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The Benjamin-Ono-Burgers Equation: New Ideas and New Results

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Abstract Consider the Cauchy problem for the Benjamin-Ono-Burgers equation. There exists a unique global weak solution under appropriate conditions on the initial function and the external force. Here are many very important and interesting questions.

• Can we accomplish the exact limits for all order derivatives of the global smooth solution of the Benjamin-Ono-Burgers equation, in terms of some given information, representing certain physical mechanisms?

• What are the influences of various physical mechanisms (represented by the initial function, the external force, the order of the derivatives and the diffusion coefficient) on the exact limits?

• Can we establish improved decay estimates with sharp rates for all order derivatives of the solution, so that the most important constants $\mathcal{A}$ and $\mathcal{C}$ are independent of any norm of any order derivatives of the initial function, the external force and the solution, for all sufficiently large $t > 0$? Other positive constants $\mathcal{B}$ and $\mathcal{D}$ in the estimates are much less important because $\mathcal{B}t^{-1}$ and $\mathcal{D}t^{-1}$ becomes arbitrarily small as $t \to \infty$. This kind of decay estimates may play a substantial role in long time, accurate numerical simulations.

• Can we use the solution of the corresponding linear equation to approximate the solution of the Benjamin-Ono-Burgers equation?

• Can we couple together classical ideas (such as the Fourier transformation, the Parseval's identity, Lebesgue's dominated convergence theorem, squeeze theorem, etc) in an appropriate way to establish important and interesting results for the Benjamin-Ono-Burgers equation?

• For very similar dissipative dispersive wave equations, such as the Korteweg-de Vries-Burgers equation and the Benjamin-Bona-Mahony-Burgers equation, can we apply the same ideas developed in this paper to establish the same or very similar results?

We will couple together a few novel ideas, several existing ideas and existing results and use rigorous mathematical analysis to provide positive solutions to these important and interesting questions.

Keywords Benjamin-Ono-Burgers equation, global smooth solution, all order derivatives, exact limits, improved decay estimates with sharp rates

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1. Introduction

1.1. The Mathematical model equations and known related results

Consider the Cauchy problem for the Benjamin-Ono-Burgers equation

$$\frac{\partial}{\partial t}u + \mathcal{H}\frac{\partial^2}{\partial x^2}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t), \tag{1.1}$$

$$u(x,0) = u_0(x). \tag{1.2}$$

Also consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v + \mathcal{H}\frac{\partial^2}{\partial x^2}v - \alpha\frac{\partial^2}{\partial x^2}v = f(x,t), \tag{1.3}$$

$$v(x,0) = u_0(x). \tag{1.4}$$

In these equations, the positive constant $\alpha > 0$ represents the diffusion coefficient, the Hilbert operator $\mathcal{H}$ is defined by the principal value of the following singular integral

$$[\mathcal{H}\phi](x) = \frac{1}{\pi}\ \text{P. V.} \int_{\mathbb{R}} \frac{\phi(y)}{x-y}\mathrm{d}y, \tag{1.5}$$

for all functions $\phi \in L^2(\mathbb{R})$. The Fourier transformation of the Hilbert operator is given by

$$\widehat{\mathcal{H}\phi}(\xi) = \mathrm{i}\mathcal{S}(\xi)\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R}, \tag{1.6}$$

where $\mathcal{S} = \mathcal{S}(\xi)$ represents the standard sign function

$$\mathcal{S}(\xi) = -1 \text{ for } \xi < 0, \qquad \mathcal{S}(0) = 0, \qquad \mathcal{S}(\xi) = +1 \text{ for } \xi > 0.$$

It is well known that there exists a global smooth solution

$$u \in C^\infty(\mathbb{R} \times \mathbb{R}^+), \tag{1.7}$$

$$u \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x}u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \forall m > 0, \tag{1.8}$$

to the Cauchy problem for the Benjamin-Ono-Burgers equation, if the initial function and the external force satisfy the following assumptions

$$u_0 \in H^{2m}(\mathbb{R}), \tag{1.9}$$

$$f \in C^\infty(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R})) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \tag{1.10}$$

for all positive constants $m > 0$.

It is also well known that there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0}\left\{t^{1/2}\int_{\mathbb{R}} |u(x,t)|^2\mathrm{d}x\right\} < \infty, \tag{1.11}$$

if the initial function and the external force satisfy the additional conditions

$$u_0 \in L^1(\mathbb{R}) \cap H^{2m}(\mathbb{R}), \tag{1.12}$$

$$f \in C^{\infty}(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+, L^2(\mathbb{R})), \tag{1.13}$$

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x \neq 0, \qquad \int_0^{\infty}\int_{\mathbb{R}} f(x,t)\mathrm{d}x\mathrm{d}t \neq 0, \tag{1.14}$$

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x + \int_0^{\infty}\int_{\mathbb{R}} f(x,t)\mathrm{d}x\mathrm{d}t \neq 0. \tag{1.15}$$

This decay estimate may be improved to be

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\} < \infty, \tag{1.16}$$

if the initial function and the external force satisfy the following additional conditions

$$u_0(x) = \frac{\partial}{\partial x}\phi(x), \qquad f(x,t) = \frac{\partial}{\partial x}\psi(x,t), \tag{1.17}$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, where ϕ and ψ are real scalar smooth functions,

$$\phi \in C^1(\mathbb{R})\cap L^1(\mathbb{R}), \qquad \psi \in C^{\infty}(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+). \tag{1.18}$$

See Bona and Luo [8], Dix [11], Gassot [18], Guo and Zhang [19], Otani [29], and Zhang [35].

However, if the initial function and the external force

$$u_0 \in L^1(\mathbb{R})\cap L^2(\mathbb{R}),$$
$$f \in C^{\infty}(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+, L^2(\mathbb{R})),$$

and that

$$u_0(x) = \frac{\partial}{\partial x}\phi(x), \qquad f(x,t) = \frac{\partial}{\partial x}\psi(x,t),$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, where ϕ and ψ are real scalar smooth functions,

$$\phi \in C^1(\mathbb{R})\cap L^1(\mathbb{R}), \qquad \psi \in C^{\infty}(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+),$$

then many very important and interesting problems have been open. Here is a very important list.

(I) The existence and uniqueness of the global smooth solution

$$u \in C^{\infty}(\mathbb{R}\times\mathbb{R}^+), \tag{1.19}$$

$$u \in L^{\infty}(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x}u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \forall m>0, \tag{1.20}$$

have been open.

(II) The primary decay estimates with sharp rates

$$\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2\mathrm{d}x\right\} < \infty,$$

for all positive constants $m>0$, have been open.

(III) The computations and explicit representations of the following limits

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x\right\}, \tag{1.21}$$

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right\}, \tag{1.22}$$

in terms of certain physical mechanisms, have been open.

(IV) The improved decay estimates with sharp rates

$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x \le \mathcal{A}(\alpha,\delta,\varepsilon,m)+\mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},$$

$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x \le \mathcal{C}(\alpha,\delta,\varepsilon,m)+\mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},$$

for all constants $m \ge 0$, for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, have been open.

The author will study these important problems in this paper.

See the next Subsection for the main purposes of this paper.

Let us review a few popular concepts.

Definition 1.1. Define the Fourier transformation by

$$\widehat{\phi}(\xi) = [\mathcal{F}\phi](\xi) = \int_{\mathbb{R}} \exp(-\mathrm{i}x\xi)\phi(x)\mathrm{d}x, \tag{1.23}$$

for all functions $\phi \in L^1(\mathbb{R})$, where $\xi \in \mathbb{R}$ and $\mathrm{i}=\sqrt{-1}$.

Definition 1.2. Let the positive constant $m>0$. Define the differential operator $(-\frac{\partial^2}{\partial x^2})^m$ by using the Fourier transformation

$$\mathcal{F}\left[\left(-\frac{\partial^2}{\partial x^2}\right)^m \phi\right](\xi) = |\xi|^{2m}\widehat{\phi}(\xi), \qquad \phi \in L^1(\mathbb{R}). \tag{1.24}$$

1.2. The main purposes

The main purpose of this paper is to couple together a few novel ideas, several existing ideas and existing results to accomplish the following exact limits

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x\right\},$$

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right\},$$

in terms of some given information, such as the diffusion coefficient α, the order m of the derivative, and the following integrals

$$\int_{\mathbb{R}}\phi(x)\mathrm{d}x, \qquad \int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t, \qquad \int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t,$$

where the auxiliary functions ϕ and ψ will be made clearly in the next subsection.

Moreover, we will establish the following improved decay estimates with sharp rates

$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x \leq \mathcal{A}(\alpha,\delta,\varepsilon,m)+\mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},$$
$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x \leq \mathcal{C}(\alpha,\delta,\varepsilon,m)+\mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},$$

for all constants $m \geq 0$ and for all sufficiently large t.

The positive constants

$$\mathcal{A}=\mathcal{A}(\alpha,\delta,\varepsilon,m),\ \ \mathcal{B}=\mathcal{B}(\alpha,\delta,\varepsilon,m),$$
$$\mathcal{C}=\mathcal{C}(\alpha,\delta,\varepsilon,m),\ \ \mathcal{D}=\mathcal{D}(\alpha,\delta,\varepsilon,m),$$

will be represented explicitly in terms of known information. The key point is that they are independent of any norm of any order derivatives of the initial function and the solution.

1.3. The mathematical assumptions

Let the initial function and the external force satisfy the following assumptions

$$u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$
$$f \in C^\infty(\mathbb{R}\times\mathbb{R}) \cap L^1(\mathbb{R}\times\mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R})).$$

Suppose that there exist real scalar smooth functions

$$\phi \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}), \qquad \psi \in C^\infty(\mathbb{R}\times\mathbb{R}^+) \cap L^1(\mathbb{R}\times\mathbb{R}^+),$$

such that the initial function and the external force are given by

$$u_0(x)=\frac{\partial}{\partial x}\phi(x), \qquad f(x,t)=\frac{\partial}{\partial x}\psi(x,t),$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$.

Suppose that there exist the following exact limits

$$\lim_{t\to\infty}\left\{t^{m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m \psi(x,t)\right|\mathrm{d}x\right\} \equiv \mathcal{L}(m),$$

for all constants $m \geq 0$.

Suppose that there exists a global weak solution

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R})), \qquad \frac{\partial}{\partial x}u \in L^2(\mathbb{R}\times\mathbb{R}^+).$$

Suppose that there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty.$$

1.4. The main results

There are two parts in the results: the main results (for the Benjamin-Ono-Burgers equation) and the minor results (for the corresponding linear equation).

Let the positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$. Here are the main results of this paper.

Theorem 1.1. *There hold the following exact limits*

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x\right\}\\&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\\&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right\}\\&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\end{aligned}$$

for all order derivatives of the global smooth solution of the Cauchy problem for the Benjamin-Ono-Burgers equation.

Theorem 1.2. *The ratios of the exact limits of the Benjamin-Ono-Burgers equation are the same as the ratios of the exact limits of the corresponding linear equation, for each fixed constant $m \geq 0$. That is,*

$$\begin{aligned}&\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2} u(x,t)\right|^2 \mathrm{d}x\right]\right\}\\&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} u(x,t)\right|^2 \mathrm{d}x\right]\right\}\\&=\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2} v(x,t)\right|^2 \mathrm{d}x\right]\right\}\\&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} v(x,t)\right|^2 \mathrm{d}x\right]\right\}=\frac{4m+3}{4\alpha},\\&\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1} u(x,t)\right|^2 \mathrm{d}x\right]\right\}\\&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} u(x,t)\right|^2 \mathrm{d}x\right]\right\}\\&=\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1} v(x,t)\right|^2 \mathrm{d}x\right]\right\}\end{aligned}$$

$$/ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m v(x,t)\right|^2 \mathrm{d}x\right]\right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},$$

and

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2}[u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\}\\
&/ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m}[u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\}\\
&= \left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2} v(x,t)\right|^2 \mathrm{d}x\right]\right\}\\
&/ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} v(x,t)\right|^2 \mathrm{d}x\right]\right\} = \frac{4m+3}{4\alpha},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1}[u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\}\\
&/ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m}[u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\}\\
&= \left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1} v(x,t)\right|^2 \mathrm{d}x\right]\right\}\\
&/ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} v(x,t)\right|^2 \mathrm{d}x\right]\right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},
\end{aligned}$$

for all constants $m \geq 0$.

Theorem 1.3. *There hold the primary decay estimates with sharp rates*

$$\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x\right\} < \infty,$$

for all order derivatives of the global smooth solution.

Theorem 1.4. *There hold the following improved decay estimates with sharp rates*

$$\begin{aligned}
&t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x \leq \mathcal{A}(\alpha,\delta,\varepsilon,m) + \mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},\\
&t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x \leq \mathcal{C}(\alpha,\delta,\varepsilon,m) + \mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},
\end{aligned}$$

for all order derivatives of the global smooth solution to the Cauchy problem for the Benjamin-Ono-Burgers equation, for all positive constants $0 < \delta < 4$ *and* $0 < \varepsilon < 1$, *for all constants* $m \geq 0$ *and for all sufficiently large* t.

The positive constants are given by

$$\mathcal{A}(\alpha,\delta,\varepsilon,m)$$

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$$
\begin{aligned}
&= \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}|\phi(x)|\mathrm{d}x\right\}^2\\
&+ \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2\\
&+ \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\quad \mathcal{B}(\alpha,\delta,\varepsilon,m)\\
&= 5\left\{\frac{1}{2\pi}\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\mathcal{L}(m+\frac{1}{4}(\delta-1))\\
&+ 5\left\{\frac{1}{2\pi}\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^4,\\
&\quad \mathcal{C}(\alpha,\delta,\varepsilon,m)\\
&= \frac{2}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\quad \mathcal{D}(\alpha,\delta,\varepsilon,m)\\
&= 4^m\left\{\frac{1}{2\pi}\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\int_{\mathbb{R}}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\int_{\mathbb{R}}|\eta|^{2m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^4.
\end{aligned}
$$

Here are the minor results of this paper.

Theorem 1.5. *There hold the following exact limits*

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m v(x,t)\right|^2\mathrm{d}x\right\}\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
&\cdot\ \left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}
$$

for all order derivatives of the global smooth solution of the Cauchy problem for the corresponding linear equation.

Theorem 1.6. *The ratios of the exact limits of the global smooth solution are given by*

$$\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2}v(x,t)\right|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m}v(x,t)\right|^2\mathrm{d}x\right]\right\}=\frac{4m+3}{4\alpha},$$
$$\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1}v(x,t)\right|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m}v(x,t)\right|^2\mathrm{d}x\right]\right\}=\frac{(4m+3)(4m+5)}{(4\alpha)^2},$$

for all real constants $m\geq 0$.

2. The mathematical analysis and the proofs of the main results

The main purposes of this section are to accomplish the exact limits and to establish the improved decay estimates with sharp rates for all order derivatives of the global smooth solution of the Benjamin-Ono-Burgers equation. We will couple together a few novel ideas, several existing ideas and existing results to establish the main results.

Let the positive constants $0<\delta<4$ and $0<\varepsilon<1$.

The Fourier transformation of the global smooth solution may be represented as

$$\begin{aligned}\widehat{u}(\xi,t)&=\mathrm{i}\xi\exp[(-\alpha\xi^2+\mathrm{i}|\xi|\xi)t]\widehat{\phi}(\xi)\\&+\mathrm{i}\xi\int_0^t\exp[(-\alpha\xi^2+\mathrm{i}|\xi|\xi)(t-\tau)]\widehat{\psi}(\xi,\tau)\mathrm{d}\tau\\&-\mathrm{i}\xi\int_0^t\exp[(-\alpha\xi^2+\mathrm{i}|\xi|\xi)(t-\tau)]\widehat{u^2}(\xi,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $(\xi,t)\in\mathbb{R}\times\mathbb{R}^+$.

What we will really be using is the following representation

$$\begin{aligned}t^{1/2}\widehat{u}(t^{-1/2}\eta,t)&=\mathrm{i}\eta\exp[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)t]\widehat{\phi}(t^{-1/2}\eta)\\&+\mathrm{i}\eta\int_0^t\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\&-\mathrm{i}\eta\int_0^t\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

Definition 2.1. Define the following complex auxiliary functions

$$\Lambda_1(\eta,t)=\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)$$

$$+\int_0^t \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$-\int_0^t \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$
$$\Lambda_2(\eta,t)=\int_0^t \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

and

$$\Gamma_1(\eta,t)=\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)$$
$$+\int_0^{(1-\varepsilon)t} \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$-\int_0^{(1-\varepsilon)t} \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$
$$\Gamma_2(\eta,t)=\int_0^{(1-\varepsilon)t} \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

Remark 2.1. It is easy to see that

$$\mathrm{i}\eta\Lambda_1(\eta,t)=t^{1/2}\widehat{u}(t^{-1/2}\eta,t),$$
$$\Lambda_1(\eta,t)-\Gamma_1(\eta,t)=\int_{(1-\varepsilon)t}^t \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$-\int_{(1-\varepsilon)t}^t \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$
$$\Lambda_2(\eta,t)-\Gamma_2(\eta,t)=\int_{(1-\varepsilon)t}^t \exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

Remark 2.2. Note that the Fourier transformations

$$\widehat{\phi}(t^{-1/2}\eta),\qquad \widehat{\psi}(t^{-1/2}\eta,\tau),\qquad \widehat{u^2}(t^{-1/2}\eta,\tau)$$

are continuous functions of η and t, for each fixed $\tau>0$. It is easy to apply Lebesgue's dominated convergence theorem to show that

$$\lim_{t\to\infty}\Gamma_1(\eta,t)=\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)$$
$$\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\},$$
$$\lim_{t\to\infty}\Gamma_2(\eta,t)=\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\},$$

for all $\eta\in\mathbb{R}$.

2.1. The elementary estimates

First of all, we will establish a series of elementary estimates.

Lemma 2.1. *There hold the following elementary estimates*

$$\begin{aligned}
&\int_{\mathbb{R}} |\eta|^{4m} \left|\eta \exp(-\alpha|\eta|^2 + \mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right|^2 \mathrm{d}\eta \\
\le &\int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta \left\{\int_{\mathbb{R}} |\phi(x)|\mathrm{d}x\right\}^2, \\
&\int_{\mathbb{R}} |\eta|^{4m} \left|\eta \int_0^{(1-\varepsilon)t} \exp\left[(-\alpha|\eta|^2 + \mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right] \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
\le &\int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty \int_{\mathbb{R}} |\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2, \\
&\int_{\mathbb{R}} |\eta|^{4m} \left|\eta \int_0^{(1-\varepsilon)t} \exp\left[(-\alpha|\eta|^2 + \mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right] \widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
\le &\int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty \int_{\mathbb{R}} |u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2, \\
&\int_{\mathbb{R}} |\eta|^{4m} \left|\eta \int_{(1-\varepsilon)t}^{t} \exp\left[(-\alpha|\eta|^2 + \mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right] \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
\le &\, t^{-1} \left\{\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{m+1+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} \left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+\frac{1}{4}(\delta-1)} \psi(x,\tau)\right| \mathrm{d}x\right]\right\}^2, \\
&\int_{\mathbb{R}} |\eta|^{4m} \left|\eta \int_{(1-\varepsilon)t}^{t} \exp\left[(-\alpha|\eta|^2 + \mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right] \widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
\le &\, 4^m t^{-1} \left\{\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2\mathrm{d}x\right]\right\} \\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} \left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+\frac{1}{4}(\delta-1)} u(x,\tau)\right|^2 \mathrm{d}x\right]\right\}.
\end{aligned}$$

These elementary estimates are true for all positive constants $0 < \delta < 4$ *and* $0 < \varepsilon < 1$*, for all constants* $m \ge 0$*, and for all* $t > 0$*.*

Proof. The proof follows from a few simple properties of the Fourier transformation and the change of variables $\eta = t^{1/2}\xi$, where $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$. The proofs of the first three estimates are skipped because they are very simple. Let us prove the fourth estimate. There hold the following estimates

$$\begin{aligned}
&\int_{\mathbb{R}} |\eta|^{4m} \left|\eta \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right\} \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
= &\int_{\mathbb{R}} \frac{1}{\alpha^2|\eta|^{1+\delta}} \left|\int_{(1-\varepsilon)t}^{t} \alpha|\eta|^{2m+1+\frac{1}{2}(1+\delta)}\right.
\end{aligned}$$

$$
\begin{aligned}
&\cdot\ \exp\left\{(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\bigg|^2\mathrm{d}\eta\\
&=\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\int_{(1-\varepsilon)t}^{t}\left\{\alpha|\eta|^2\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\right\}\right.\\
&\cdot\ \left.\left\{|\eta|^{2m-1+\frac{1}{2}(1+\delta)}\widehat{\psi}(t^{-1/2}\eta,\tau)\right\}\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&=\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\int_{(1-\varepsilon)t}^{t}\left\{\alpha|\eta|^2\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\right\}\right.\\
&\cdot\ \left.\left\{t^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\mathcal{F}\left[\left(-\frac{\partial^2}{\partial x^2}\right)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi\right](t^{-1/2}\eta,\tau)\right\}\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&\le\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\frac{1}{t}\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{-\alpha|\eta|^2(1-\frac{\tau}{t})\right\}\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&\cdot\ \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{m+\frac{1}{2}+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi(x,\tau)\right|\mathrm{d}x\right]\right\}^2\\
&\le t^{-1}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\ \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{m+1+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+\frac{1}{4}(\delta-1)}\psi(x,\tau)\right|\mathrm{d}x\right]\right\}^2,
\end{aligned}
$$

for all constants $m\ge 0$ and for all $t>0$.

The next estimate may be proved very similarly. The details are skipped. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *There hold the following estimates*

$$
\begin{aligned}
&\left|\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right|\\
&\le\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\\
&+\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},\\
&\left|\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right|\\
&\le\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\\
&+\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},
\end{aligned}
$$

for all constants $m\ge 0$ and for all $t>0$.

Proof. The estimates follow from the standard Cauchy-Schwartz's inequality and the definitions of the auxiliary functions $\Lambda_i(\eta,t)$ and $\Gamma_i(\eta,t)$, for $i=1,2$. □

2.2. The comprehensive analysis

The main purpose of this subsection is to make use of the representation of the Fourier transformation of the global smooth solution and the elementary estimates to establish estimates for the following quantities

$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x,$$
$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x,$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all constants $m\geq 0$ and for all $t>0$. The comprehensive analysis will play a very important role when we accomplish the primary and the improved decay estimates with sharp rates for all order derivatives of the global smooth solution.

Lemma 2.3. *There hold the following estimates*

$$\begin{aligned}
& t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x \\
\leq{}& \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}|\phi(x)|\mathrm{d}x\right\}^2 \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+\frac{5}{2\pi t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{m+1+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+\frac{1}{4}(\delta-1)}\psi(x,\tau)\right|\mathrm{d}x\right]\right\}^2 \\
&+\frac{5}{2\pi t}4^m\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right]\right\} \\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+\frac{1}{4}(\delta-1)}u(x,\tau)\right|^2\mathrm{d}x\right]\right\}.
\end{aligned}$$

These estimates are true for all positive constants $0<\delta<4$ *and* $0<\varepsilon<1$*, for all constants* $m\geq 0$ *and for all* $t>0$.

Proof. The main idea in the proof is the appropriate coupling of the Parseval's identity, a few properties of the Fourier transformation, the representation of the Fourier transformation of the global smooth solution, a simple change of variables and the elementary estimates. In fact, we have

$$t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{t^{2m+3/2}}{2\pi}\int_{\mathbb{R}} |\xi|^{4m}|\widehat{u}(\xi,t)|^2 \mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}|t^{1/2}\widehat{u}(t^{-1/2}\eta,t)|^2 \mathrm{d}\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta) \\
&+ \eta\int_0^t \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \eta\int_0^t \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta) \\
&+ \eta\int_0^{(1-\varepsilon)t} \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \eta\int_0^{(1-\varepsilon)t} \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \eta\int_{(1-\varepsilon)t}^t \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \eta\int_{(1-\varepsilon)t}^t \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta \\
&\le \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\int_0^{(1-\varepsilon)t} \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\int_0^{(1-\varepsilon)t} \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\int_{(1-\varepsilon)t}^t \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta \\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}\Big|\eta\int_{(1-\varepsilon)t}^t \exp\Big[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\Big]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2 \mathrm{d}\eta \\
&\le \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}|\phi(x)|\mathrm{d}x\right\}^2 \\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+ \frac{5}{2\pi t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}
\end{aligned}
$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{m+1+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^{m+\frac{1}{4}(\delta-1)} \psi(x,\tau) \right| \mathrm{d}x \right] \right\}^2$$
$$+ \frac{5}{2\pi t} 4^m \left\{ \int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^{m+\frac{1}{4}(\delta-1)} u(x,\tau) \right|^2 \mathrm{d}x \right] \right\}.$$

The proof of Lemma 2.3 is finished now. □

Lemma 2.4. *There holds the following estimate*

$$t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m [u(x,t) - v(x,t)] \right|^2 \mathrm{d}x$$
$$\le \frac{2}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t \right\}^2$$
$$+ \frac{2}{2\pi t} 4^m \left\{ \int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^{m+\frac{1}{4}(\delta-1)} u(x,\tau) \right|^2 \mathrm{d}x \right] \right\}.$$

These estimates are true for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all constants $m \ge 0$ and for all $t > 0$.

Proof. The proof of this lemma is very similar to the proof of Lemma 2.3 and the details are skipped. □

2.3. The primary decay estimates with sharp rates

The main purpose of this subsection is to couple together the comprehensive analysis and the following elementary decay estimate with the sharp rate $r = \frac{3}{2}$

$$\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty,$$

to establish the primary decay estimates with sharp rates for all order derivatives.

Lemma 2.5. *There hold the following primary decay estimates with sharp rates*

$$\sup_{t>0} \left\{ t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m u(x,t) \right|^2 \mathrm{d}x \right\} < \infty,$$

for all positive constants $m > 0$.

Proof. Let us use a very simple iteration technique to prove the result. For any positive constant $m > 0$, there exists an integer $k \geq 1$ and a positive constant $0 < \delta_0 < 1$, such that $m = k\frac{1-\delta_0}{4}$. Let

$$\begin{aligned}
&m_1 = \frac{1-\delta_0}{4},\\
&m_2 = 2\frac{1-\delta_0}{4},\\
&m_3 = 3\frac{1-\delta_0}{4},\\
&\cdots\cdots\cdots\cdots\\
&m_k = k\frac{1-\delta_0}{4}.
\end{aligned}$$

Now in the estimate of Lemma 2.3, letting $m = m_1$, noting that

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\} < \infty,$$

we immediately obtain the first estimate

$$\sup_{t>0}\left\{t^{2m_1+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m_1}u(x,t)\right|^2\mathrm{d}x\right\} < \infty.$$

Then letting $m = m_2$, we obtain the second estimate

$$\sup_{t>0}\left\{t^{2m_2+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m_2}u(x,t)\right|^2\mathrm{d}x\right\} < \infty.$$

Repeating this procedure for finitely many times by letting $m = m_1$, $m = m_2$, $m = m_3$, $\cdots\cdots$, $m = m_k$, we obtain the estimate

$$\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m}u(x,t)\right|^2\mathrm{d}x\right\} < \infty.$$

The proof is finished now. □

2.4. The fundamental limits

The main purpose of this subsection is to couple together the elementary estimates and the primary decay estimates to establish several fundamental limits. These limits are true for all constants $m \geq 0$.

Lemma 2.6. *There hold the following limits*

$$\begin{aligned}
&\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}\\
=&\left\{\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}$$

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$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}$$
$$=\left\{\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all constants $m\geq 0$.

Proof. The limits follow from Lebesgue's dominated convergence theorem and the first three elementary estimates in Lemma 2.1. □

Lemma 2.7. *There hold the following limits*

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}=0,$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}=0,$$

for all constants $m\geq 0$.

Proof. The proof follows from the assumption and the last two elementary estimates in Lemma 2.1 and the primary decay estimates in Lemma 2.5. In fact, we have

$$\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\leq C_1(m)t^{-1/2},$$
$$\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\leq C_2(m)t^{-1/2},$$

for all constants $m\geq 0$ and for all $t>0$, where $C_i=C_i(m)$ are positive constants, independent of t, for all $i=1,2$. Other details are omitted because they are trivial. □

Lemma 2.8. *There hold the following limits*

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)|^2\mathrm{d}\eta\right\}$$
$$=\left\{\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}$$
$$\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)|^2\mathrm{d}\eta\right\}$$
$$=\left\{\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all constants $m\geq 0$.

Proof. The results follow from the squeeze theorem, the elementary estimates in Lemma 2.2 and the limits in Lemma 2.6 and Lemma 2.7. The details are skipped because they are very simple. □

2.5. The exact limits

The main purpose of this subsection is to make complete use of the fundamental limits and several traditional ideas to accomplish the exact limits for all order derivatives of the global smooth solution. More precisely, the main purpose of this subsection is to establish the exact limits

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}$$

for all constants $m\geq 0$.

Let us make the change of variables $\eta=t^{1/2}\xi$, where $\xi\in\mathbb{R}$ and $\eta\in\mathbb{R}$. Now by coupling together the Parseval's identity, a few simple properties of the Fourier transformation and the representation of the Fourier transformation of the global smooth solution, we have the following computations

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2\mathrm{d}x\right\}\\
&=\lim_{t\to\infty}\left\{\frac{t^{2m+3/2}}{2\pi}\int_{\mathbb{R}}|\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi\right\}\\
&=\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}|t^{1/2}\widehat{u}(t^{-1/2}\eta,t)|^2\mathrm{d}\eta\right\}\\
&=\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}\left|\eta\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right.\right.\\
&+\eta\int_0^t\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\left.\left.\eta\int_0^t\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
&=\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}\left|\eta\exp(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right.\right.\\
&+\eta\int_0^{(1-\varepsilon)t}\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\left.\left.\eta\int_0^{(1-\varepsilon)t}\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta
\end{aligned}$$

$$\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} |u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all constants $m \geq 0$.

Very similarly, we have the following computations

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m[u(x,t)-v(x,t)]\right|^2\mathrm{d}x\right\}\\
=&\lim_{t\to\infty}\left\{\frac{t^{2m+3/2}}{2\pi}\int_{\mathbb{R}}|\xi|^{4m}|\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|^2\mathrm{d}\xi\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}|t^{1/2}[\widehat{u}(t^{-1/2}\eta,t)-\widehat{v}(t^{-1/2}\eta,t)]|^2\mathrm{d}\eta\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}\left|\eta\int_0^t\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
=&\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}\left|\eta\int_0^{(1-\varepsilon)t}\exp\left[(-\alpha|\eta|^2+\mathrm{i}|\eta|\eta)(1-\frac{\tau}{t})\right]\widehat{u^2}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}$$

for all constants $m \geq 0$.

2.6. The ratios of the exact limits

The main purpose of this subsection is to compute the ratios of the exact limits for each fixed $m \geq 0$.

Lemma 2.9. *There hold the following results*

$$\begin{aligned}
&\int_{\mathbb{R}}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\frac{4m+3}{4\alpha}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,\\
&\int_{\mathbb{R}}|\eta|^{4m+6}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta=\frac{(4m+3)(4m+5)}{(4\alpha)^2}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,
\end{aligned}$$

for all constants $m \geq 0$.

Proof. It follows from the integration by parts. The details are skipped. □

Lemma 2.10. *There hold the following results*

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2}u(x,t)\right|^2\mathrm{d}x\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m}u(x,t)\right|^2\mathrm{d}x\right]\right\}=\frac{4m+3}{4\alpha},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1}u(x,t)\right|^2\mathrm{d}x\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x\right]\right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2}[u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\} = \frac{4m+3}{4\alpha},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1}[u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m [u(x,t)-v(x,t)]\right|^2 \mathrm{d}x\right]\right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},
\end{aligned}
$$

for all constants $m \geq 0$.

Proof. It follows from Theorem 1.1 and Lemma 2.9. □

2.7. The improved decay estimates with sharp rates

For convenience, let us define $\mathcal{I} = \mathcal{I}(m)$ and $\mathcal{J}$ by

$$
\begin{aligned}
&\mathcal{I}(m) = \frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,\\
&\mathcal{J} = \left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2.
\end{aligned}
$$

Recall that we have the exact limits

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m \psi(x,t)\right|\mathrm{d}x\right\} = \mathcal{L}(m),\\
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2\mathrm{d}x\right\} = \mathcal{I}(m)\mathcal{J},
\end{aligned}
$$

for all constants $m \geq 0$.

By using the squeeze theorem, we have the exact limits

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m \psi(x,\tau)\right|\mathrm{d}x\right]\right\} = \mathcal{L}(m),\\
&\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,\tau)\right|^2\mathrm{d}x\right]\right\} = \mathcal{I}(m)\mathcal{J},
\end{aligned}
$$

for all $m \geq 0$.

Therefore, there exists a sufficiently large positive constant $T \gg 1$, such that

$$
\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m \psi(x,\tau)\right|\mathrm{d}x\right\} \leq 2\mathcal{L}(m),
$$

$$\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,\tau)\right|^2\mathrm{d}x\right\}\le 2\mathcal{I}(m)\mathcal{J},$$

for all $t > T$. In particular, there hold the following estimates

$$\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right\}\le 2\mathcal{I}(0)\mathcal{J},$$

$$\sup_{(1-\varepsilon)t\le\tau\le t}\left\{\tau^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+\frac{1}{4}(\delta-1)} u(x,\tau)\right|^2\mathrm{d}x\right\}$$
$$\le 2\mathcal{I}(m+\frac{1}{4}(\delta-1))\mathcal{J},$$

for all $t > T$.

2.8. The linear results

The main purpose of this subsection is to establish the exact limits for all order derivatives of the global smooth solution of the Cauchy problem for the corresponding linear equation.

Lemma 2.11. *There hold the following exact limits*

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m v(x,t)\right|^2\mathrm{d}x\right\}$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all order derivatives of the global smooth solution of the corresponding linear equation.

Proof. The proof is very similar to that of Theorem 1.1. The details are skipped. □

Lemma 2.12. *The ratios of the exact limits are given by*

$$\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1/2} v(x,t)\right|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} v(x,t)\right|^2\mathrm{d}x\right]\right\}=\frac{4m+3}{4\alpha},$$

$$\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m+1} v(x,t)\right|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^{m} v(x,t)\right|^2\mathrm{d}x\right]\right\}=\frac{(4m+3)(4m+5)}{(4\alpha)^2},$$

for all constants $m \ge 0$.

Proof. It follows from the results of Lemma 2.9 and Lemma 2.11 below. The details are skipped. □

2.9. The proofs of the theorems

The Proof of Theorem 1.1: The proof of Theorem 1.1 may be finished by using the mathematical analysis and the results in Subsection 2.5. ☐

The Proof of Theorem 1.2: The proof of Theorem 1.2 may be finished by coupling together the results of Theorem 1.1, Lemma 2.9 and Lemma 2.10. ☐

The Proof of Theorem 1.3: The proof of Theorem 1.3 may be finished by using Lemma 2.5. ☐

The Proof of Theorem 1.4: The proof of Theorem 1.4 may be finished by coupling together the results of Lemma 2.3, Lemma 2.4 and Subsection 2.7.

The Proof of Theorem 1.5: The proof of Theorem 1.5 may be finished by using Lemma 2.11. ☐

The Proof of Theorem 1.6: The proof of Theorem 1.6 may be finished by using Lemma 2.12. ☐

3. Conclusion and remarks

3.1. Summary

Consider the Cauchy problem for the Benjamin-Ono-Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}u+\mathcal{H}\frac{\partial^2}{\partial x^2}u-\alpha\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}(u^2)=f(x,t),\\&u(x,0)=u_0(x),\end{aligned}$$

and the Cauchy problem for the corresponding linear equation

$$\begin{aligned}&\frac{\partial}{\partial t}v+\mathcal{H}\frac{\partial^2}{\partial x^2}v-\alpha\frac{\partial^2}{\partial x^2}v=f(x,t),\\&v(x,0)=u_0(x).\end{aligned}$$

Let the initial function and the external force satisfy the following conditions

$$\begin{aligned}&u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\\&f\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R})).\end{aligned}$$

Suppose that there exist real scalar smooth functions

$$\phi\in C^1(\mathbb{R})\cap L^1(\mathbb{R}),\qquad \psi\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+),$$

such that the initial function and the external force are given by

$$u_0(x)=\frac{\partial}{\partial x}\phi(x),\qquad f(x,t)=\frac{\partial}{\partial x}\psi(x,t),$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$.

Suppose that there exist the following exact limits

$$\lim_{t\to\infty}\left\{t^{m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m\psi(x,t)\right|\mathrm{d}x\right\}\equiv\mathcal{L}(m),$$

for all constants $m\geq 0$.

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Suppose that there exists a global weak solution

$$u \in L^{\infty}(\mathbb{R}^+, L^2(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R} \times \mathbb{R}^+),$$

to the Cauchy problem for the Benjamin-Ono-Burgers equation.

Suppose that there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty.$$

We have accomplished the following exact limits

$$\begin{aligned}
&\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m u(x,t) \right|^2 \mathrm{d}x \right\} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \\
&\cdot \left\{ \int_{\mathbb{R}} \phi(x) \mathrm{d}x + \int_0^{\infty} \int_{\mathbb{R}} \psi(x,t) \mathrm{d}x \mathrm{d}t - \int_0^{\infty} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \mathrm{d}t \right\}^2, \\
&\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m [u(x,t) - v(x,t)] \right|^2 \mathrm{d}x \right\} \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ \int_0^{\infty} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \mathrm{d}t \right\}^2,
\end{aligned}$$

for all order derivatives of the global smooth solution.

We have also established the following improved decay estimates with sharp rates

$$\begin{aligned}
&t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m u(x,t) \right|^2 \mathrm{d}x \le \mathcal{A}(\alpha,\delta,\varepsilon,m) + \mathcal{B}(\alpha,\delta,\varepsilon,m) t^{-1}, \\
&t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m [u(x,t) - v(x,t)] \right|^2 \mathrm{d}x \le \mathcal{C}(\alpha,\delta,\varepsilon,m) + \mathcal{D}(\alpha,\delta,\varepsilon,m) t^{-1},
\end{aligned}$$

for all constants $m \ge 0$ and for all sufficiently large t.

The four positive constants are given by

$$\begin{aligned}
&\mathcal{A}(\alpha,\delta,\varepsilon,m) \\
&= \frac{5}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \left\{ \int_{\mathbb{R}} |\phi(x)| \mathrm{d}x \right\}^2 \\
&+ \frac{5}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^{\infty} \int_{\mathbb{R}} |\psi(x,t)| \mathrm{d}x \mathrm{d}t \right\}^2 \\
&+ \frac{5}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^{\infty} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \mathrm{d}t \right\}^2, \\
&\mathcal{B}(\alpha,\delta,\varepsilon,m) \\
&= 5 \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} \frac{[1 - \exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2 |\eta|^{1+\delta}} \mathrm{d}\eta \right\} \mathcal{L}(m + \frac{1}{4}(\delta - 1))
\end{aligned}$$

$$
\begin{aligned}
&+5\left\{\frac{1}{2\pi}\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{2m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^4,\\
&\mathcal{C}(\alpha,\delta,\varepsilon,m)\\
&=\frac{2}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\mathcal{D}(\alpha,\delta,\varepsilon,m)\\
&=4^m\left\{\frac{1}{2\pi}\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{2m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^4.
\end{aligned}
$$

3.2. Remarks

Remark 3.1. Let the initial function and the external force satisfy the following conditions

$$
\begin{aligned}
&u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\\
&f\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R})).
\end{aligned}
$$

Suppose that there exist real scalar smooth functions

$$
\phi\in C^1(\mathbb{R})\cap L^1(\mathbb{R}),\qquad \psi\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+),
$$

such that

$$
u_0(x)=\frac{\partial}{\partial x}\phi(x),\qquad f(x,t)=\frac{\partial}{\partial x}\psi(x,t),
$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$. Then there exists a unique global smooth solution

$$
\begin{aligned}
&u\in C^\infty(\mathbb{R}\times\mathbb{R}^+),\\
&u\in L^\infty(\mathbb{R}^+,H^{2m}(\mathbb{R})),\qquad \frac{\partial}{\partial x}u\in L^2(\mathbb{R}^+,H^{2m}(\mathbb{R})),
\end{aligned}
$$

to the Cauchy problem for the Benjamin-Ono-Burgers equations, where $m>0$ is any appropriate positive constant.

Moreover, there hold the following decay estimates with sharp rates

$$
\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2\mathrm{d}x\right\}<\infty,
$$

for all positive constants $m > 0$.

Remark 3.2. (A) Consider the Cauchy problem for the nonlinear Korteweg-de Vries-Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}u+\frac{\partial^3}{\partial x^3}u-\alpha\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}(u^2)=f(x,t),\\&u(x,0)=u_0(x),\end{aligned}$$

and the Cauchy problem for the corresponding linear equation

$$\begin{aligned}&\frac{\partial}{\partial t}v+\frac{\partial^3}{\partial x^3}v-\alpha\frac{\partial^2}{\partial x^2}v=f(x,t),\\&v(x,0)=u_0(x).\end{aligned}$$

(B) Consider the Cauchy problem for the Benjamin-Bona-Mahony-Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}u-\frac{\partial^3}{\partial x^2\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial}{\partial x}u+\frac{\partial}{\partial x}(u^2)=f(x,t),\\&u(x,0)=u_0(x),\end{aligned}$$

and the Cauchy problem for the corresponding linear equation

$$\begin{aligned}&\frac{\partial}{\partial t}v-\frac{\partial^3}{\partial x^2\partial t}v-\alpha\frac{\partial^2}{\partial x^2}v+\beta\frac{\partial}{\partial x}v=f(x,t),\\&v(x,0)=u_0(x).\end{aligned}$$

We have strong reasons to believe that there hold the same results

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2\mathrm{d}x\right\}\\&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\&\quad\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\\&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m[u(x,t)-v(x,t)]\right|^2\mathrm{d}x\right\}\\&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\right\}^2,\end{aligned}$$

for all constants $m \geq 0$, under the same assumptions as in this paper.

Remark 3.3. Let us see why the integral

$$\int_0^\infty\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t$$

exists.

First of all, from the elementary energy equation

$$\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x+2\alpha\int_0^t\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}u(x,\tau)\right|^2\mathrm{d}x\mathrm{d}\tau$$

$$= \int_{\mathbb{R}} |u_0(x)|^2 \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}} u(x,\tau) f(x,\tau) \mathrm{d}x\mathrm{d}\tau,$$

for all $t > 0$, we may easily obtain the uniform energy estimate

$$\left\{ \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x + 2\alpha \int_0^t \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} u(x,\tau) \right|^2 \mathrm{d}x\mathrm{d}\tau \right\}^{1/2}$$
$$\leq \left\{ \int_{\mathbb{R}} |u_0(x)|^2 \mathrm{d}x \right\}^{1/2} + \int_0^\infty \left\{ \int_{\mathbb{R}} |f(x,t)|^2 \mathrm{d}x \right\}^{1/2} \mathrm{d}t.$$

Secondly, we have the assumption

$$\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty.$$

Overall, we have the following estimate

$$\sup_{t>0} \left\{ (1+t)^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty.$$

Therefore, the improper integral

$$\int_0^\infty \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t < \infty,$$

exists.

Remark 3.4. The results in Theorem 1.1 and Theorem 1.4 indicate that we may use the solution of the corresponding linear equation to approximate the solution of the Benjamin-Ono-Burgers equation very well.

Remark 3.5. We do not make the assumption that

$$\sup_{t>0} \left\{ \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m u(x,t) \right|^2 \mathrm{d}x \right\} < \infty,$$

for all order derivatives of the global smooth solution of the Cauchy problem for the Benjamin-Ono-Burgers equation. We did not use such strong conditions in the analysis.

Remark 3.6. The exact limits and the improved decay estimate with sharp rates for all order derivatives of the global smooth solution may have strong influence in accurate long time numerical simulations.

3.3. Open Problems and Further Research Directions

Open problem 1: (A) Consider the Cauchy problem for the general Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t} u + \frac{\partial^3}{\partial x^3} u - \alpha \frac{\partial^2}{\partial x^2} u + \frac{\partial}{\partial x} \mathcal{N}(u) = f(x,t), \tag{3.1}$$
$$u(x,0) = u_0(x), \tag{3.2}$$

and the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v+\frac{\partial^3}{\partial x^3}v-\alpha\frac{\partial^2}{\partial x^2}v=f(x,t), \tag{3.3}$$
$$v(x,0)=u_0(x). \tag{3.4}$$

See [1], [2], [3], [4], [9], [13], [16], [17], [21], [22], [23], [24], [25], [26], [27], [30], [31], [32], [36], [37], [38], [39], [40], [41].
(B) Consider the Cauchy problem for the general Benjamin-Ono-Burgers equation

$$\frac{\partial}{\partial t}u+\mathcal{H}\frac{\partial^2}{\partial x^2}u-\alpha\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t), \tag{3.5}$$
$$v(x,0)=u_0(x), \tag{3.6}$$

and the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v+\mathcal{H}\frac{\partial^2}{\partial x^2}v-\alpha\frac{\partial^2}{\partial x^2}v=f(x,t), \tag{3.7}$$
$$v(x,0)=u_0(x), \tag{3.8}$$

See [8], [10], [14], [15], [18], [19], [20], [28], [29], [34], [33], [35].
(C) Consider the Cauchy problem for the general Benjamin-Bona-Mahony-Burgers equation

$$\frac{\partial}{\partial t}u-\frac{\partial^3}{\partial x^2\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial}{\partial x}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t), \tag{3.9}$$
$$v(x,0)=u_0(x), \tag{3.10}$$

and the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v-\frac{\partial^3}{\partial x^2\partial t}v-\alpha\frac{\partial^2}{\partial x^2}v+\beta\frac{\partial}{\partial x}v=f(x,t), \tag{3.11}$$
$$u(x,0)=u_0(x). \tag{3.12}$$

See [5], [6], [7], [41], [42].
These are the most popular nonlinear dissipative dispersive wave equations in one-dimensional space. The diffusion coefficient $\alpha>0$ is a positive constant, β is a real nonzero constant, the nonlinear function

$$\mathcal{N}(u)=au+bu^2+cu^3+du^4+eu^5,$$

where $a\in\mathbb{R}$, $b\in\mathbb{R}$, $c\in\mathbb{R}$, $d\in\mathbb{R}$ and $e\in\mathbb{R}$ are real constants. In the Benjamin-Ono-Burgers equation, $\mathcal{H}$ represents the Hilbert operator, defined by the principal value of the singular integral

$$[\mathcal{H}\phi](x)=\frac{1}{\pi}\text{ P. V. }\int_{\mathbb{R}}\frac{\phi(y)}{x-y}\mathrm{d}y,$$

for all functions $\phi\in L^2(\mathbb{R})$.

Suppose that the initial function and the external force satisfy

$$u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),$$

$$
\begin{aligned}
& f \in C^{\infty}(\mathbb{R} \times \mathbb{R}^{+}) \cap L^{1}(\mathbb{R} \times \mathbb{R}^{+}) \cap L^{1}(\mathbb{R}^{+}, L^{2}(\mathbb{R})), \\
& \int_{\mathbb{R}} u_0(x) \mathrm{d}x \neq 0, \qquad \int_0^{\infty} \int_{\mathbb{R}} f(x,t) \mathrm{d}x \mathrm{d}t \neq 0, \\
& \int_{\mathbb{R}} u_0(x) \mathrm{d}x + \int_0^{\infty} \int_{\mathbb{R}} f(x,t) \mathrm{d}x \mathrm{d}t \neq 0.
\end{aligned}
$$

However, we do not make the following assumption

$$
u_0 \in H^{2m}(\mathbb{R}), \qquad f \in L^2(\mathbb{R}^{+}, H^{2m}(\mathbb{R})),
$$

for some positive constant $m > 0$. Can we establish the following exact limits

$$
\begin{aligned}
& \lim_{t \to \infty} \left\{ t^{2m+1/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m u(x,t) \right|^2 \mathrm{d}x \right\}, \\
& \lim_{t \to \infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m [u(x,t) - v(x,t)] \right|^2 \mathrm{d}x \right\},
\end{aligned}
$$

in terms of α, m and the following integrals

$$
\int_{\mathbb{R}} u_0(x) \mathrm{d}x, \qquad \int_0^{\infty} \int_{\mathbb{R}} f(x,t) \mathrm{d}x \mathrm{d}t,
$$

for all constants $m \geq 0$?

The main difficulty is that the integral

$$
\int_0^{\infty} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \mathrm{d}t
$$

is divergent - it does not exist.

Open Problem 2: Consider the Cauchy problem for the Burgers equation

$$
\begin{aligned}
& \frac{\partial}{\partial t} u - \alpha \frac{\partial^2}{\partial x^2} u + \beta \frac{\partial}{\partial x} (u^2) = 0, \\
& u(x,0) = u_0(x).
\end{aligned}
$$

In this problem, the positive constant $\alpha > 0$ represents the diffusion coefficient, and the real constant $\beta \neq 0$. The initial function $u_0 \in L^1(\mathbb{R})$. It is well known that this problem is equivalent to the Cauchy problem for the heat equation

$$
\begin{aligned}
& \frac{\partial}{\partial t} w - \alpha \frac{\partial^2}{\partial x^2} w = 0, \\
& w(x,0) = w_0(x),
\end{aligned}
$$

under the famous Hopf-Cole transformation

$$
w(x,t) = \exp\left\{ -\frac{\beta}{\alpha} \int_{-\infty}^{x} u(y,t) \mathrm{d}y \right\}, w_0(x) = \exp\left\{ -\frac{\beta}{\alpha} \int_{-\infty}^{x} u_0(y) \mathrm{d}y \right\}.
$$

Can we accomplish the following exact limits

$$
\lim_{t \to \infty} \left\{ t^{2m+1/2} \int_{\mathbb{R}} \left| \left(-\frac{\partial^2}{\partial x^2} \right)^m u(x,t) \right|^2 \mathrm{d}x \right\},
$$

for all real constants $m \geq 0$, in terms of α, β, m and the integral

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x?$$

Suppose that $u_0 \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x \neq 0.$$

This problem has been open for a long time.

Open problem 3: For the Cauchy problem (1)-(2), can we represent the integral

$$\int_0^\infty \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t$$

in terms of the diffusion coefficient α and the following integrals

$$\int_{\mathbb{R}} \phi(x)\mathrm{d}x, \qquad \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t?$$

3.4. Some Technical Lemmas Used

Lemma 3.1. *There holds the Parseval's identity*

$$\int_{\mathbb{R}} |\phi(x)|^2 \mathrm{d}x = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi,$$

for all functions $\phi \in L^2(\mathbb{R})$.

Lemma 3.2. *Let* $m > 0$ *be a positive constant. There holds the following estimate*

$$\begin{aligned} &\left\{ \int_{\mathbb{R}} |(-\triangle)^m [\phi(x)\psi(x)]| \mathrm{d}x \right\}^2 \\ &\leq 4^m \left\{ \int_{\mathbb{R}} |\phi(x)|^2 \mathrm{d}x \right\} \left\{ \int_{\mathbb{R}} |(-\triangle)^m \psi(x)|^2 \mathrm{d}x \right\} \\ &+ 4^m \left\{ \int_{\mathbb{R}} |(-\triangle)^m \phi(x)|^2 \mathrm{d}x \right\} \left\{ \int_{\mathbb{R}} |\psi(x)|^2 \mathrm{d}x \right\}, \end{aligned}$$

for all functions $\phi \in H^{2m}(\mathbb{R})$ *and* $\psi \in H^{2m}(\mathbb{R})$.

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