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Exact Solutions of a (4+1)-Dimensional Boiti-Leon-Manna-Pempinelli (BLMP) Equation

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Abstract This paper focuses on constructing the exact solutions of a (4+1)-dimensional BLMP equation via the two variables $(G'/G, 1/G)$-expansion method and extended generalized Riccati equation mapping method. Firstly, the main ideas of the methods are described. Then, the methods are applied to the equation to derive the exact solutions including singular, kink (or anti-kink) and periodic solutions. Finally, the 3D plots of some exact solutions are observed graphically and intuitively by assigning the values of unknown parameters. The results prove that the methods are powerful, enriching the diversity of forms of exact solutions.

Keywords (4+1)-dimensional BLMP equation, exact solution, the two variables $(G'/G, 1/G)$-expansion method, the extended generalized Riccati equation mapping method

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1. Introduction

Nonlinear evolution equations (NLEEs) are frequently adopted to explicate complicated physical phenomena in numerous fields such as mathematical physics, chaos, quantum field theory, plasma physics, oceanography, etc. The completely integrable systems are claimed to be exactly solvable models among NLEEs. High-dimensional NLEEs are closer to actual natural phenomena and have more complex behavior. In order to study in depth the dynamic processes described by high-dimensional models, it is growing increasingly compelling to establish exact solutions that imply many physical properties of high-dimensional NLEEs. With the progress of technology and the efforts of researchers, extensive well-validated methods for solving fascinating nonlinear models are successively adapted, for instance, Darboux transformation [1], mETF method [2, 3], bifurcation analysis [4, 5], extended generalized Riccati equation mapping method [6–9], Hirota bilinear method [10], two variables $(G'/G, 1/G)$-expansion method [11–13], linear superposition method [14, 15], Lie symmetry method [16–18], etc. [19–21].

A (4+1)-dimensional BLMP equation [22] proposed by Xu and Wazwaz shall be studied, which reads

$$\begin{aligned} &\omega_t\left(\omega_y+\omega_z+\omega_s\right)+\sigma(\omega_y+\omega_z+\omega_s)_{xxx} \\ &+\mu\left(\omega_x\left(\omega_y+\omega_z+\omega_s\right)+\omega_{xx}\left(\omega_y+\omega_z+\omega_s\right)\right)=0, \end{aligned} \tag{1.1}$$

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where σ, μ are non-zero parameters and $\omega = \omega(x, y, z, s, t)$. Here, x, y, z, s represent spatial variables while t means time. Eq. (1.1) can be regarded as being evolved from the KdV equation in (4+1)-dimensions.

In recent years, breakthroughs have been gained in the research of Eq. (1.1). Among them, Painlevé properties held for Eq. (1.1) and Lax pair, bilinear Bäcklund transformation and infinite conservation laws were first considered by Xu and Wazwaz [22]. Hao [23] revealed block solitons, block kinks, periodic block solutions through the heuristic function method. Resonant multi-solitons and rational solutions were constructed by Kuo [24] and Hoessini et al. [25] via the linear superposition method, respectively. Raheel et al. [26] explored new solutions including periodic cross-kink wave solutions as well as interaction between kink solitary and rogue wave and secured these solutions. Moreover, the generalized exponential rational function method was utilized to derive explicit solitary wave solutions by Rasool et al. [27]. Motivated by them, we aim to explore exact solutions of Eq. (1.1) via the two variables $(G'/G, 1/G)$-expansion method and extended generalized Riccati equation mapping method in this paper.

It is worth pointing out that Eq. (1.1) can be converted into the following (2+1) and (3+1)-dimensional BLMP equations which catch people's eyes.

(i) When $\sigma = \mu = -1$ and $\omega = \omega(x, y, t)$, Eq. (1.1) is reduced to a (2+1)-dimensional BLMP equation presented by Boiti et al. [28] as follows

$$\omega_{yt} - \omega_{xxxy} - \omega_{xx}\omega_y - \omega_x\omega_{xy} = 0, \tag{1.2}$$

which is widely accepted to study incompressible liquids.

(ii) When $\sigma = 1, \mu = -3$ and $\omega = \omega(x, y, t)$, another form provided by Gilson et al. [29] is

$$\omega_{yt} + \omega_{xxxy} - 3\omega_{xx}\omega_y - 3\omega_x\omega_{xy} = 0, \tag{1.3}$$

whose integrability properties were verified by Luo [30] and various types of solutions were offered in [29–31].

(iii) When $\sigma = 1, \mu = -3$ and $\omega = \omega(x, y, z, t)$, a (3+1)-dimensional BLMP equation is formed as follows

$$\omega_{yt} + \omega_{zt} + \omega_{xxxy} + \omega_{xxxz} - 3\left(w_x\left(w_{xy} + w_{xz}\right) + w_{xx}\left(w_y + w_z\right)\right) = 0, \tag{1.4}$$

which describes the propagation of fluid. Numerous methods have been applied to construct lump-kink, multi-soliton, breather wave solutions, Painlevé analysis, Hirotas bilinear representation and so on [32–36].

The remaining plots are programmed as follows. Section 2 concisely provides the central thoughts of the two variables $(G'/G, 1/G)$-expansion method and extended generalized Riccati equation mapping method. The two methods are successively applied to Eq. (1.1) to summarize exact solutions including singular, kink (or anti-kink) and periodic solutions in Section 3. Section 4 performs some solutions graphically by using suitable parametric selections. Section 5 provides the discussion and comparisons. A summary is placed in Section 6.

2. Description of the methods

This section gives the brief steps of the methods considered. Discuss a NLPDE which is shown as

$$\Upsilon\left(\omega, \omega_t, \omega_{x_i}, \omega_{x_i x_j}, \omega_{x_i x_j x_\mu}, \cdots\right) = 0, \tag{2.1}$$

where t and x_i $(i = 1, 2, \cdots)$ represent independent variables, and Υ is a polynomial about ω and its partial derivatives. Through wave transformation

$$\omega(x, y, z, s, t) = \omega(\zeta), \quad \zeta = kx + my + nz + ls - vt, \tag{2.2}$$

where k, m, n, l, v are constants, and Eq. (2.1) becomes

$$\Upsilon\left(\omega, \omega', \omega'', \cdots\right) = 0. \tag{2.3}$$

2.1. The two variables $(G'/G, 1/G)$-expansion method

Such a solution of Eq. (2.3) is supposed as

$$\omega(\zeta) = \sum_{p=0}^{K} \alpha_p \varpi^p(\zeta) + \sum_{q=1}^{K} \beta_q \pi^q(\zeta), \tag{2.4}$$

where α_p, β_q $(p = 0, 1, \cdots, K; q = 1, 2, \cdots, K)$ are unsettled coefficients and $\alpha_K \neq 0$, $\beta_K \neq 0$. ϖ and π are defined as

$$\varpi(\zeta) = \frac{G^{'}(\zeta)}{G(\zeta)}, \ \pi(\zeta) = \frac{1}{G(\zeta)}. \tag{2.5}$$

Additionally, $G(\zeta)$ meets

$$G^{''} = -\rho G + \tau. \tag{2.6}$$

Along with Eq. (2.5) and Eq. (2.6), we get

$$\begin{aligned} &\varpi' = -\varpi^2 + \tau\pi - \rho, \\ &\pi' = -\varpi\pi. \end{aligned} \tag{2.7}$$

The solutions of Eq. (2.6) are divided into three cases:

(i) When $\rho < 0$, Eq. (2.6) generates a hyperbolic function solution

$$G(\zeta) = Q_1 \sinh(\sqrt{-\rho}\zeta) + Q_2 \cosh(\sqrt{-\rho}\zeta) + \frac{\tau}{\rho}, \tag{2.8}$$

and we obtain

$$\pi^2 = \frac{-\rho}{\rho^2\varrho + \tau^2}\left(\varpi^2 - 2\tau\pi + \rho\right), \tag{2.9}$$

where $\varrho = Q_1^2 - Q_2^2$.

(ii) When $\rho > 0$, the trigonometric function solution of Eq. (2.6) is generated as

$$G(\zeta) = Q_1 \sin(\sqrt{\rho}\zeta) + Q_2 \cos(\sqrt{\rho}\zeta) + \frac{\tau}{\rho}, \tag{2.10}$$

so we have

$$\pi^2 = \frac{\rho}{\rho^2\varrho - \tau^2}\left(\varpi^2 - 2\tau\pi + \rho\right), \tag{2.11}$$

where $\varrho = {Q_2}^2 + {Q_1}^2$.

(iii) When $\rho = 0$, the rational function solution of Eq. (2.6) is yielded as

$$G(\zeta) = \frac{\tau}{2}\zeta^2 + Q_1\zeta + Q_2, \tag{2.12}$$

and we find

$$\pi^2 = \frac{1}{{Q_1}^2 - 2\tau Q_2}\left(\varpi^2 - 2\tau\pi\right), \tag{2.13}$$

where Q_1 and Q_2 are arbitrary constants.

According to the homogeneous balance principle, the value of K can be determined.

When $\rho < 0$, taking (2.4) into Eq. (2.3) along with (2.7) and (2.9), we get a polynomial about ϖ and π whose degree is not larger than one on the left of Eq. (2.3). Assume that the coefficient of each term is equal to zero, which yields a set of algebraic equations that derive the values of α_p, β_q $(p = 0, 1, \cdots, K; q = 1, 2, \cdots, K)$, k, m, n, l and v. Similarly, when $\rho > 0$ and $\rho = 0$, the values of α_p, β_q $(p = 0, 1, \cdots, K; q = 1, 2, \cdots, K)$, k, m, n, l and v can also be found.

2.2. The extended generalized Riccati equation mapping method

We suppose that Eq. (2.3) has the solution

$$\omega(\zeta) = \sum_{p=0}^{K} \vartheta_p \left(\frac{G'}{G}\right)^p, \tag{2.14}$$

where ϑ_p $(p = 1, 2, \cdots, K)$ can be determined later with $\vartheta_K \neq 0$ and $G = G(\zeta)$ satisfies

$$G' = \kappa + \varepsilon G + \delta G^2, \tag{2.15}$$

where κ, ε, δ are arbitrary constants and $\delta \neq 0$.

Based on the homogeneous balance, we calculate the value of K.

Inserting (2.14) along with (2.15) into Eq. (2.3), we collect all coefficients of G^j, G^{-j} $(j = 0, 1, 2, \cdots)$ to be zero on the left of Eq. (2.3), which yields algebraic equations that determine the values of κ, ε, δ, ϑ_p $(p = 0, 1, \cdots, K)$, k, m, n, l and v.

Additionally, for Eq. (2.15), it has twenty-seven solutions which are not listed here, but detailed in [6,7,9].

3. Application of the methods

Consider wave transformation (2.2), which converts Eq. (1.1) into an ODE that reads

$$(l + m + n)\left(k^3\alpha g^{(4)} + 2\beta k^2 (g')(g'') - v g''\right) = 0. \tag{3.1}$$

Integrate once with the integral constant is equal to zero, which leads to

$$(l + m + n)\left(k^3\alpha g^{(3)} + \beta k^2 (g')^2 - v g'\right) = 0. \tag{3.2}$$

Subsequently, balancing $g^{(3)}$ and $(g')^2$, we obtain

$$K + 3 = 2(K + 1) \Rightarrow K = 1.$$

3.1. Application of the two variables $(G'/G, 1/G)$-expansion method

Firstly, Eq. (1.1) shall be explored by executing the two variables $(G'/G, 1/G)$-expansion method. Since (2.4), we reach easily

$$\omega(\zeta) = \alpha_0 + \alpha_1 \varpi + \beta_1 \pi, \quad \alpha_1, \ \beta_1 \neq 0, \tag{3.3}$$

where α_0, α_1 and β_1 are constants to be confirmed later.

Case I: When $\rho < 0$, determining equations are enumerated through taking (3.3) along with (2.7), (2.9) into Eq. (3.2), which leads to the following results:

$$\begin{aligned} &\alpha_0 = \alpha_0, \ \alpha_1 = \frac{3\sigma k}{\mu}, \ \beta_1 = \frac{3k\sigma\sqrt{-\rho\left(\rho^2\varrho + \tau^2\right)}}{\rho\mu}, \\ &k = k, \ m = m, \ n = n, \ l = l, \ v = -\sigma k^3 \rho. \end{aligned}$$

Substituting the values of the above parameters into Eq. (3.3), we have the solution of Eq. (1.1)

$$\begin{aligned} \omega_1(\zeta) = &\frac{3\sigma k\sqrt{-\rho}\left(Q_1 \cosh\left(\sqrt{-\rho}\zeta\right) + Q_2 \sinh\left(\sqrt{-\rho}\zeta\right)\right)}{\mu\left(Q_1 \sinh\left(\sqrt{-\rho}\zeta\right) + Q_2 \cosh\left(\sqrt{-\rho}\zeta\right) + \frac{\tau}{\rho}\right)} \\ &+ \frac{3k\sigma\sqrt{-\rho\left(\rho^2\varrho + \tau^2\right)}}{\rho\mu\left(Q_1 \sinh\left(\sqrt{-\rho}\zeta\right) + Q_2 \cosh\left(\sqrt{-\rho}\zeta\right) + \frac{\tau}{\rho}\right)} + \alpha_0. \end{aligned} \tag{3.4}$$

Alternatively, if $\tau = 0$ and $Q_1 = 0$, $Q_2 \neq 0$ or $Q_2 = 0$, $Q_1 \neq 0$ are chosen, solution (3.4) respectively becomes

$$\omega_2(\zeta) = \frac{3\sigma k\sqrt{-\rho}}{\mu} \tanh\left(\sqrt{-\rho}\zeta\right) - \frac{3k\sigma\sqrt{\rho\varrho}}{\mu Q_2} \operatorname{sech}\left(\sqrt{-\rho}\zeta\right) + \alpha_0, \tag{3.5}$$

and

$$\omega_3(\zeta) = \frac{3\sigma k\sqrt{-\rho}}{\mu} \coth\left(\sqrt{-\rho}\zeta\right) - \frac{3k\sigma\sqrt{-\rho\varrho}}{\mu Q_1} \operatorname{csch}\left(\sqrt{-\rho}\zeta\right) + \alpha_0, \tag{3.6}$$

where $\zeta = kx + my + nz + ls - \sigma\rho k^3 t$ and $\varrho = Q_1^2 - Q_2^2$.

Case II: When $\rho > 0$, we work out the following results:

$$\begin{aligned} &\alpha_0 = \alpha_0, \quad \alpha_1 = \frac{3\sigma k}{\mu}, \quad \beta_1 = \frac{3k\sigma\sqrt{\rho\left(\rho^2\varrho - \tau^2\right)}}{\rho\mu}, \\ &k = k, \quad m = m, \quad n = n, \quad l = l, \quad v = -\sigma k^3 \rho. \end{aligned}$$

Therefore, the periodic solution of Eq. (1.1) is derived as

$$\begin{aligned} \omega_4(\zeta) = &\frac{3\sigma k\sqrt{\rho}\left(Q_1 \cos\left(\sqrt{\rho}\zeta\right) - Q_2 \sin\left(\sqrt{\rho}\zeta\right)\right)}{\mu\left(Q_1 \sin\left(\sqrt{\rho}\zeta\right) + Q_2 \cos\left(\sqrt{\rho}\zeta\right) + \frac{\tau}{\rho}\right)} \\ &+ \frac{3k\sigma\sqrt{\rho\left(\rho^2\varrho - \tau^2\right)}}{\rho\mu\left(Q_1 \sin\left(\sqrt{\rho}\zeta\right) + Q_2 \cos\left(\sqrt{\rho}\zeta\right) + \frac{\tau}{\rho}\right)} + \alpha_0, \end{aligned} \tag{3.7}$$

where $\zeta = kx + my + nz + ls - \sigma\rho k^3 t$ and $\varrho = Q_1^2 + Q_2^2$.

When we sign $\tau=0$ and $Q_1=0$, $Q_2\neq 0$ or $Q_2=0$, $Q_1\neq 0$ in (3.7), we receive

$$\omega_5(\zeta)=-\frac{3\sigma k\sqrt{\rho}}{\mu}\tan\left(\sqrt{\rho}\zeta\right)+\frac{3k\sigma\sqrt{\rho\varrho}}{\mu Q_2}\sec\left(\sqrt{\rho}\zeta\right)+\alpha_0, \tag{3.8}$$

and

$$\omega_6(\zeta)=\frac{3\sigma k\sqrt{\rho}}{\mu}\cot\left(\sqrt{\rho}\zeta\right)+\frac{3k\sigma\sqrt{\rho\varrho}}{\mu Q_1}\csc\left(\sqrt{\rho}\zeta\right)+\alpha_0, \tag{3.9}$$

where $\zeta=kx+my+nz+ls-\sigma\rho k^3t$ and $\varrho=Q_1^2+Q_2^2$.

3.2. Application of the extended generalized Riccati equation mapping method

Due to (2.14), we judge

$$\omega(\zeta)=\vartheta_0+\vartheta_1\left(\frac{G'}{G}\right),\quad \vartheta_1\neq 0, \tag{3.10}$$

where ϑ_1 and ϑ_0 are parameters to be fixed.

Take (3.10) along with (2.15) into Eq. (3.2) and gather coefficients of G^j, G^{-j} ($j=0,1,2,3,4$) to be zero, which generates algebraic equations about κ, ε, δ, ϑ_1, ϑ_0, k, m, n, l and v. Solving them, we receive

$$\begin{aligned}&\vartheta_0=\vartheta_0,\quad \vartheta_1=-\frac{6k\sigma}{\mu},\quad \kappa=0,\quad \varepsilon=\frac{\sqrt{k\sigma v}}{k^2\sigma},\\&k=k,\quad m=m,\quad n=n,\quad l=l,\quad v=v.\end{aligned}$$

Thus, it demonstrates that Eq. (1.1) contains periodic, kink, singular and soliton solutions as follows:

When $\varepsilon^2-4\delta\kappa>0$ and $\varepsilon\delta\neq 0$, Eq. (1.1) has solutions

$$\omega_{\tilde{1}}=\vartheta_0-\frac{3v}{\mu k^2\cosh^2\left(\sqrt{\frac{v}{4k^3\sigma}}\zeta\right)\left(\frac{\sqrt{k\sigma v}}{k^2\sigma}+\sqrt{\frac{v}{k^3\sigma}}\tanh\left(\sqrt{\frac{v}{4k^3\sigma}}\zeta\right)\right)},$$

$$\omega_{\tilde{2}}=\vartheta_0+\frac{3v}{\mu k^2\left(\cosh^2\left(\sqrt{\frac{v}{4k^3\sigma}}\zeta\right)-1\right)\left(\frac{\sqrt{k\sigma v}}{k^2\sigma}+\sqrt{\frac{v}{k^3\sigma}}\coth\left(\sqrt{\frac{v}{4k^3\sigma}}\zeta\right)\right)},$$

$$\omega_{\tilde{3}}=\vartheta_0\pm\frac{6v\left(i\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\mp 1\right)}{k^2\mu\sqrt{\frac{v}{k^3\sigma}}\cosh^2\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\left(1+\left(\tanh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\pm i\operatorname{sech}\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)\right)},$$

$$\begin{aligned}\omega_{\tilde{4}}=&\frac{\pm 6v}{k^2\mu\left(\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\mp 1\right)\left(\frac{\sqrt{k\sigma v}}{k^2\sigma}+\sqrt{\frac{v}{k^3\sigma}}\left(\coth\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\pm\operatorname{csch}\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)\right)}\\&+\vartheta_0,\end{aligned}$$

$$\begin{aligned}\omega_{\tilde{5}}=&\vartheta_0+\frac{3v}{2k^2\mu\cosh^2\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\left(\cosh^2\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-1\right)}\\&\cdot\frac{1}{\left(\frac{2\sqrt{k\sigma v}}{k^2\sigma}+\sqrt{\frac{v}{k^3\sigma}}\left(\frac{1}{4}\tanh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+\coth\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)\right)},\end{aligned}$$

$$\omega_{\tilde{6}}=\frac{6k\sigma A\left(\frac{Bv}{k^3\sigma}\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+\frac{v\sqrt{A^2+B^2}}{k^3\sigma}\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\frac{Av}{k^3\sigma}\right)}{\mu\left(A\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+B\right)^2\left(-\frac{\sqrt{k\sigma v}}{k^2\sigma}+\frac{\sqrt{\frac{v\left(A^2+B^2\right)}{k^3\sigma}}-A\sqrt{\frac{v}{k^3\sigma}}\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)}{A\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+B}\right)}+\vartheta_0,$$

$$\omega_{\tilde{7}}=\frac{6k\sigma A\left(-\frac{Bv}{k^3\sigma}\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+\frac{v\sqrt{B^2-A^2}}{k^3\sigma}\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\frac{Av}{k^3\sigma}\right)}{\mu\left(A\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+B\right)^2\left(-\frac{\sqrt{k\sigma v}}{k^2\sigma}-\frac{\sqrt{\frac{v\left(B^2-A^2\right)}{k^3\sigma}}+A\sqrt{\frac{v}{k^3\sigma}}\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)}{A\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+B}\right)}+\vartheta_0,$$

where A and B satisfy $B^2-A^2>0$ and are non-zero real constants.

$$\omega_{\tilde{8}}=\frac{3v}{k^2\mu\left(\sqrt{\frac{v}{k^3\sigma}}\sinh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\frac{\sqrt{k\sigma v}}{k^2\sigma}\cosh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)\cosh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)}+\vartheta_0,$$

$$\omega_{\tilde{9}}=\frac{3v\left(-\sqrt{\frac{v}{k^3\sigma}}\cosh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+\frac{\sqrt{k\sigma v}}{k^2\sigma}\sinh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)}{k^2\mu\left(\sqrt{\frac{v}{k^3\sigma}}\cosh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\frac{\sqrt{k\sigma v}}{k^2\sigma}\sinh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)^2\sinh\left(\frac{1}{2}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)}+\vartheta_0,$$

$$\omega_{\tilde{10}}=\vartheta_0\pm\frac{6v\left(i\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\pm 1\right)}{k^2\mu\sqrt{\frac{v}{k^3\sigma}}\left(\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\pm i\right)\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)},$$

$$\omega_{\tilde{11}}=\vartheta_0\pm\frac{-6v\left(\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\pm 1\right)}{k^2\mu\sqrt{\frac{v}{k^3\sigma}}\left(\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\pm 1\right)\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)},$$

$$\omega_{\tilde{12}}=-\frac{6\sqrt{\frac{v\sigma}{k}}\left(\cosh^2\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\sinh\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\cosh\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)}{\mu\left(\cosh\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\sinh\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)^2\sinh\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\cosh\left(\frac{1}{4}\sqrt{\frac{v}{k^3\sigma}}\zeta\right)}+\vartheta_0,$$

where $\zeta=kx+my+nz+ls-vt$.

When $\kappa=0$ and $\varepsilon\delta\neq 0$, Eq. (1.1) has solutions

$$\omega_{\tilde{13}}=\vartheta_0-\frac{6\sqrt{\sigma v}\left(\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)}{\sqrt{k}\mu\left(C_1+\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)-\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)},$$

$$\omega_{\tilde{14}}=\vartheta_0-\frac{6C_1\sqrt{\sigma v}}{\sqrt{k}\mu\left(C_1+\cosh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)+\sinh\left(\sqrt{\frac{v}{k^3\sigma}}\zeta\right)\right)},$$

where C_1 is an arbitrary constant and $\zeta=kx+my+nz+ls-vt$.

When $\kappa=\varepsilon=0$ and $\delta\neq 0$, Eq. (1.1) has a solution

$$\omega_{\tilde{15}}=\vartheta_0+\frac{6k\sigma\delta}{\mu\left(\delta\zeta+D_1\right)},$$

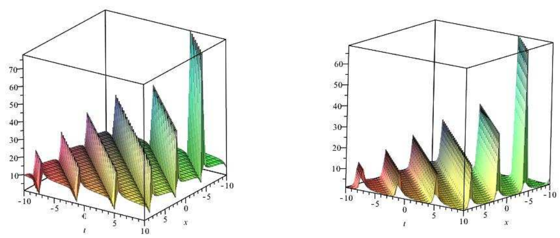
where D_1 is an arbitrary constant and $\zeta=kx+my+nz+ls-vt$.

4. Graphical representations

This section concentrates on illustrating some exact solutions graphically. By choosing appropriate values of parameters, we point out 3D plots and classify the solutions.

4.1. Periodic solutions

As presented in Figure. 1, exact solutions ω_2 and $\omega_{\tilde{3}}$ of Eq. (1.1) are revealed graphically for the range of $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$.



(a) solution ω_2 with $Q_1 = 2$, $Q_2 = 1$, $\sigma = -1$, $\mu = 3$, $k = 1$, $l = 1$, $m = -2$, $n = 0.1$, $\tau = 2$, $\alpha_0 = 10$, $\rho = 1$, $y = z = s = 0$.

(b) solution $\omega_{\tilde{3}}$ with $k = 5$, $v = l = 1$, $\sigma = 10$, $\mu = -3$, $m = -2$, $n = -0.1$, $\vartheta_0 = 1$, $y = z = s = 0$.

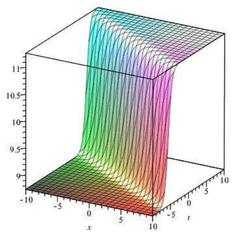
Figure 1. Periodic solutions of Eq. (1.1).

4.2. Kink (or anti-kink) solutions

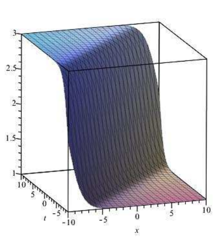
Figure. 2 indicates the 3D plots of the solutions ω_1, $\omega_{\tilde{1}}$, $\omega_{\tilde{4}}$, $\omega_{\tilde{6}}$, $\omega_{\tilde{7}}$, $\omega_{\tilde{8}}$, $\omega_{\tilde{10}}$ and $\omega_{\tilde{14}}$ by choosing suitable parameters, which shows that they are kink (or anti-kink) solutions. Among them, ω_1, $\omega_{\tilde{1}}$, $\omega_{\tilde{6}}$, $\omega_{\tilde{7}}$, and $\omega_{\tilde{14}}$ are anti-kink solutions while $\omega_{\tilde{4}}$, $\omega_{\tilde{8}}$ and $\omega_{\tilde{10}}$ are kink solutions.

4.3. Singular solutions

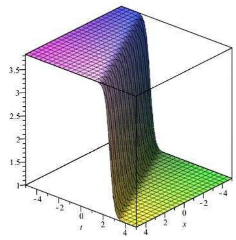
As shown in Figure. 3, we graphically present the singular solutions $\omega_{\tilde{2}}$, $\omega_{\tilde{5}}$, $\omega_{\tilde{9}}$, $\omega_{\tilde{11}}$, $\omega_{\tilde{12}}$, $\omega_{\tilde{13}}$ and $\omega_{\tilde{15}}$ for the range of $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ by selecting the appropriate values of parameters.



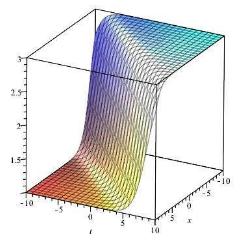
(a) solution ω_1 with $Q_1 = 1$, $Q_2 = -2$, $\sigma = -1$, $\mu = -3$, $k = -2$, $l = 1$, $m = -2$, $n = 0.1$, $\tau = 2$, $\alpha_0 = 10$, $\rho = -0.4$, $y = z = s = 0$.



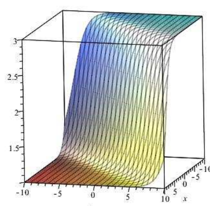
(b) solution $\omega_{\tilde{1}}$ with $k = 0.5$, $v = 0.5$, $l = 1$, $m = -2$, $n = 0.1$, $\sigma = 1$, $\mu = -3$, $\vartheta_0 = 1$, $y = z = s = 0$.



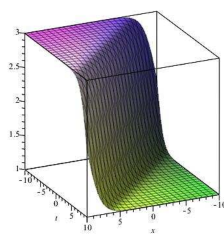
(c) solution $\omega_{\tilde{4}}$ with $k = -0.5$, $v = -l$, $\sigma = l = 1$, $\mu = 3$, $m = -2$, $n = 0.1$, $\vartheta_0 = 1$, $y = z = s = 0$.



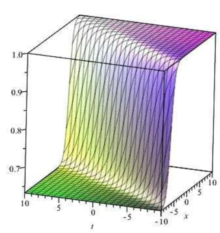
(d) solution $\omega_{\tilde{6}}$ with $k = 0.1$, $v = 0.l$, $\sigma = l = 1$, $\mu = -3$, $m = -2$, $n = 0.1$, $\vartheta_0 = 1$, $y = z = s = 0$, $A = 1$, $B = 2$.



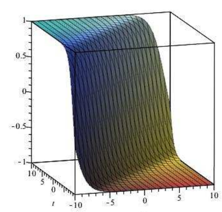
(e) solution $\omega_{\tilde{7}}$ with $k = 0.1$, $v = 0.l$, $\sigma = l = 1$, $\mu = -3$, $m = -2$, $n = 0.1$, $\vartheta_0 = 1$, $y = z = s = 0$, $A = 1$, $B = 2$.



(f) solution $\omega_{\tilde{8}}$ with $k = 0.1$, $v = 0.l$, $\sigma = l = 1$, $\mu = -3$, $m = -2$, $n = 0.1$, $\vartheta_0 = 1$, $y = z = s = 0$.

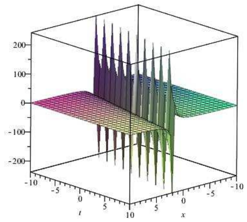


(g) solution $\omega_{\tilde{10}}$ with $k = -1.5$, $v = -l$, $\sigma = l = 1$, $\mu = -3$, $m = -2$, $n = 0.1$, $\vartheta_0 = 1$, $y = z = s = 0$.

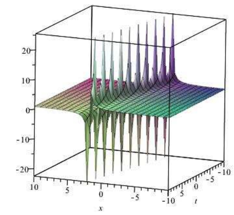


(h) solution $\omega_{\tilde{14}}$ with $k = -0.5$, $v = -0.5$, $\sigma = l = 1$, $\mu = -3$, $m = -2$, $n = 0.1$, $\vartheta_0 = 1$, $y = z = s = 0$, $C_1 = 2$.

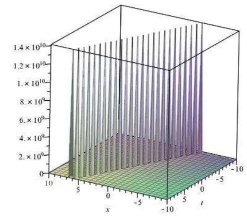
Figure 2. Kink (or anti-kink) solutions of Eq. (1.1).



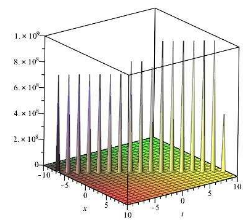
(a) solution $\omega_{\bar{2}}$ with $k = 5,\ v = l = 1,\ m = -2,\ n = -0.1,\ \sigma = 10,\ \mu = -3,\ \vartheta_0 = 1,\ y = z = s = 0$.



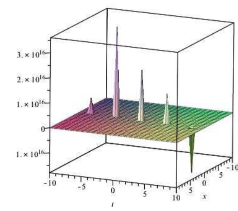
(b) solution $\omega_{\bar{5}}$ with $k = 0.5,\ v = 0.1,\ \sigma = l = 1,\ m = -2,\ n = 0.1,\ \mu = -3,\ \vartheta_0 = 1,\ y = z = s = 0$.



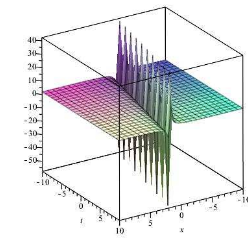
(c) solution $\omega_{\bar{9}}$ with $k = -0.8,\ v = -0.4,\ m = -2,\ n = 0.1,\ \sigma = l = 1,\ \mu = 3,\ \vartheta_0 = 1,\ s = 1,\ y = z = 0$.



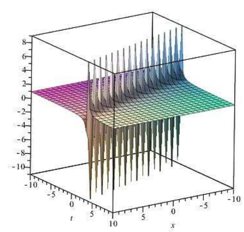
(d) solution $\omega_{\bar{11}}$ with $k = 1.5,\ \sigma = v = l = 1,\ \mu = -3,\ m = -2,\ n = 0.1,\ \vartheta_0 = 1,\ y = z = s = 0$.



(e) solution $\omega_{\bar{12}}$ with $k = 2,\ v = 0.2,\ \sigma = l = 1,\ \mu = -3,\ m = -2,\ n = 0.1,\ \vartheta_0 = 1,\ y = z = s = 0$.



(f) solution $\omega_{\bar{13}}$ with $k = 5,\ v = 1,\ C_1 = -1,\ l = 1,\ m = -2,\ n = 0.1,\ \sigma = 1,\ \mu = -3,\ \vartheta_0 = 1,\ s = 1,\ y = z = 0$.



(g) solution $\omega_{\bar{15}}$ with $v = 3,\ k = \sigma = l = 1,\ \mu = -3,\ m = -2,\ n = 0.1,\ \vartheta_0 = 1,\ y = z = 0,\ s = 2,\ D_1 = -1,\ \delta = 1$.

Figure 3. Singular solutions of Eq. (1.1).

5. Discussion and comparisons

The two methods chosen are skilled at constructing more types of new solutions with different physical structures, including soliton solutions, periodic solutions and singular solutions. The newly obtained solutions are described in the forms of hyperbolic, trigonometric, rational functions, etc. Since they are both based on the homogeneous balance principe, we can roughly know the forms of the solutions in advance. And the methods are simple, effective and reliable. However, the two methods are limited by the auxiliary equations to extract the above solutions and are only applicable to solving the equations that contain the highest-order derivative term and nonlinear terms. And sometimes fewer types of solitons are constructed.

By comparing our results with those in other literature, our methods can obtain solutions in more diverse forms, including trigonometric, hyperbolic, rational and complex function solutions, but are limited by the auxiliary equations, whereas the generalized exponential rational function method applied by Rasool et al. [27] didn't rely on the auxiliary equation and constructed more general solutions and novel multiple soliton solutions. The Hirota bilinear method and the extended (G'/G)-expansion method used by Raheel et al. [26] established interaction solutions and obtained more solutions than ours, while we get multi-types of exact solutions by two simpler methods, which proves that our methods have restrictions on constructing multiple soliton solutions.

6. Conclusion

This paper focuses on establishing some exact solutions of a (4+1)-dimensional BLMP equation. By using the two variables $(G'/G, 1/G)$-expansion method and extended generalized Riccati equation mapping method, the parametric expressions of exact solutions that contain singular, kink (or anti-kink) and periodic solutions are provided, enriching the diversity of exact solutions. For some solutions obtained, we have illustrated the 3D plots of exact solutions graphically by fixing the values of parameters. We compare our results with other studies and prove the validity of our methods.

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References

[1] X. H. Wu, Y. T. Gao, X. Yu, C. C. Ding and L. Q. Li, *Modified generalized Darboux transformation and solitons for a Lakshmanan-Porsezian-Daniel equation*, Chaos, Solitons & Fractals, 2022, 162, 112399.

[2] S. Rani, S. Kumar and N. Mann, *On the dynamics of optical soliton solutions, modulation stability, and various wave structures of a (2+1)-dimensional complex modified Korteweg-de-Vries equation using two integration mathematical methods*, Optical and Quantum Electronics, 2023, 55(8), 731, 27 pages.

[3] L. M. B. Alam and X. Jiang, *Exact and explicit traveling wave solution to the time-fractional phi-four and (2+1) dimensional CBS equations using the modified extended tanh-function method in mathematical physics*, Partial Differential Equations in Applied Mathematics, 2021, 4, 100039.

[4] J. Li and Z. Liu, *Smooth and non-smooth traveling waves in a nonlinearly dispersive equation*, Applied Mathematical Modelling, 2000, 25(1), 41–56.

[5] S. Kumar, N, Mann, H. Kharbanda and M. Inc, *Dynamical behavior of analytical soliton solutions, bifurcation analysis, and quasi-periodic solution to*

the (2+1)-dimensional Konopelchenko-Dubrovsky (KD) system, Analysis and Mathematical Physics, 2023, 13(3), 40, 30 pages.

[6] E. M. E. Zayed and A. G. Al-Nowehy, *Solitons and other solutions to the nonlinear Bogoyavlenskii equations using the generalized Riccati equation mapping method*, Optical and Quantum Electronics, 2017, 49, 1–23.

[7] I. Hamid and S. Kumar, *Symbolic computation and Novel solitons, traveling waves and soliton-like solutions for the highly nonlinear (2+1)-dimensional Schrödinger equation in the anomalous dispersion regime via newly proposed modified approach*, Optical and Quantum Electronics, 2023, 55(9), 755, 23 pages.

[8] S. Kumar, I. Hamid and M. A. Abdou, *Dynamic frameworks of optical soliton solutions and soliton-like formations to Schrödinger-Hirota equation with parabolic law non-linearity using a highly efficient approach*, Optical and Quantum Electronics, 2023, 55(14), 1261, 31 pages.

[9] H. Naher, F. A. Abdullah, *The modified Benjamin-Bona-Mahony equation via the extended generalized Riccati equation mapping method*, Applied Mathematical Sciences, 2012, 6(111), 5495–5512

[10] A. M. Wazwaz, *Multiple-soliton solutions for the KP equation by Hirotas bilinear method and by the tanh-coth method*, Applied Mathematics and Computation, 2007, 190(1), 633–640.

[11] M. M. Mamun, A. H. M. Shahadat, A. M. Ali and A. M. Wazwaz, *Some applications of the $(G'/G, 1/G)$-expansion method to find new exact solutions of NLEEs*, The European Physical Journal Plus, 2017, 132(6), 1–15.

[12] D. Kumar and G. C. Paul, *Solitary and periodic wave solutions to the family of nonlinear conformable fractional Boussinesq-like equations*, Mathematical Methods in the Applied Sciences, 2021, 44(4), 3138–3158.

[13] S. Duran, *Extractions of travelling wave solutions of (2+1)-dimensional Boiti-Leon-Pempinelli system via $(G'/G, 1/G)$-expansion method*, Optical and Quantum Electronics, 2021, 53(6), 299.

[14] C. K. Kuo, *Novel resonant multi-soliton solutions and inelastic interactions to the (3+1)- and (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equations via the simplified linear superposition principle*, The European Physical Journal Plus, 2021, 136(1), 1–11.

[15] E. M. E. Zayed and A. G. Al-Nowehy, *The multiple exp-function method and the linear superposition principle for solving the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation*, Zeitschrift fr Naturforschung A, 2015, 70(9), 775–779.

[16] S. Kumar, W. X. Ma, S. K. Dhiman and A. Chauhan, *Lie group analysis with the optimal system, generalized invariant solutions, and an enormous variety of different wave profiles for the higher-dimensional modified dispersive water wave system of equations*, The European Physical Journal Plus, 2023, 138(5), 434, 13 pages.

[17] S. Kumar and M. Niwas, *Analyzing multi-peak and lump solutions of the variable-coefficient Boiti-Leon-Manna-Pempinelli equation: a comparative study of the Lie classical method and unified method with applications*, Nonlinear Dynamics, 2023, 1–19.

[18] S. Kumar, D. Kumar and A. Kumar, *Lie symmetry analysis for obtaining the abundant exact solutions, optimal system and dynamics of solitons for a higher-dimensional Fokas equation*, Chaos, Solitons & Fractals, 2021, 142, 110507.

[19] S. Kumar, M. Niwas, M. S. Osman and M. A. Abdou, *Abundant different types of exact soliton solution to the (4+1)-dimensional Fokas and (2+1)-dimensional breaking soliton equations*, Communications in Theoretical Physics, 2021, 73(10), 105007.

[20] M. Niwas and S. Kumar, *Multi-peakons, lumps, and other solitons solutions for the (2+1)-dimensional generalized Benjamin-Ono equation: an inverse* (G'/G)*-expansion method and real-world applications*, Nonlinear Dynamics, 2023, 1–14.

[21] M. A. Abdou, L. Ouahid and S. Kumar, *Plenteous specific analytical solutions for new extended deoxyribonucleic acid (DNA) model arising in mathematical biology*, Modern Physics Letters B, 2023, 37(34), 2350173.

[22] G. Q. Xu and A. M. Wazwaz, *Integrability aspects and localized wave solutions for a new (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Nonlinear Dynamics, 2019, 98, 1379–1390.

[23] Q. Hao, *Exact Solution of (4+1)-Dimensional Boiti-Leon-Manna-Pempinelli Equation*, Advances in Mathematical Physics, 2023, 2023.

[24] C. K. Kuo, *Novel resonant multi-soliton solutions and inelastic interactions to the (3+1)- and (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equations via the simplified linear superposition principle*, The European Physical Journal Plus, 2021, 136(1), 1–11.

[25] K. Hosseini, W. X. Ma, R. Ansari, M. Mirzazadeh, R. Pouyanmehr and F. Samadani, *Evolutionary behavior of rational wave solutions to the (4+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Physica Scripta, 2020, 95(6), 065208.

[26] M. Raheel, A. Zafar, A. Bekir and K. U. Tariq, *Interaction between kink solitary wave and rogue wave, new periodic cross-kink wave solutions and other exact solutions to the (4+1)-dimensional BLMP model*, Journal of Ocean Engineering and Science, 2022. (https://doi.org/10.1016/j.joes.2022.05.020)

[27] T. Rasool, R. Hussain, H. Rezazadeh and D. Gholami, *The plethora of exact and explicit soliton solutions of the hyperbolic local (4+1)-dimensional BLMP model via GERF method*, Results in Physics, 2023, 46, 106298.

[28] M. Boiti, J. J. P. Leon, M. Manna and F. Pempinelli, *On the spectral transform of a Korteweg-de Vries equation in two spatial dimensions*, Inverse problems, 1986, 2(3), 271.

[29] C. R. Gilson, J. J. C. Nimmo and R. Willox, *A (2+1)-dimensional generalization of the AKNS shallow water wave equation*, Physics Letters A, 1993, 180(4-5), 337–345.

[30] L. Luo, *New exact solutions and Bäcklund transformation for Boiti-Leon-Manna-Pempinelli equation*, Physics Letters A, 2011, 375(7), 1059–1063.

[31] M. Kumar and A. K. Tiwari, *Soliton solutions of BLMP equation by Lie symmetry approach*, Computers & Mathematics with Applications, 2018, 75(4), 1434–1442.

[32] J. G. Liu, *Double-periodic soliton solutions for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation in incompressible fluid*, Computers & Mathematics with Applications, 2018, 75(10), 3604–3613.

[33] G. Xu, *Painlevé analysis, lump-kink solutions and localized excitation solutions for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Applied Mathematics Letters, 2019, 97, 81–87.

[34] J. Liu, Y. Zhang and I. Muhammad, *Resonant soliton and complexiton solutions for (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Computers & Mathematics with Applications, 2018, 75(11), 3939–3945.

[35] B. Q. Li and Y. L. Ma, *Multiple-lump waves for a (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation arising from incompressible fluid*, Computers & Mathematics with Applications, 2018, 76(1), 204–214.

[36] H. Ma, Y. Bai and A. Deng, *Exact three-wave solutions for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Advances in Difference Equations, 2013, 2013(1), 1–11.