_2) \end{vmatrix}.$$

Proof. Employing the definitions of the functions $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$, $i = 1,2$, we obtain

$$\begin{aligned} W_F(x,\lambda) &= \begin{vmatrix} 0 & -\omega_1(x,\lambda) & 0 & 0 \\ 0 & 0 & -\omega_2(x,\lambda) & 0 \\ M_{\phi_1}(-0) & M_{\chi_1}(-0) & -kM_{\phi_2}(+0) & -kM\chi_2(+0) \\ N_{\phi_1}(-0) & N\chi_1(-0) & -\frac{1}{k}N\phi_2(+0) & -\frac{1}{k}N\chi_2(+0) \end{vmatrix} \\ &= \omega_1(x,\lambda)\omega_2(x,\lambda) \begin{vmatrix} M\phi_1(-0) & -kM\chi_2(+0) \\ N\phi_1(-0) & -\frac{1}{k}N\chi_2(+0) \end{vmatrix} \\ &= \omega_1(x,\lambda)\omega_2(x,\lambda) \begin{vmatrix} kM\phi_2(+0) & -kM\chi_2(+0) \\ \frac{1}{k}N\phi_2(+0) & -\frac{1}{k}N\chi_2(+0) \end{vmatrix} \\ &= -\omega_1(x,\lambda)\omega_2^2(x,\lambda) \\ &= -\omega^3(x,\lambda), \end{aligned}$$

where

$$\begin{aligned} M\phi_1(-0) &= I_{-1^+}^{(1-\beta)(1-\alpha)}\phi_1(-0) & M\chi_1(-0) &= I_{-1^+}^{(1-\beta)(1-\alpha)}\chi_1(-0) \\ M\phi_2(+0) &= I_{-1^+}^{(1-\beta)(1-\alpha)}\phi_2(+0) & M\chi_2(+0) &= I_{-1^+}^{(1-\beta)(1-\alpha)}\chi_2(+0) \\ N\phi_1(-0) &= I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}\phi_1(-0) & N\chi_1(-0) &= I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}\chi_1(-0) \\ N\phi_2(+0) &= I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}\phi_2(+0) & N\chi_2(+0) &= I_{1^-}^{\beta(1-\alpha)}D_{-1^+}^{\alpha,\beta}\chi_2(+0) \end{aligned}$$

□

Corollary 4.3. *The zeros of the function* $W_F(\lambda)$ *consist of the zeros of the characteristic function* $\omega(\lambda)$.

Theorem 4.4. *The eigenvalues of fractional boundary value problems (3.1)-(3.5) are the same as the roots of the characteristic function* $\omega(\lambda)$.

Proof. Let $\lambda = \lambda_0$ be a root of the characteristic function $\omega(\lambda)$. Hence by (4.40), $\omega(x,\lambda_0) = \omega_2(x,\lambda_0) = 0$. According to (4.38), it follows that ϕ_2 and χ_2 are linearly dependent, that is

$$\phi_2(x,\lambda_0) = c\chi_2(x,\lambda_0), \quad x \in (0,1],$$

for some $c \neq 0$. As a result, the function $\phi_2(x, \lambda_0)$ satisfies the fractional boundary condition (3.3). So, $\phi(x, \lambda_0)$ which is given by

$$\phi(x,\lambda_0) = \begin{cases} \phi_1(x,\lambda_0) & x \in [-1,0) \\ \phi_2(x,\lambda_0) & x \in (0,1] \end{cases},$$

satisfies the main problems (3.1)-(3.5). So the function $\phi(x, \lambda_0)$ is an eigenfunction of problems (3.1)-(3.5) corresponding to the eigenvalue λ_0.

Assume $\lambda = \lambda_0$ is an eigenvalue and $u_0(x, \lambda_0)$ is the corresponding eigenfunction. Suppose that $\omega(x, \lambda_0) \neq 0$. Then there exist constants c_i, $i = 1, 2, 3, 4$, at least one of which is not zero, such that

$$u_0(x,\lambda_0) = \begin{cases} c_1\phi_1(x,\lambda_0) + c_2\chi_1(x,\lambda_0) & x \in [-1,0) \\ c_3\phi_2(x,\lambda_0) + c_4\chi_2(x,\lambda_0) & x \in (0,1] \end{cases},$$

since $\omega_1(x, \lambda_0) \neq 0$ and $\omega_2(x, \lambda_0) \neq 0$.

Since the eigenfunction $u_0(x, \lambda)$ satisfies both fractional boundary and fractional transmission conditions (3.2)-(3.5), we have

$$L_i u_0(., \lambda_0) = 0, \text{ for } \quad i = 1, 2, 3, 4.$$

Also with at least one of the constants c_i, $i = 1, 2, 3, 4$, is not zero,

$$det(L_i u_0(., \lambda_0)) = 0,$$

that is, $W_F(\lambda) = 0$. But, by Lemma 4.4, $W_F(\lambda) \neq 0$. This contradiction completes the proof. □

5. Conclusions

This article employs the Hilfer fractional operator as a tool to analyze a specific set of S-L boundary value problems with discontinuities. The research focuses on investigating the eigenvalues and eigenfunctions of the fractional S-L problems and demonstrates that the corresponding operator is symmetric. Moreover, we establish the uniqueness of the solution to the problem by employing an iterative method and delve into how the roots of characteristic functions relate to the eigenvalues.

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