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Mathematical Formulation and Investigation of Contamination in the System of Lakes: A Comparative Study

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Abstract. The contamination in water bodies is a big threat to the environment to control the water pollution, a model study is conducted. The pollution model for a system of three lakes that are interconnected by channels is taken into account. Three input models (periodic, exponentially decaying, and linear) were solved by employing a variational iteration technique coupled with various types of multipliers. Identification of exact multipliers for nth-order differential equations is also presented. All said models were examined mathematically. We noticed that exact Lagrange multipliers in the correction functional bring more accurate and efficient results that have been deliberated graphically. The results obtained via variational iteration method (VIM)-λ_E, already published work and the Runge-Kutta method of order four provides an excellent agreement. Additionally, the use of exact Lagrange multipliers with other techniques may be extended to other dynamical problems.

AMS subject classifications: 65L06, 34C20

Key words: Pollution, System of lakes model, Exact Langrage multipliers, Variational iteration method, Differential transform method, RK-4.

1 Introduction

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Nowadays, a very severe issue for our environment is pollution. Pollution is the presentation of impurities into the indigenous habitat that cause antagonistic change. Pollution can appear as synthetic substances or vitality, for example, heat, light, and noise. Pollutions, the segments of contamination, can be either remote energies/substances or normally happening contaminants. The contamination in water bodies like rivers, lakes, aquifers, canals, oceans, and groundwater is known as water pollution. This type of environmental degradation happens when contaminations are indirectly or directly released into water forms without sufficient treatments to eliminate dangerous compounds. It is important to save our environment through proper planning for this threat. Monitoring of this issue using differential equations is probable now. A model [1] was introduced to study the pollution of a system of lakes. Three altered input mathematical simulations were presented to observe the pollution in a lake.

Theoretical investigation of physical and dynamical systems has huge importance in engineering and sciences. Mathematical modeling and solutions of these modeled problems via mathematical algorithms are very significant. In the past, many researchers devoted their attention to develop new algorithms or to extend existing techniques. Analytical [1-5], numerical [6-8] and wavelets [9-10] based algorithms have been utilized to explore differential equations. A lot of research work has been done using analytical techniques, which is the evidence analytical techniques [2-5] provide better approximations for physical models. An analytical technique that provides rapid convergence is presented by a Chinese mathematician He [11]. A careful survey of the literature revealed that VIM is more effective and proficient in getting the numerical and analytic solutions of nonlinear models. Ramous [12], Tatari, and Dehghan [12] proved the convergence of the variational iteration method. Later, various scholars used this method to find the solution to nonlinear problems. He constructs the approximate solution of strongly nonlinear equations [14-15]. The cited work [16-19] endorses that VIM is a comparatively better technique for obtaining the solution of nonlinear problems.

Many modifications were made in VIM to find appropriate solutions to various types of physical problems. Noor and Mohyud-Din [20] developed the elegant coupling of Adomian's polynomials with correction functional named it as modified variational iteration method (MVIM). Nadjafi and Ghorbani [21] introduced another modification in VIM. Herisanu and Marinca's modification was much more attractive, where the variational iteration method coupled with the least squares technology [22]. Yilmaz and Inc constructed a variational iteration algorithm, wherein they introduced an auxiliary parameter [23]. Hosseini et al. [24] proposed a new algorithm using an auxiliary parameter in the variational iteration method. Coupling of Taylor's series with correction functional cited in [25]. Recently, the solution of delay differential equations was tackled by VIM [26].

J. L. Lagrange was the pioneer who introduced the concept of Lagrange multiplier. He solved an isoperimetric problem in the calculus of variations. It is a constrained variational principle to maximize a closed area with a fixed perimeter. Lagrange premeditated Euler's result on the said problem and referred him to his results. Later, in

1755 based on Lagrange's results Lagrange multiplier was introduced. The Lagrange multipliers provide many advantages for iterative technique. Inokuti [27] was a pioneer who used Lagrange multipliers in iterative methods. Hamid et. al. [28] coupled the exact Lagrange multipliers with various analytical techniques. Recently, He [29] provided a review of Lagrange multipliers. Detailed study about Lagrange multipliers can be found in [29-30]

In the present study, we investigate the solution of the pollution model for a system of lakes. The purpose of the model is to designate the pollution of a system of three lakes [1]. Each lake is supposed to be like a huge section and there connecting channels as pipes between the sections with a given flow direction. A pollutant familiarized into the first lake at some variable or constant rate. After that, we need to calculate the pollution level at the time for every lake. We supposed that the volume of water is constant and the distribution of pollutants in individual lakes is uniform. Next, we considered that the sort of pollutant is not degrading and persistent to other forms. Literature reveals that the periodic and exponential decaying input models have never been reported as potential behaviors for the pollution source. We applied VIM (λ_E) to explore the solution of discussed problems. Obtained results compared with already published work [1, 31-33] which shows that coupling of Lagrange multipliers with analytical techniques provides more worthy results as compared to traditional VIM. Moreover, graphical plots and error analysis also reveal lesser error nature while coupling with iterative methods. Additionally, the same concept may be extended to other dynamical problems.

2 Analysis of Variational Iteration Method

To see how the variational iteration method works we consider the following system of differential equations.

$$L_1 u + N_1 u + R_1 v = g_1(x), \tag{1}$$

$$L_2 v + N_2 v + R_2 w = g_2(x), \tag{2}$$

$$L_3 w + N_3 w + R_3 u = g_3(x), \tag{3}$$

In above equations (1-3); L is a linear operator, N is a nonlinear operator and $g(x)$ is the forcing term. According to He's Variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda_1(s)[L_1 u_n(s) + N_1 \widetilde{u_n}(s) + R_1 \widetilde{v_n}(s) - g_1(s)]\mathrm{ds}, \tag{4}$$

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda_2(s)[L_2 v_n(s) + N_2 \widetilde{v_n}(s) + R_2 \widetilde{w_n}(s) - g_2(s)]\mathrm{ds}, \tag{5}$$

$$w_{n+1}(x) = w_n(x) + \int_0^x \lambda_3(s)[L_3 w_n(s) + N_3 \widetilde{w_n}(s) + R_3 \widetilde{u_n}(s) - g_3(s)]\mathrm{ds}, \tag{6}$$

In above $\lambda_i, i = 1,2,3$ are the Lagrange multipliers [6]. The Lagrange multiplier can be calculated with the help of variational theory. The subscripts n denote the nth approximation, $\widetilde{u_n}, \widetilde{v_n}, \widetilde{w_n}$ are considered restricted variations. i.e., $\delta\widetilde{u_n} = \delta\widetilde{v_n} = \delta\widetilde{w_n} = 0$. The Eqs. (4-6) are known as the correction functional. The beauty of the variational iteration method is in many cases the solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier.

The successive approximations $u_{n+1}, n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0. Consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3 Identification of Lagrange Multipliers

To know how to find the exact Lagrange multiplier we consider the linear nth order ordinary differential equation as follows:

$$a_n \frac{\mathrm{d}^n y(x)}{\mathrm{d}x^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y(x)}{\mathrm{d}x^{n-1}} + \cdots + a_2 \frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + a_1 \frac{\mathrm{d}y(x)}{\mathrm{d}x} + a_0 y(x) = f(x), \tag{7}$$

with, $y^n(0) = \alpha_n, n = 0,1,2,3, \dots, n-1$.

The correction functional of Eq. (7) under the traditional Variational iteration method which deliberated in section 2 is:

$$y_{n+1} = y_n + \int_0^x \lambda(s) \left[a_n \frac{\mathrm{d}^n y}{\mathrm{d}s^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}s^{n-1}} + a_{n-2} \frac{\mathrm{d}^{n-2} y}{\mathrm{d}s^{n-2}} + \cdots + a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + a_1 \frac{\mathrm{d}y}{\mathrm{d}s} + a_0 y(s) - f(s)\right] \mathrm{d}s. \tag{8}$$

To find the optimal value of $\lambda(s)$ follow the subsequent steps:

Step 1. First, restrict the non-linear terms we get and following integrate:

$$y_{m+1} = y_m + a_n \int_0^x \lambda(s) y_m^n(s) \mathrm{d}s + a_{n-1} \int_0^x \lambda(s) y_m^{n-1}(s) \mathrm{d}s + \cdots + a_1 \int_0^x \lambda(s) y_m'(s) \mathrm{d}s + a_0 \int_0^x \lambda(s) y_m(s) \mathrm{d}s, \tag{9}$$

Step 2. Using the following relation for integration:

$$\int_0^x \lambda(s) y_m^n(s) \mathrm{d}s = \sum_{k=1}^m (-1)^{k-1} \lambda^{k-1}(s) \delta {y_m}^{m-k}(s)|_{s=x} + (-\delta)^m + \int_0^x \lambda^m(s) y_m(s) \mathrm{d}s,$$

we get the following system of equations:

$$\delta y_m : 1 + \sum_{k=0}^{n-1} (-1)^k a_{k+1} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \lambda(s) = 0, \tag{10}$$

$$\delta {y_m}^{n-1-i} : \sum_{k=0}^{i} (-1)^k a_{i-k} \frac{\mathrm{d}^k}{\mathrm{d}s^k} \lambda(s) = 0, 0 \leq i \leq n-2 \tag{11}$$

and associated with ODE

$$\sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \lambda(s) = 0, \tag{12}$$

Step 3. Solve the ordinary differential equation (12) subject to the boundary conditions (10-11) we get the following optimal values of the Lagrange multiplier

$$\lambda(s) = c_1 \mathrm{e}^{h_1(s-t)} + c_2 \mathrm{e}^{h_2(s-t)} + c_3 \mathrm{e}^{h_3(s-t)} + \cdots + c_n \mathrm{e}^{h_m(s-t)} \tag{13}$$

where $c_i, h_i, i = 1,2,3, \dots, n$ are the constants depend upon $a_i, i = 0,1,2,3, \dots, n$.

4 Description of the Problem and Solution Procedure

A system of lakes is a set of lakes interrelated by channels [1]. Each lake is supposed to be like a huge section and there connecting channels as pipes between the sections with a given flow direction. A pollutant familiarized into the first lake at some variable or constant rate. The geometry of the problem is provided in Figure 1. At $t = 0$, a pollutant is dropped, for instance, from a factory into one of the lakes (we supposed to Lake 1) at a rate $p(t)$. Then the flow of polluted water moves to other lakes through the channels.

We also assumed that the pollutant is uniformly distributed and persistent to each lake and the volume of water in every lake is constant. Due to these conventions, we need to calculate the level of pollution in every lake for $t \geq 0$. To model the dynamic performance of the said problem, let $x_i(t)$ and V_i, $i = 1,2,3$ respectively indicate the amount of pollutant and volume of water in lake i. Then the concentration of the pollutant in lake i at time $t \geq 0$ is given by

$$c_i(t) = \frac{x_i(t)}{V_i}.$$

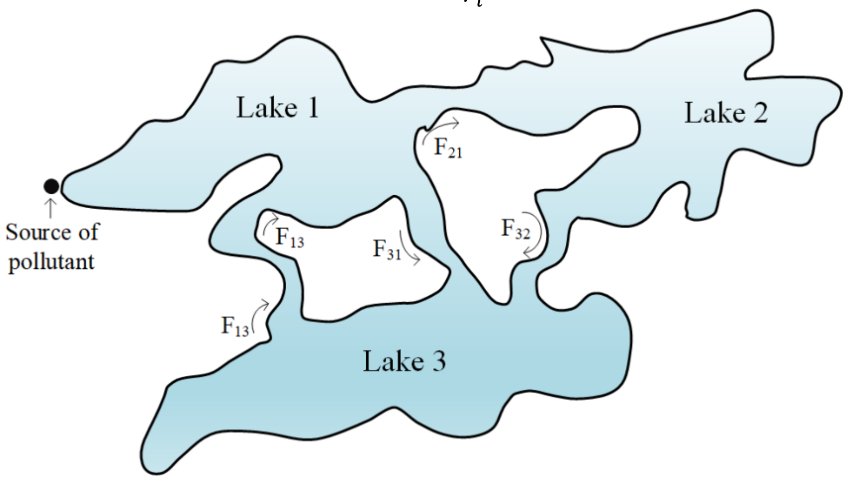


Figure 1: Three lakes system with interconnecting channels

If we assume further that the flow rate F_{ji} from lake i to lake j is constant, then the flux (t) of the pollutant flowing from lake i into lake j for $t \geq 0$ is given by

$$r_{ji}(t) = F_{ji} c_i(t) = \frac{F_{ji} x_i(t)}{V_i}.$$

Therefore, (t) measures the rate at which the concentration of the pollutant in Lake i flows into Lake j at any time t. Applying the principle

Rate of change of pollutant = Input rate − Output rate,

to each lake, we obtain the following system,

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{F_{13}}{V_3} x_3(t) - \frac{F_{31}}{V_1} x_1(t) - \frac{F_{21}}{V_1} x_1(t) + p(t), \tag{14}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{F_{21}}{V_1} x_1(t) - \frac{F_{32}}{V_2} x_2(t), \tag{15}$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \frac{F_{31}}{V_1} x_1(t) + \frac{F_{32}}{V_2} x_2(t) - \frac{F_{13}}{V_3} x_3(t). \tag{16}$$

If we assume that the lakes are initially free from pollutants, then the initial conditions for (14-16) are

$$x_1(0) = x_2(0) = x_3(0) = 0. \tag{17}$$

Since we have a constant volume of water in each lake for $t \geq 0$, then we equal the rate of (incoming and outgoing) flow for each lake. This leads to conditions on flow rates stated as:

Lake 1: $F_{13} = F_{21} + F_{31}$, Lake 2: $F_{21} = F_{32}$, Lake 1: $F_{31} + F_{32} = F_{13}$.

For results comparison, we consider $V_1 = 2900\ \mathrm{mi}^3$, $V_2 = 850\ \mathrm{mi}^3$, $V_3 = 1180\ \mathrm{mi}^3$, $F_{21} = 18\ \mathrm{mi}^3/\mathrm{year}$, $F_{32} = 18\ \mathrm{mi}^3/\mathrm{year}$, $F_{31} = 20\ \mathrm{mi}^3/\mathrm{year}$, $F_{13} = 38\ \mathrm{mi}^3/\mathrm{year}$.

The correction functional of Eqs. (14-16) using variational iteration method are given below:

$$x_{1_{k+1}} = x_{1_k} + \int_0^t \lambda_1(s)\left(\frac{\mathrm{d}}{\mathrm{d}s}x_{1_k}(s) - \frac{F_{13}}{V_3}x_3(t) + \frac{F_{31}}{V_1}x_1(t) - p(s)\right)\mathrm{d}s,$$

$$x_{2_{k+1}} = x_{2_k} + \int_0^t \lambda_2(s)\left(\frac{\mathrm{d}}{\mathrm{d}s}x_{2_k}(s) - \frac{F_{21}}{V_1}x_1(t) + \frac{F_{32}}{V_2}x_2(t)\right)\mathrm{d}s,$$

$$x_{3_{k+1}} = x_{3_k} + \int_0^t \lambda_3(s)\left(\frac{\mathrm{d}}{\mathrm{d}s}x_{3_k}(s) - \frac{F_{31}}{V_1}x_1(t) - \frac{F_{32}}{V_2}x_2(t) + \frac{F_{13}}{V_3}x_3(s)\right)\mathrm{d}s.$$

To enhance the approximation, we must find the Lagrange multipliers $\lambda_1(s), \lambda_2(s)$ and $\lambda_3(s)$ using the methodology deliberated in section 3. According to section 3, the exact Lagrange multipliers are given as:

$$\lambda_1(s) = -\mathrm{e}^{\frac{F_{13}}{V_1}(s-t)}, \lambda_2(s) = -\mathrm{e}^{\frac{F_{32}}{V_2}(s-t)}, \lambda_3(s) = -\mathrm{e}^{\frac{F_{13}}{V_3}(s-t)}.$$

Using these exact Lagrange multipliers in the above correction functional and obtain the subsequent approximation for $k = 0,1,2, \ldots$. In the present study, we discuss the following three models, their solutions, and their comparison with already existing results [1, 31-33].

4.1 Periodic Model

In the periodic model we used a function like sinusoidal as input for pollutants. So, $p(t)$ can be characterized as:

$$p(t) = p_i \times \left(1 + a \times \sin\left(2\pi \times \frac{t}{T}\right)\right).$$

This periodic model involves three influences i.e., a, T and p_i which indicates the normalized amplitude, period of fluctuations, and average input concentration of pollutants. For the comparison purpose, we suppose that the values of normalized amplitude and average input concentration of pollutant as unity and the period of fluctuations are 2π. Therefore, the periodic model yields

$$p(t) = 1 + \sin t.$$

Table 1 shows the comparison with the collocation method [33]. From this table, we observed that the proposed algorithm is more compatible and reliable to seeking the solutions of the above-discussed problem. In this table $\epsilon_{x_1}, \epsilon_{x_2}$ and ϵ_{x_3} are calculated from the following relations:

$$\epsilon_{x_1} = \frac{\mathrm{d}x_1}{\mathrm{d}t} - \left(\frac{F_{13}}{V_3}x_3(t) - \frac{F_{31}}{V_1}x_1(t) - \frac{F_{21}}{V_1}x_1(t) + p(t)\right),$$

$$\epsilon_{x_2} = \frac{\mathrm{d}x_2}{\mathrm{d}t} - \left(\frac{F_{21}}{V_1}x_1(t) - \frac{F_{32}}{V_2}x_2(t)\right),$$

$$\epsilon_{x_3} = \frac{\mathrm{d}x_3}{\mathrm{d}t} - \left(\frac{F_{31}}{V_1}x_1(t) + \frac{F_{32}}{V_2}x_2(t) - \frac{F_{13}}{V_3}x_3(t)\right).$$

Table 2 is generated to compare the obtained result with Adomian's decomposition method [1] for different approximations. Clearly, our obtained solutions show excellent agreement with the published results [1, 33] and the Runge-Kutta method.

Table 1: Error analysis in the obtained results with collocation method [33] for different approximations and t

N	t_i	$\epsilon_{x_1}(t_i)$		$\epsilon_{x_2}(t_i)$		$\epsilon_{x_3}(t_i)$	
		CM [33]	VIM-λ_E	CM [33]	VIM-λ_E	CM [33]	VIM-λ_E
3	0.0	0	0	0	0	0	0
	0.2	1.96E-003	5.91E-009	4.57E-006	1.92E-009	4.97E-006	2.14E-009
	0.4	1.10E-003	4.93E-008	2.61E-006	1.61E-008	2.84E-006	1.79E-008
	0.6	1.62E-003	1.73E-007	3.91E-006	5.65E-008	4.26E-006	6.28E-008
	0.8	7.36E-003	4.26E-007	1.83E-005	1.39E-007	1.99E-005	1.55E-007
	1.0	3.19E-002	9.62E-007	8.15E-005	2.81E-007	8.88E-005	3.13E-007
6	0.0	0	0	0	0	0	0
	0.2	4.23E-008	2.63E-018	6.08E-010	1.06E-018	6.78E-010	1.77E-018
	0.4	4.12E-008	1.72E-016	5.54E-010	6.97E-017	6.19E-010	1.16E-016
	0.6	6.57E-008	2.01E-015	8.32E-010	8.12E-016	9.28E-010	1.35E-015
	0.8	2.03E-007	1.15E-014	2.43E-009	4.67E-015	2.71E-009	7.77E-015
	1.0	1.05E-005	4.49E-014	1.19E-007	1.82E-014	1.33E-007	3.02E-014
10	0.0	0	0	0	0	0	0
	0.2	1.25E-013	1.67E-031	1.76E-015	4.64E-032	1.52E-015	1.06E-031
	0.4	3.54E-013	1.74E-028	3.46E-015	4.82E-029	2.96E-015	1.10E-028
	0.6	3.91E-013	1.01E-026	5.09E-015	2.82E-027	4.28E-015	6.41E-027
	0.8	5.42E-013	1.82E-025	6.65E-015	5.07E-026	5.52E-015	1.15E-025
	1.0	4.97E-011	1.72E-024	5.39E-013	4.78E-025	6.18E-013	1.08E-024

Table 2: Comparison of the VIM-λ_E results with Adomian's decomposition method [1] for different approximations

N	$x_1\left(\frac{N}{100}\right)$			$x_2\left(\frac{N}{100}\right)$			$x_3\left(\frac{N}{100}\right)$		
	[1]	VIM-λ_E	RK-4	[1]	VIM-λ_E	RK-4	[1]	VIM-λ_E	RK-4
1	0.010 05	0.010 05	0.010 05	0.3×10^{-6}	0.3×10^{-6}	0.3×10^{-6}	0.4×10^{-6}	0.4×10^{-6}	0.4×10^{-6}
2	0.020 20	0.020 20	0.020 20	0.1×10^{-5}	0.3×10^{-5}	0.1×10^{-5}	0.1×10^{-5}	0.1×10^{-5}	0.1×10^{-5}
3	0.030 44	0.030 44	0.030 44	0.3×10^{-5}	0.3×10^{-5}	0.3×10^{-5}	0.3×10^{-5}	0.3×10^{-5}	0.3×10^{-5}
4	0.040 79	0.040 79	0.040 79	0.5×10^{-5}	0.5×10^{-5}	0.5×10^{-5}	0.6×10^{-5}	0.6×10^{-5}	0.6×10^{-5}
5	0.051 23	0.051 23	0.051 23	0.8×10^{-5}	0.8×10^{-5}	0.8×10^{-5}	0.9×10^{-5}	0.9×10^{-5}	0.9×10^{-5}
6	0.061 78	0.061 78	0.061 78	0.000 01	0.000 01	0.000 01	0.000 01	0.000 01	0.000 01
7	0.072 42	0.072 42	0.072 42	0.000 02	0.000 02	0.000 02	0.000 02	0.000 02	0.000 02
8	0.083 16	0.083 16	0.083 16	0.000 02	0.000 02	0.000 02	0.000 02	0.000 02	0.000 02
9	0.093 99	0.093 99	0.093 99	0.000 02	0.000 03	0.000 03	0.000 03	0.000 03	0.000 03

4.2 Exponentially Decaying Input Model

For the exponential case, we assume the pollutant input to have the form $p(t) = a\mathrm{e}^{-bt}$, where a and b are the physical parameters like in the periodic case. To show the efficiency of the suggested technique we take $a = 200$, $b = 10$ and find the solutions for different orders of approximations. Tables 3 demonstrate how the error decay gradually as we increase the number of approximants. In the viewing of table 3 we say increase the order of approximants error goes to zero.

4.3 Linear Input Model

When pollutant is introduced into the lake model with a linear concentration; that is $p(t) = ct$, where c is a positive constant. For simplicity, we take $c = 100$ and apply the proposed method to seek the optimal solutions to the problem. Error analysis is presented in Table 3 for numerous approximations. Large value of N enhance the solutions of the problem related to lake model.

Table 3: Decay of error for linear, exponential, and periodic models with differential approximation

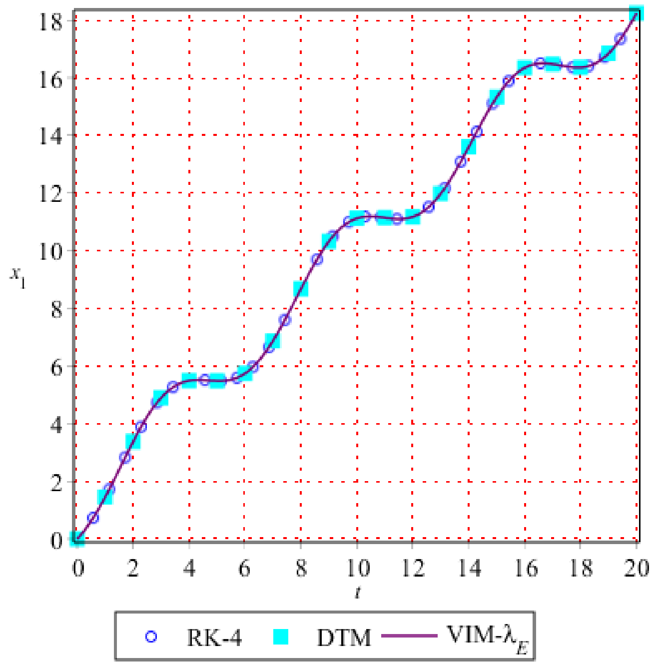
N	Linear Model			Periodic Model			Exponential Model		
	ϖ_{x_1}	ϖ_{x_2}	ϖ_{x_3}	ϖ_{x_1}	ϖ_{x_2}	ϖ_{x_3}	ϖ_{x_1}	ϖ_{x_2}	ϖ_{x_3}
10	1.9E-24	3.3E-25	7.6E-25	1.6E-25	4.3E-26	9.8E-26	1.5E-23	4.3E-24	9.7E-24
15	5.9E-39	1.8E-39	3.8E-39	1.1E-39	3.2E-40	6.9E-40	1.3E-37	2.1E-36	9.8E-36
20	7.1E-54	2.1E-58	5.4E-54	1.6E-54	4.9E-55	2.9E-55	2.1E-52	2.5E-51	3.5E-52
25	2.8E-69	8.3E-70	1.8E-69	7.7E-70	2.3E-70	4.9E-70	1.1E-67	1.5E-67	1.4E-68
30	4.3E-85	1.3E-85	2.8E-85	1.4E-85	4.3E-86	9.1E-86	2.1E-83	1.1E-82	2.9E-82

where $\varpi_{x_k} = \int_0^1 \epsilon_{x_k}\, \mathrm{d}t, k = 1,2,3$.

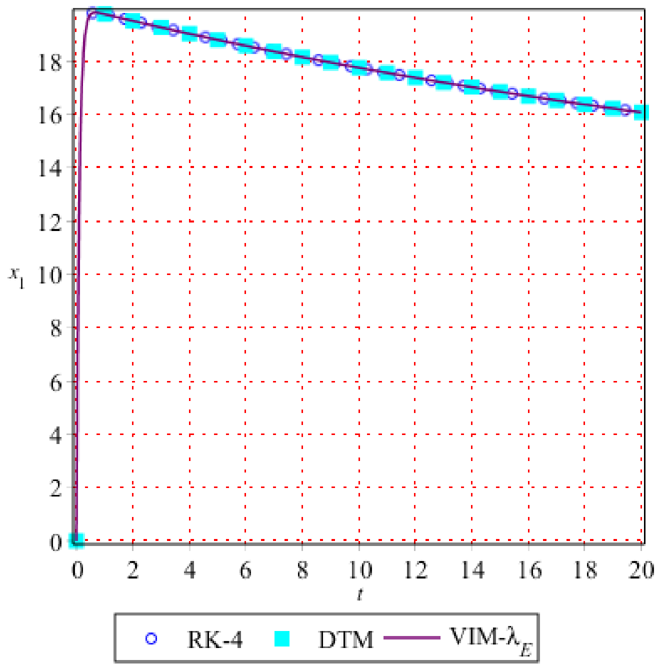
5 Results and Discussions

In this work, we presented and applied the variational iteration method coupled with various types of Lagrange multipliers to solve a pollution model for a system of three lakes. An algorithm to compute exact Lagrange multipliers is also provided. All the tables and figures evidence that the variational iteration method using exact Lagrange multipliers (VIM-λ_E) is a convenient analytical tool to solve pollution models for a system of lakes. Hence very efficient and encouraging solutions of all the systems are successfully obtained.

In this work, we compute exact Lagrange multipliers for input models and then use these multipliers to solve these models. This work shows that the use of exact Lagrange multipliers is user-friendly. We compare our solution with the variational iteration method using approximate Lagrange multipliers [31]. Figures (2-4) show the graphical comparison among the VIM-λ_E, RK-4 and DTM [32] for the input models: (periodic, exponential, and linear). In general terms, the absolute error is low for all approximations. If required, the error can be reduced by using more iterations. Graphical representation and error tables show that the appropriateness of VIM using ELM is much better than VIM using ALM.



(a) periodic input



(b) exponential input

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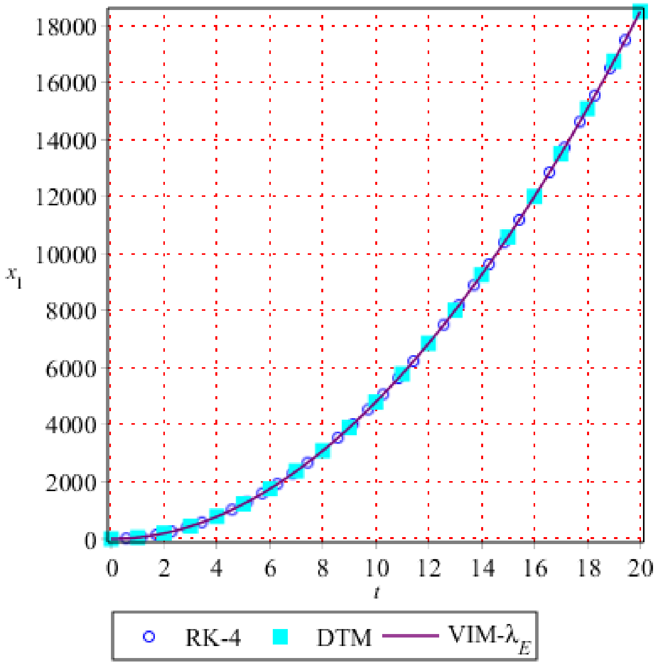
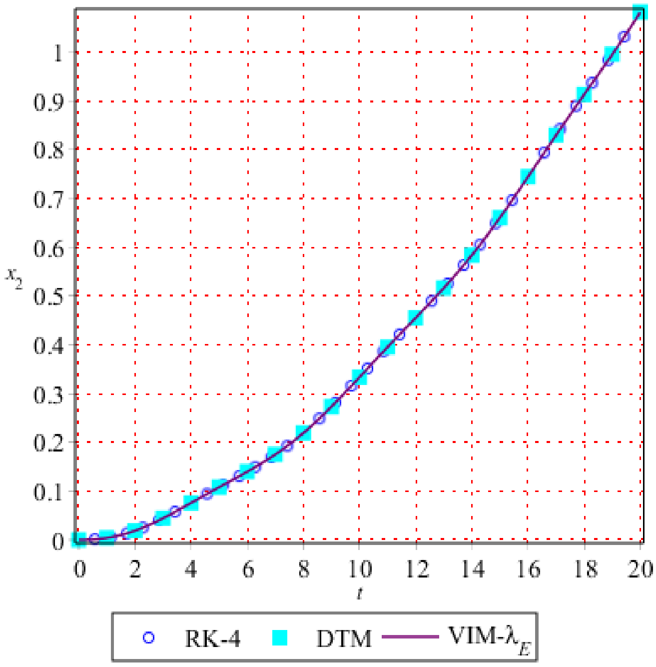


(c) linear input

Figure 2: Comparison of the solution x_1 obtained from the proposed method with DTM [32] and RK-4 for (a) periodic input (b) exponential input and (c) linear input



(a) periodic input

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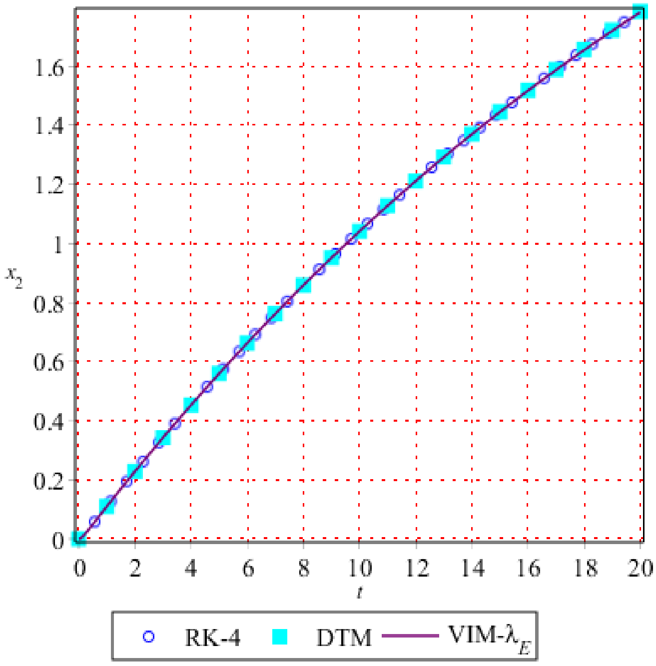
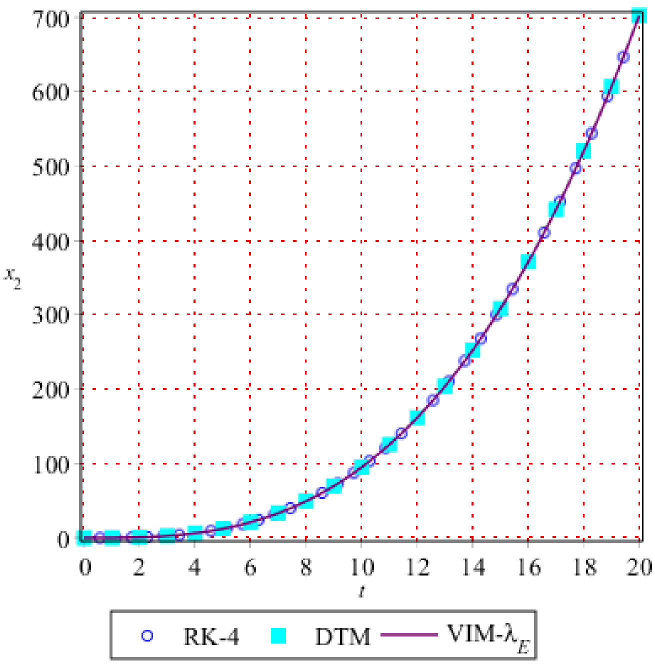


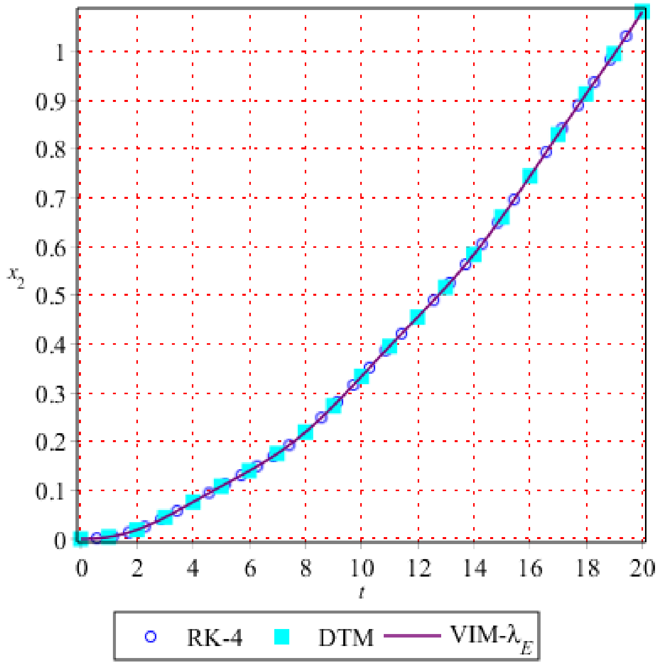
(b) exponential input



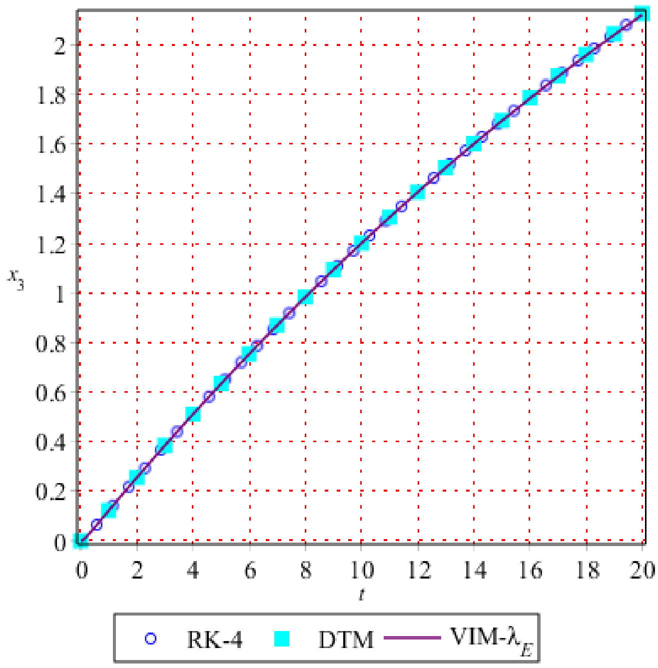
(c) linear input

Figure 3: Comparison of the solution x_1 obtained from the proposed method with DTM [32] and RK-4 for (a) periodic input (b) exponential input and (c) linear input

DOI https://doi.org/10.4208/JICS-2023-007 | Generated on 2025-04-05 21:21:23

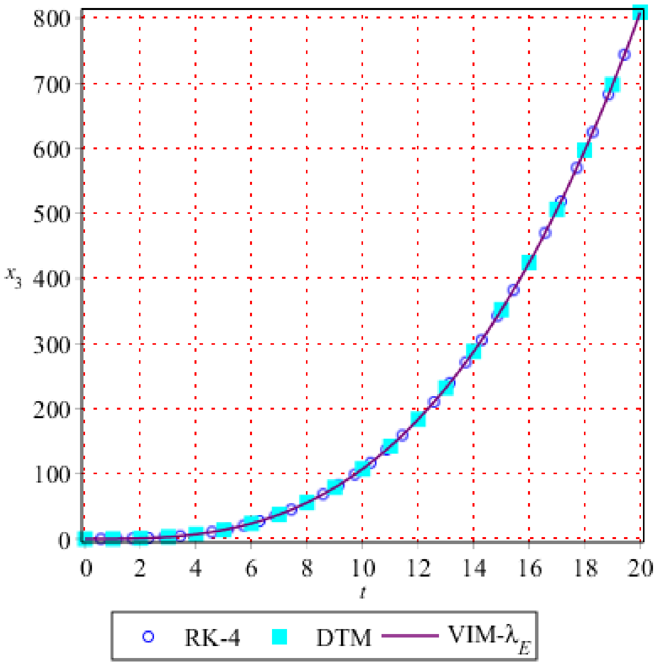


(a) periodic input



(b) exponential input

DOI https://doi.org/10.4208/JICS-2023-007 | Generated on 2025-04-05 21:21:23



(c) linear input

Figure 4: Comparison of the solution x_1 obtained from the proposed method with DTM [32] and RK-4 for (a) periodic input (b) exponential input and (c) linear input

Table 4: Growth of error w.r.t time and decay w.r.t order of approximation in VIM using approximate Lagrange multiplier [31] and exact Lagrange multiplier for periodic input model

t	Error $x_1(t)$ ALM [31]	Error $x_2(t)$ ALM [31]	Error $x_3(t)$ ALM [31]	Error $x_1(t)$ VIM-λ_E	Error $x_2(t)$ VIM-λ_E	Error $x_3(t)$ VIM-λ_E
0	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00
3	4.707 02E-52	9.624 99E-53	3.744 52E-52	1.790 31E-98	5.378 39E-99	1.147 96E-98
6	3.436 92E-44	7.018 04E-45	2.735 12E-44	3.934 64E-86	1.182 14E-86	2.521 26E-86
9	1.392 54E-39	2.839 55E-40	1.108 59E-39	6.478 46E-79	1.946 57E-79	4.148 56E-79
12	2.595 63E-36	5.285 46E-37	2.067 09E-36	8.466 45E-74	2.544 11E-74	5.418 03E-74
15	8.899 13E-34	1.809 60E-34	7.089 52E-34	7.792 09E-70	2.341 66E-70	4.983 20E-70
18	1.041 99E-31	2.115 90E-32	8.303 96E-32	1.335 76E-66	4.014 52E-67	8.536 82E-67
21	5.803 43E-30	1.176 83E-30	4.626 60E-30	7.167 64E-64	2.154 36E-64	4.577 81E-64
24	1.875 43E-28	3.797 76E-29	1.495 65E-28	1.641 76E-61	4.935 00E-62	1.047 86E-61
27	3.999 52E-27	8.087 82E-28	3.190 74E-27	1.961 27E-59	5.895 92E-60	1.250 97E-59
30	6.147 58E-26	1.241 44E-26	4.906 13E-26	1.401 44E-57	4.213 30E-58	8.932 91E-58

6 Conclusions

A model study was conducted to investigate the contamination in water bodies. The pollution model for a system of three lakes that are interconnected by channels is taken into account. The model aims to describe the pollution of a system of three lakes. We utilized an analytical method with λ_E to obtain the solution of said system. Identification of exact Lagrange multipliers for nth-order-ordinary differential equations presented. The fact is that coupling of exact Lagrange multipliers with analytical methods provides better solutions for physical models and brings more accurate, efficient results. A worthy comparison among results obtained by VIM-λ_E, RK-4, CM, VIM-λ_A, ADM and DTM are also provided. The results asserted graphically representing that the coupling may become a powerful tool to solve dynamical models. Similarly, the tables also endorse the lesser error nature of the proposed algorithms. Further same concept may be extended to other dynamic or mathematical models.

Table 5. Growth of error w.r.t time and decay w.r.t order of approximation in VIM using approximate Lagrange multiplier [31] and exact Lagrange multiplier for exponential input model

t	Error $x_1(t)$ ALM [31]	Error $x_2(t)$ ALM [31]	Error $x_3(t)$ ALM [31]	Error $x_1(t)$ VIM-λ_E	Error $x_2(t)$ VIM-λ_E	Error $x_3(t)$ VIM-λ_E
0	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00
3	3.975 73E-50	8.129 37E-51	3.162 79E-50	1.940 55E-96	5.829 77E-97	1.244 29E-96
6	1.727 20E-42	3.526 54E-43	1.374 54E-42	2.812 24E-84	8.449 20E-85	1.802 00E-84
9	4.821 83E-38	9.830 78E-39	3.838 76E-38	3.368 84E-77	1.012 23E-77	2.157 19E-77
12	6.750 66E-35	1.374 33E-35	5.376 33E-35	3.404 33E-72	1.022 99E-72	2.178 44E-72
15	1.843 29E-32	3.747 25E-33	1.468 57E-32	2.528 05E-68	7.597 32E-69	1.616 61E-68
18	1.793 86E-30	3.641 51E-31	1.429 70E-30	3.610 33E-65	1.085 07E-65	2.307 14E-65
21	8.575 48E-29	1.738 32E-29	6.837 16E-29	1.655 03E-62	4.974 55E-63	1.056 91E-62
24	2.437 96E-27	4.934 90E-28	1.944 47E-27	3.305 17E-60	9.935 25E-61	2.109 27E-60
27	4.661 92E-26	9.423 25E-27	3.719 60E-26	3.500 82E-58	1.052 42E-58	2.232 61E-58
30	6.521 48E-25	1.316 34E-25	5.205 14E-25	2.249 21E-56	6.762 19E-57	1.433 44E-56

Table 6. Growth of error w.r.t time and decay w.r.t order of approximation in VIM using approximate Lagrange multiplier [31] and exact Lagrange multiplier for linear input model

t	Error $x_1(t)$ ALM [31]	Error $x_2(t)$ ALM [31]	Error $x_3(t)$ ALM [31]	Error $x_1(t)$ VIM-λ_E	Error $x_2(t)$ VIM-λ_E	Error $x_3(t)$ VIM-λ_E
0	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00	0.000 00E+00
3	4.712 84E-51	9.637 34E-52	3.749 11E-51	1.195 56E-97	3.591 66E-98	7.666 12E-98
6	6.299 42E-43	1.286 42E-43	5.013 00E-43	4.942 66E-85	1.484 98E-85	3.167 26E-85
9	3.564 34E-38	7.268 98E-39	2.837 44E-38	1.155 93E-77	3.473 19E-78	7.402 44E-78
12	8.385 80E-35	1.707 86E-35	6.677 94E-35	1.920 91E-72	5.772 16E-73	1.229 33E-72
15	3.454 15E-32	7.025 32E-33	2.751 62E-32	2.122 68E-68	6.378 98E-69	1.357 58E-68
18	4.725 70E-30	9.598 65E-31	3.765 84E-30	4.223 40E-65	1.269 30E-65	2.699 38E-65
21	3.021 81E-28	6.129 61E-29	2.408 85E-28	2.573 96E-62	7.736 40E-63	1.644 08E-62
24	1.107 37E-26	2.243 27E-27	8.830 46E-27	6.598 54E-60	1.983 44E-60	4.211 99E-60
27	2.652 54E-25	5.366 31E-26	2.115 91E-25	8.730 64E-58	2.624 54E-58	5.569 37E-58
30	4.543 28E-24	9.179 34E-25	3.625 35E-24	6.855 87E-56	2.061 12E-56	4.370 61E-56

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Conflicts of Interest

The authors declare no conflict of interest.

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