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Square-Mean Pseudo S-Asymptotically (ω, c)-Periodic Mild Solutions to Some Stochastic Fractional Evolution Systems*

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Abstract In this paper, we introduce the concept of square-mean pseudo S-asymptotically (ω, c)-periodic for stochastic processes and establish some composition and convolution theorems for such stochastic processes. In addition, we investigate the existence and uniqueness of square-mean pseudo S-asymptotically (ω, c)-periodic mild solutions to some stochastic fractional integrodifferential equations. We illustrate our main results with an application to stochastic Weyl fractional integrodifferential equations.

Keywords Stochastic processes, stochastic evolution equations, Brownian motion, pseudo S-asymptotically (ω, c)-periodic functions

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1. Introduction

Many publications have studied the problem of periodicity of stochastic and deterministic evolution equations, because of its importance for both pure and applied mathematics. Many real-world phenomena do not satisfy conditions of strict periodicity, which are often hard to meet. Over the recent decades, researchers have developed some generalized quasi-periodic functions, such as almost periodic functions, asymptotic periodic functions, asymptotic almost periodic functions, pseudo almost periodic functions and S-asymptotic periodic functions and so on, to better investigate and represent these periodic behaviours and their mathematical models. These types of functions are not exactly periodic, but posses some periodic characteristics. They are helpful for modeling complex systems that have fluctuations or perturbations. For more details on these subjects, see [7, 12, 13, 16, 19, 20, 25] and references therein.

The topic of quasi-periodicity is very attractive and interesting to researchers because it includes several and diverse fascinating untreated problems. Inspired by

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the well-known differential equation of Mathieu

$$\varphi''(\tau) + [a - 2q\cos(2t)]\varphi(\tau) = 0,$$

with solution verifying $\varphi(t+\omega) = c\varphi(t)$, $\omega \in \mathbb{R}$, $c \in \mathbb{C}$ that appear as simulations in different scenarios, such as the firmness of train rails with passing locomotives and the cyclic fluctuations in population growth, Alvarez, Gómez and Pinto [3] introduced the category of (ω, c)-periodic functions. Several authors have been interested in the theory, such as Abadias et al. [1], Mophou and N'Guérékata [17], Kéré et al. [15], Khalladi et al. [14]. On the other hand, some mathematicians have also examined how small changes can affect (ω, c)-periodic functions in abstract spaces. For example, Alvarez, Castillo and Pinto [4,5] defined the concepts of (ω, c)-asymptotically periodic functions and (ω, c)-pseudo periodic functions in abstract spaces and applied them to the abstract Cauchy problem of first order and the Lasota-Wazewska model with unbounded and ergodic production of red cells. Recently, the concept of pseudo S-asymptotically (ω, c)-periodic was introduced by Chang et al. [8], which extends the S-asymptotically ω-periodic functions. A study on fundamental properties and applications of S-asymptotically ω-periodic functions can be found in [6,9,12,13,19,20].

It cannot be overlooked that the presence of random noise emanating from natural sources frequently makes physical phenomena fluctuate or perturb. Hence, to get a more precise model, it is required to include some stochastic terms in the systems. The existence of quasi-periodic solutions for the stochastic evolution equations is very limited (see [10,11,18,23,26] and references therein). It is natural to study the stochastic versions of the deterministic concepts mentioned before. According to our understanding and search, there is no previous work on the idea of square-mean pseudo S-asymptotically (ω, c)-periodic for stochastic processes, which is the main reason for this research. This issue is interesting and new, and hence, the question of whether there exists a (pseudo) S-asymptotically (ω, c)-periodic mild solution in square-mean sense is still untreated for stochastic evolution systems. The primary novelties and major contributions of this paper are listed as follows :

(i) We introduce a new concept of square-mean S-asymptotically and pseudo S-asymptotically (ω, c)-periodic for stochastic processes.

(ii) We establish some completeness, composition and convolution theorems for such stochastic processes.

(iii) We also investigate the existence and uniqueness of square-mean pseudo S-asymptotically (ω, c)-periodic mild solutions to the following class of stochastic fractional evolution equations:

$$\partial_\tau^\alpha \phi(\tau) = A\phi(\tau) + \int_{-\infty}^{\tau} b(\tau - s)A\phi(s)ds + g(\tau, \phi(\tau)) + f(\tau, \phi(\tau)\,(dW(\tau)/d\tau), \tag{1.1}$$

where $\tau \in \mathbb{R}$, ∂_τ^α denotes the Weyl fractional derivative of order $\alpha > 0$, $A : D(A) \subseteq \mathbb{L}^2(\Omega,\mathbb{H}) \to \mathbb{L}^2(\Omega,\mathbb{H})$ is a closed linear operator on a complex separable Hilbert space $\mathbb{L}^2(\Omega,\mathbb{H})$ (where $\mathbb{L}^2(\Omega,\mathbb{H})$ is an appropriate function space specified in Section 2) and generate an α-resolvent family $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}$ on $\mathbb{H}$. g, f are $\mathbb{H}$-valued appropriate functions to be define later. Here $(W(\tau))_{\tau\in\mathbb{R}}$ represents a two-sided and standard one-dimensional Brownian motion on $\mathbb{H}$.

The obtained outcomes show that for each pseudo S-asymptotically (ω, c)-periodic input, the output is still a bounded and continuous mild solution to the stochastic evolution equation (1.1), which is also pseudo S-asymptotically (ω, c)-periodic. Furthermore, we think that the obtained outcomes in this paper could contribute to studying the existence and other qualitative properties of (pseudo) S-asymptotically (ω, c)-periodic solutions of other kinds of stochastic evolution equations, especially the fractional problems under different situations, without Lipschitz conditions on nonlinear terms.

This paper is organized as follows: Section 2 is concerned with some basic definitions, lemmas and notations. Section 3 mainly focuses on properties of square-mean S-asymptotically (ω, c)-periodic processes. Section 4 is devoted to applications to some stochastic fractional evolution equations in Hilbert spaces. To end this work, we give some illustrations in Section 5.

2. Background

Throughout this paper, we suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ represents a complete probability space with some filtration $\{\mathcal{F}_\tau\}_{\tau \geq 0}$ satisfying the usual conditions, and that $\mathbb{H}$ and $\mathbb{K}$ denote complex separable Hilbert spaces. For convenience, the same notations $\|\cdot\|$ and $(\cdot,\cdot)$ are applied to denote the norms and the inner products in $\mathbb{H}$ and $\mathbb{K}$. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the Banach space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$ endowed with the topology defined by the operator norm, and $\mathbb{L}^2(\Omega, \mathbb{H})$ stands for the collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, which is a complex Hilbert space endowed with the norm

$$\|\phi\|_{\mathbb{L}^2} = (\mathsf{E}\|\phi\|^2)^{1/2},\ \phi \in \mathbb{L}^2(\Omega,\mathbb{H}),$$

where $\mathsf{E}(\cdot)$ is the expectation defined by $\mathsf{E}\|\phi\|^2 = \int_\Omega \|\phi\|^2 d\mathbb{P}$. For each $t \in \mathbb{R}$, we denote by $\mathcal{F}_\tau$ the σ-field generated by the random variables $\{W(s), s \leq t\}$ and the $\mathbb{P}$-null sets.

Definition 2.1. A stochastic process $\phi : \mathbb{R} \to \mathbb{L}^2(\Omega,\mathbb{H})$ is said to be

(i) stochastically bounded if there exists a constant $M > 0$ such that

$$\mathsf{E}\|\phi(\tau)\|^2 = \int_\Omega \|\phi(\tau)\|^2 d\mathbb{P} < M \text{ for all } \tau \in \mathbb{R};$$

(ii) stochastically continuous if

$$\lim_{\tau \to s} \mathsf{E}\|\phi(\tau) - \phi(s)\|^2 = 0 \text{ for all } s \in \mathbb{R}.$$

We denote by $\mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ (resp. $\mathcal{C}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$) the complex Banach space of all bounded and continuous (resp. continuous) stochastic processes ϕ from $\mathbb{R}$ into $\mathbb{L}^2(\Omega,\mathbb{H})$ equipped with the norm $\|\phi\|_\infty = \left(\sup_{\tau\in\mathbb{R}} \mathsf{E}\|\phi(\tau)\|^2\right)^{1/2}$. Furthermore, the set $\mathcal{C}\left(\mathbb{R} \times \mathbb{L}^2(\Omega,\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H})\right)$ represents the collection of all continuous processes $g : \mathbb{R} \times \mathbb{L}^2(\Omega,\mathbb{H}) \to \mathbb{L}^2(\Omega,\mathbb{H})$.

By using the principal branch of the complex Logarithm, we define $c^{\frac{\tau}{\omega}} := \exp\left(\frac{\tau}{\omega}\log(c)\right)$. In what follow, we use the following notations

$$c^\wedge(\tau) = c^{\tau/\omega} \text{ and } |c|^\wedge(\tau) := |c^\wedge(\tau)| = \exp\left(\frac{\tau}{\omega}\log(|c|)\right).$$

We recall some definitions and properties on (ω, c)-periodic functions. Let $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ be a complex Banach space and $C(\mathbb{R}, \mathbf{X})$ be the set of all continuous functions $f : \mathbb{R} \to \mathbf{X}$ equipped with sup-norm $\|f\|_\infty = \sup_{\tau\in\mathbb{R}} \|f(\tau)\|_{\mathbf{X}}$.

Definition 2.2 ([3]). For given $(\omega, c) \in \mathbb{R}^+\backslash\{0\} \times \mathbb{C}\backslash\{0\}$, a function $\phi \in C(\mathbb{R}, \mathbf{X})$ is said to be (ω, c)-periodic if $\phi(\tau+\omega) = c\,\phi(\tau)$ for all $\tau \in \mathbb{R}$. ω is called the c-period of ϕ.

We denote by $\mathcal{P}_{\omega,c}(\mathbb{R}, \mathbf{X}))$ the set of all (ω, c)-periodic functions from $\mathbb{R}$ to $\mathbf{X}$.

Lemma 2.1 ([3]). *$\mathcal{P}_{\omega,c}(\mathbb{R}, \mathbf{X}))$ is a Banach space equipped with norm*

$$\|\phi\|_{\omega c} = \sup_{\tau\in[0,\omega]} \|c^\wedge(-\tau)\phi(\tau)\|_{\mathbf{X}}.$$

3. Main results and proofs

In this section, we introduce and establish some fundamental results on the new concepts of square-mean S-asymptotically (ω, c)-periodic and square-mean pseudo S-asymptotically (ω, c)-periodic process.

3.1. On square-mean S-asymptotically (ω, c)-periodicity

We will define a square-mean S-asymptotically (ω, c)-periodic process. First, for given $(\omega, c) \in \mathbb{R}\backslash\{0\} \times \mathbb{C}\backslash\{0\}$, we define the following set

$$\mathcal{BC}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right) := \left\{\phi \in \mathcal{C}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right) \;:\; \sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)\phi(\tau)\|^2 < \infty\right\}.$$

We get the following outcomes.

Proposition 3.1. *Let $\phi \in \mathcal{C}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. Then $\phi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ if and only if*

$$\phi(\tau) = c^\wedge(\tau)g(\tau), \quad g \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right).$$

Proof. It's clear that if $\phi(\tau) = c^\wedge(\tau)g(\tau)$ with $g \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ then $\phi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. Inversely, suppose that $\phi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. If we put $g(\tau) = c^\wedge(-\tau)\phi(\tau)$, then we have

$$\|g\|_\infty^2 = \sup_{t\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)\phi(\tau)\|^2 < \infty.$$

We derive that $g \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ and $\phi(\tau) = c^\wedge(\tau)g(\tau)$. □

Lemma 3.1. *Let $\varphi_1, \varphi_2, \varphi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. The following results hold:*

(i) $\varphi_1 + \varphi_2 \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$, and $k\varphi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ for each $k \in \mathbb{C}$.

(ii) $\mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ *is a Banach space equipped with the norm*

$$\|\phi\|_{b,\omega,c} = \left(\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\phi(\tau)\|^2\right)^{1/2}.$$

Proof.

(i) Let $\varphi_1, \varphi_2 \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. Then $\varphi_1, \varphi_2 \in \mathcal{C}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$,

$$\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\varphi_1(\tau)\|^2 < \infty \text{ and } \sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\varphi_2(\tau)\|^2 < \infty.$$

We get

$$\begin{aligned}\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)[\varphi_1(\tau) + \varphi_2(\tau)]\|^2 \leq & 2\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\varphi_1(\tau)\|^2 \\ & + 2\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\varphi_2(\tau)\|^2 < \infty.\end{aligned}$$

Then $\varphi_1 + \varphi_2 \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$.
For $k \in \mathbb{C}$ and $\varphi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$, we have $k\varphi \in \mathcal{C}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ and

$$\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)k\varphi(\tau)\|^2 = |k|^2 \sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\varphi(\tau)\|^2 < \infty.$$

Then $k\varphi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$.
Thus $\varphi_1 + \varphi_2$, $kf \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, for each $k \in \mathbb{C}$.

(ii) Let $\{\varphi_n\}_n \subseteq \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ be a Cauchy sequence. Using Proposition 3.1, we can write $\varphi_n(\tau) = c^{\wedge}(\tau)g_n(\tau)$ with $g_n \in \mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$, $\tau \in \mathbb{R}$. Since $\|g_n - g_m\|_\infty = \|\varphi_n - \varphi_m\|_{b,\omega,c}$, then $\{g_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in the Banach space $\mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$. Thus, there exists $g \in \mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$ such that $\|g_n - g\|_\infty \to 0$ as $n \to \infty$. Let $\varphi(\tau) = c^{\wedge}(\tau)g(\tau)$. Then by Proposition 3.1, $g \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ and consequently,

$$\begin{aligned}\lim_{n\to\infty} \|\varphi_n - \varphi\|^2_{b,\omega,c} &= \lim_{n\to\infty} \sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^{\wedge}(-\tau)\varphi_n(\tau) - c^{\wedge}(-\tau)\varphi(\tau)\|^2 \\ &= \lim_{n\to\infty} \|g_n - g\|^2_\infty = 0.\end{aligned}$$

That is $\varphi_n(\tau) \to \varphi(\tau) = c^{\wedge}(\tau)g(\tau)$ in $\big(\mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big); \|\cdot\|_{b,\omega,c}\big)$. This implies that the space $\mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ is Banach space equipped with the norm $\|\cdot\|_{b,\omega,c}$.

□

Definition 3.1. A stochastic process $\phi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ is said to be square-mean S-asymptotically (ω, c)-periodic if for given $\omega \in \mathbb{R}, c \in \mathbb{C}\backslash\{0\}$,

$$\lim_{|\tau|\to\infty} \mathbb{E}\|c^{\wedge}(-\tau)[\phi(\tau + \omega) - c\,\phi(\tau)]\|^2 = 0.$$

The collection of all square-mean S-asymptotically (ω, c)-periodic stochastic processes will be denoted by $\mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

In Definition 3.1, when $c = 1$ and $c = -1$, S-asymptotically (ω, c)-periodic process reduces to the standard S-asymptotically ω- periodic and ω-antiperiodic process, respectively. The above notions have been studied in [10, 11, 23, 26].

We now present the following basic properties.

Lemma 3.2. *Let $\varphi_1, \varphi_2, \varphi \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. Then the following results hold:*

(i) $\varphi_1 + \varphi_2 \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, *and* $k\varphi \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ *for each* $k \in \mathbb{C}$.

(ii) For each $b \in \mathbb{R}$, *the process* $\varphi_b(\tau) := \varphi(\tau + b) \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

(iii) $\mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ *is a Banach space equipped with the norm*

$$\|\phi\|_{b,\omega,c} = \left(\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)\phi(\tau)\|^2\right)^{1/2}, \quad \phi \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right).$$

Proof.

(i) Let $k \in \mathbb{C}$ and $\epsilon > 0$. Then from Definition 3.1, there exists a constant $\delta_\epsilon > 0$ such that for each $|\tau| > \delta_\epsilon$, we have :

$$\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - c\varphi(\tau)]\|^2 < \frac{\epsilon}{|k|^2+1},$$
$$\mathsf{E}\|c^\wedge(-\tau)[\varphi_i(\tau+\omega) - c\varphi_i(\tau)]\|^2 < \frac{\epsilon}{4}, \quad i = 1, 2.$$

Hence,

$$\mathsf{E}\|c^\wedge(-\tau)[k\varphi(\tau+\omega) - ck\varphi(\tau)]\|^2 = |k|^2\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - \varphi(\tau)]\| < \epsilon,$$

and

$$\begin{aligned}
&\mathsf{E}\|c^\wedge(-\tau)[(\varphi_1+\varphi_2)(\tau+\omega) - c(\varphi_1+\varphi_2)(\tau)]\|^2 \\
=&\mathsf{E}\|c^\wedge(-\tau)[\varphi_1(\tau+\omega) - c\varphi_1(\tau) + \varphi_2(\tau+\omega) - c\varphi_2(\tau)]\|^2 \\
\leq&2\mathsf{E}\|c^\wedge(-\tau)[\varphi_1(\tau+\omega) - c\varphi_1(\tau)]\|^2 + 2\mathsf{E}\|c^\wedge(-\tau)[\varphi_2(\tau+\omega) - c\varphi_2(\tau)]\|^2 \\
<&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $k\varphi, \varphi_1 + \varphi_2 \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

(ii) Let $b \in \mathbb{R}$ and $\varphi \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. Definition 3.1 implies that, for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - c\varphi(\tau)]\|^2 \leq \frac{\epsilon}{|c^\wedge(b)|^2}, \text{ for } |\tau| > \delta_\epsilon.$$

We derive that

$$\begin{aligned}
&\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+b+\omega) - c\varphi(\tau+b)]\|^2 \\
&\qquad=\mathsf{E}\|c^\wedge(-\tau-b)c^\wedge(b)[\varphi(\tau+b+\omega) - c\varphi(\tau+b)]\|^2 \\
&\qquad=|c^\wedge(b)|^2\, \mathsf{E}\|c^\wedge(-(\tau+b))[\varphi(\tau+b+\omega) - c\varphi(\tau+b)]\|^2 \\
&\qquad<\epsilon
\end{aligned}$$

for $|\tau| > \delta = \max\{\delta_\epsilon - b, \delta_\epsilon + b\}$. Thus $\varphi_b \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

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(iii) From assertion **(i)**, we deduce that $\mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ is a vector space. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ such that $\varphi_n \to \varphi$ as $n \to \infty$. Then for any $\epsilon > 0$, there exist constants $N > 0$ and $\delta_\epsilon > 0$ such that

$$\|\varphi_n - \varphi\|_{b,\omega,c}^2 = \sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)[\varphi_n(\tau) - \varphi(\tau)]\|^2 \leq \frac{\epsilon}{9|c|^2} \text{ for } n \geq N,$$

and

$$\mathsf{E}\|c^\wedge(-\tau)[\varphi_N(\tau+\omega) - c\varphi_N(\tau)]\|^2 \leq \frac{\epsilon}{9}, \quad \text{for } \tau > \delta_\epsilon.$$

We obtain that for $\tau > \delta_\epsilon$

$$\begin{aligned}
&\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - c\varphi(\tau)]\|^2 \\
=&\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - \varphi_N(\tau+\omega) + \varphi_N(\tau+\omega) - c\varphi(\tau) + c\varphi_N(\tau) - c\varphi_N(\tau)]\|^2 \\
\leq&3\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - \varphi_N(\tau+\omega)]\|^2 + 3\mathsf{E}\|c^\wedge(-\tau)[\varphi_N(\tau+\omega) - c\varphi_N(\tau)]\|^2 \\
&+ 3\mathsf{E}\|c^\wedge(-\tau)[c\varphi_N(\tau) - c\varphi(\tau)]\|^2 \\
\leq&3|c|^2\,\mathsf{E}\|c^\wedge(-(\tau+\omega))[\varphi(\tau+\omega) - \varphi_N(\tau+\omega)]\|^2 \\
&+ 3\mathsf{E}\|c^\wedge(-\tau)[\varphi_N(\tau+\omega) - c\varphi_N(\tau)]\|^2 \\
&+ 3|c|^2\,\mathsf{E}\|c^\wedge(-\tau)[\varphi_N(\tau) - \varphi(\tau)]\|^2 \\
\leq&\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

This implies that the space $\mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ is a closed sub-space of $\mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. Thus it's a Banach space equipped with $\|\cdot\|_{b,\omega,c}$. □

3.2. On square-mean pseudo-S-asymptotically (ω, c)-periodicity

Definition 3.2. A stochastic process $\phi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ is said to be square-mean pseudo-S-asymptotically (ω, c)-periodic if for given $\omega \in \mathbb{R}, c \in \mathbb{C}\backslash\{0\}$,

$$\lim_{q\to\infty} \frac{1}{2q} \int_{-q}^{q} \mathbb{E}\|c^\wedge(-\tau)[\phi(\tau+\omega) - c\,\phi(\tau)]\|^2 d\tau = 0.$$

We denote by $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ the set of all square-mean S-asymptotically (ω, c)-periodic stochastic processes.

We have the following basic properties.

Lemma 3.3. *Let $\varphi_1, \varphi_2, \varphi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. The following results hold:*

(i) $\varphi_1 + \varphi_2 \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$*, and* $k\varphi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ *for each* $k \in \mathbb{C}$*.*

(ii) For each $b \in \mathbb{R}$, the process $\varphi_b(\tau) := \varphi(\tau+b) \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

(iii) $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ *is a Banach space equipped with the norm* $\|\cdot\|_{b,\omega,c}$*.*

Proof.

(i) From Definition 3.2, we have :

$$\lim_{q\to\infty} \frac{1}{2q} \int_{-q}^{q} \mathsf{E}\|c^\wedge(-\tau)[\varphi_i(\tau+\omega) + c\varphi_i(\tau)]\|^2 d\tau = 0, \quad i = 1, 2.$$

Hence, for any $k \in \mathbb{C}$, we get

$$\begin{aligned}&\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[k\varphi(\tau+\omega)-ck\varphi(\tau)]\|^2d\tau\\ &=\lim_{q\to\infty}\frac{|k|^2}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)+\varphi(\tau)]\|^2d\tau=0,\end{aligned}$$

and

$$\begin{aligned}0\le&\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[(\varphi_1+\varphi_2)(\tau+\omega)-c(\varphi_1+\varphi_2)(\tau)]\|^2d\tau\\ =&\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi_1(\tau+\omega)-c\varphi_1(\tau)+\varphi_2(\tau+\omega)-c\varphi_2(\tau)]\|^2d\tau\\ \le&2\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi_1(\tau+\omega)-c\varphi_1(\tau)]\|^2d\tau\\ &+2\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi_2(\tau+\omega)-c\varphi_2(\tau)]\|^2d\tau\\ =&0.\end{aligned}$$

Thus $k\varphi, \varphi_1+\varphi_2 \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$.

(ii) Let $b \in \mathbb{R}$ and $\varphi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. From Definition 3.2, we have :

$$\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2d\tau=0.$$

Thus for each $b \in \mathbb{R}$, we get

$$\begin{aligned}&\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+b+\omega)-c\varphi(\tau+b)]\|^2d\tau\\ =&\lim_{q\to\infty}\frac{1}{2q}\int_{-q+b}^{q+b}\mathsf{E}\|c^\wedge(-\tau+b)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2d\tau\\ \le&|c^\wedge(b)|^2\lim_{q\to\infty}\frac{1}{2q}\int_{-q+|b|}^{q+|b|}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2d\tau\\ \le&|c^\wedge(b)|^2\lim_{q\to\infty}\frac{(q+|b|)}{q}\frac{1}{2(q+|b|)}\int_{-q+|b|}^{q+|b|}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2d\tau\\ =&0.\end{aligned}$$

Thus $\varphi_b \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$.

(iii) From assertion **(i)**, we deduce that $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ is a vector space. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ such that $\varphi_n \to \varphi$ as $n \to \infty$. Then for any $\epsilon > 0$, there exist constants $N > 0$ and $\delta_\epsilon > 0$ such that

$$\|\varphi_n-\varphi\|^2_{b,\omega,c}=\sup_{\tau\in\mathbb{R}}\mathsf{E}\|c^\wedge(-\tau)[\varphi_n(\tau)-\varphi(\tau)]\|^2d\tau\le\frac{\epsilon}{9|c|^2},\ \text{for } n\ge N$$

and

$$\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi_N(\tau+\omega)-c\varphi_N(\tau)]\|^2d\tau\le\frac{\epsilon}{9}\ \text{ for } q>\delta_\epsilon.$$

We obtain for $q > \delta_\epsilon$

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
=&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-\varphi_N(\tau+\omega)+\varphi_N(\tau+\omega)-c\varphi(\tau)\\
&+c\varphi_N(\tau)-c\varphi_N(\tau)]\|^2 d\tau\\
\leq&3\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-\varphi_N(\tau+\omega)]\|^2\\
&+3\mathsf{E}\|c^{\wedge}(-\tau)[\varphi_N(\tau+\omega)-c\varphi_N(\tau)]\|^2 d\tau\\
&+3\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[c\varphi_N(\tau)-c\varphi(\tau)]\|^2 d\tau\\
\leq&3|c|^2\,\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-(\tau+\omega))[\varphi(\tau+\omega)-\varphi_N(\tau+\omega)]\|^2 d\tau \left(\text{since } c^{\wedge}(+\omega)=c\right)\\
&+3\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi_N(\tau+\omega)-c\varphi_N(\tau)]\|^2 d\tau\\
&+3|c|^2\,\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi_N(\tau)-\varphi(\tau)]\|^2 d\tau\\
\leq&3|c|^2\,\|\phi_N-\varphi\|^2_{b,\omega,c}+3\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi_N(\tau+\omega)-c\varphi_N(\tau)]\|^2 d\tau\\
&+3|c|^2\,\|\phi_N-\varphi\|^2_{b,\omega,c}\\
\leq&\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}
$$

This implies that the space $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ is a closed sub-space of $\mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. Thus it's a Banach space equipped with $\|\cdot\|_{b,\omega,c}$. □

The following Lemma gives a characterization of the square-mean pseudo-S-asymptotically (ω, c)-periodic process

Lemma 3.4. *Let $\varphi \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. Then the following assertions are equivalent:*

(i) $\displaystyle\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau = 0.$

(ii) For each $\epsilon > 0$, $\displaystyle\lim_{q\to\infty}\frac{1}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)} d\tau = 0$ where

$$\mathcal{M}_{q,\epsilon}(\varphi) := \left\{\tau\in[-q,q] : \mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 \geq \epsilon\right\}.$$

Proof.

Claim 1. $(\mathbf{i}) \Longrightarrow (\mathbf{ii})$.

Since

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
&\quad=\frac{1}{2q}\int_{[-q,q]\setminus\mathcal{M}_{q,\epsilon}(\varphi)}\mathsf{E}\|c^{\wedge}(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
\geq&\frac{1}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
\geq&\frac{\epsilon}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)}d\tau\geq 0,
\end{aligned}
$$

we can see that if assertion **(i)** holds, then assertion **(ii)** is true.
Claim 2. (**ii**) $\Longrightarrow$ (**i**).

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
=&\frac{1}{2q}\int_{[-q,q]\setminus\mathcal{M}_{q,\epsilon}(\varphi)}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
&+\frac{1}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2 d\tau\\
\leq&\frac{\epsilon}{2q}\int_{-q}^{q}d\tau\\
&+\frac{1}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)}\left[2\mathsf{E}\|c^\wedge(+\omega)c^\wedge(-(\tau+\omega))\varphi(\tau+\omega)\|^2+2|c|^2\mathsf{E}\|c^\wedge(-\tau)\varphi(\tau)\|^2\right]d\tau\\
\leq&\epsilon+4|c|^2\|\varphi\|_{b,\omega,c}^2\frac{1}{2q}\int_{\mathcal{M}_{q,\epsilon}(\varphi)}d\tau.
\end{aligned}
$$

Hence, we deduce that if assertion **(ii)** holds, then assertion **(i)** is true. □

3.3. Some convolution theorems in $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$

We establish two convolution theorems for pseudo S-asymptotically (ω,c)-periodic stochastic processes. Our outcomes are based on Proposition 3.1 and Lemma (3.3).

Theorem 3.1. *Let $\{\mathcal{U}(\tau)\}_{\tau\geq 0}\subset\mathcal{B}(\mathbb{H})$ be strongly continuous such that $\|c^\wedge(-\tau)\mathcal{U}(\tau)\|\leq\psi(\tau)$ where $\psi\in L^1(\mathbb{R}_+)$. If $f\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, then*

$$
z(\tau):=\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)f(s)ds\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right).
$$

Proof. Since $f\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ then by Proposition 3.1, $f(\tau)=c^\wedge(\tau)u_f(\tau)$, where $u_f\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. Then,

$$
\begin{aligned}
z(\tau)=\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)f(s)ds&=\int_0^\infty\mathcal{U}(\zeta)c^\wedge(\tau-\zeta)u_f(\tau-\zeta)d\zeta\\
&=c^\wedge(\tau)\int_0^\infty c^\wedge(-\zeta)\mathcal{U}(\zeta)u_f(\tau-\zeta)d\zeta.
\end{aligned}
$$

Let $g(\tau)=\int_0^\infty c^\wedge(-\zeta)\mathcal{U}(\zeta)u_f(\tau-\zeta)d\zeta$, for $\tau\in\mathbb{R}$. We have to show that $g\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. By using the Cauchy- Schwarz inequality, we get

$$\begin{aligned}\mathsf{E}\,\|g(\tau)\|^2 = \mathsf{E}\left\|\int_0^\infty c^\wedge(-\zeta)\mathcal{U}(\zeta)u_f(\tau-\zeta)d\zeta\right\|^2 &\leq \mathsf{E}\left[\int_0^\infty c^\wedge(-\zeta)\|\mathcal{U}(\zeta)u_f(\zeta-\tau)\|d\zeta\right]^2\\ &\leq \mathsf{E}\left[\int_0^\infty \|c^\wedge(-\zeta)\mathcal{U}(\zeta)u_f(\tau-\zeta)\|d\zeta\right]^2\\ &\leq \mathsf{E}\left[\int_0^\infty \psi(\zeta)\|u_f(\tau-\zeta)\|d\zeta\right]^2\\ &\leq \int_0^\infty \psi(\zeta)d\zeta\int_0^\infty \psi(\zeta)\mathsf{E}\|u_f(\tau-\zeta)\|^2 d\zeta\\ &\leq \|u_f\|_\infty^2\|\psi\|_{L^1}^2 < \infty.\end{aligned}$$

Let $\tau, \tau_0 \in \mathbb{R}$. Then, by using the Cauchy- Schwarz inequality, we get

$$\begin{aligned}\mathsf{E}\,\|g(\tau)-g(\tau_0)\|^2 &= \mathsf{E}\left\|\int_0^\infty c^\wedge(-\zeta)\mathcal{U}(\zeta)\Big(u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\Big)d\zeta\right\|^2\\ &\leq \mathsf{E}\left[\int_0^\infty \left\|c^\wedge(-\zeta)\mathcal{U}(\zeta)\Big(u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\Big)\right\|d\zeta\right]^2\\ &\leq \mathsf{E}\left[\int_0^\infty \psi(\zeta)\|u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\|d\zeta\right]^2\\ &\leq \int_0^\infty \psi(\zeta)d\zeta\int_0^\infty \psi(\zeta)\mathsf{E}\|u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\|^2 d\zeta\\ &\leq \|\psi\|_{L^1}^2\int_0^\infty \psi(\zeta)\mathsf{E}\|u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\|^2 d\zeta.\end{aligned}$$

Since

$$\int_0^\infty \psi(\zeta)\mathsf{E}\|u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\|^2 d\zeta \leq 4\|\psi\|_{L^1}^2\|u_f\|_\infty^2 < \infty,$$

using the fact that $u_f \in \mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$ and the dominated convergence theorem, it follows that

$$\lim_{\tau\to\tau_0}\int_0^\infty \psi(\zeta)\mathsf{E}\|u_f(\tau-\zeta)-u_f(\tau_0-\zeta)\|^2 d\zeta = 0.$$

Therefore

$$\lim_{\tau\to\tau_0}\mathsf{E}\,\|g(\tau)-g(\tau_0)\|^2 = 0.$$

We conclude that $g \in \mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$. Hence by Proposition 3.1,

$$z \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)\,.$$

Now let $q > 0$. Then, we have

$$\begin{aligned}&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[z(\tau+\omega)-cz(\tau)]\|^2 d\tau\\ &\quad=\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)\left[\int_{-\infty}^{\tau+\omega}\mathcal{U}(\tau+\omega-s)f(s)ds - c\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)f(s)ds\right]\right\|^2 d\tau\\ &\quad=\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)\left[\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)f(s+\omega)ds - c\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)f(s)ds\right]\right\|^2 d\tau\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^{\wedge}(-\tau)\left[\int_0^{\infty}\mathcal{U}(s)[f(\tau-s+\omega)-cf(\tau-s)]ds\right]\right\|^2 d\tau\\
&\le\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left[\int_0^{\infty}\|c^{\wedge}(-\tau)\mathcal{U}(s)[f(\tau-s+\omega)-cf(\tau-s)]\|\,ds\right]^2 d\tau\\
&\le\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left[\int_0^{\infty}\|c^{\wedge}(-s)\mathcal{U}(s)c^{\wedge}(s-\tau)[f(\tau-s+\omega)-cf(\tau-s)]\|\,ds\right]^2 d\tau\\
&\le\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left[\int_0^{\infty}\psi(s)\,\|c^{\wedge}(s-\tau)[f(\tau-s+\omega)-cf(\tau-s)]\|\,ds\right]^2 d\tau.
\end{aligned}
$$

By using Cauchy- Schwarz inequality and Fubini's theorem, we get that

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[z(\tau+\omega)-cz(\tau)]\|^2 d\tau\\
\le&\frac{1}{2q}\int_{-q}^{q}\left[\left(\int_0^{\infty}\psi(s)ds\right)\left(\int_0^{\infty}\psi(s)\mathsf{E}\,\|c^{\wedge}(s-\tau)[f(\tau-s+\omega)-cf(\tau-s)]\|^2\,ds\right)\right]d\tau\\
\le&\frac{\|\psi\|_{L^1}}{2q}\int_{-q}^{q}\left(\int_0^{\infty}\psi(s)\mathsf{E}\,\|c^{\wedge}(s-\tau)[f(\tau-s+\omega)-cf(\tau-s)]\|^2\,ds\right)d\tau\\
\le&\|\psi\|_{L^1}\left(\int_0^{\infty}\psi(s)\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\,\|c^{\wedge}(-(\tau-s))[f(\tau-s+\omega)-cf(\tau-s)]\|^2\,d\tau\right)ds.
\end{aligned}
$$

By $f\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, Lemma 3.3-(ii) and the dominated convergence theorem, we get that

$$
\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^{\wedge}(-\tau)[z(\tau+\omega)-cz(\tau)]\|^2 d\tau=0.
$$

Thus $z\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. □

Theorem 3.2. *Let $\{\mathcal{U}(\tau)\}_{\tau\ge 0}\subset\mathcal{B}(\mathbb{H})$ be strongly continuous such that $\|c^{\wedge}(-\tau)\mathcal{U}(\tau)\|\le\psi(\tau)$ where $\psi\in L^2(\mathbb{R}_+)$. If $g\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, then*

$$
z(\tau):=\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)g(s)dW(s)\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right).
$$

Proof. Since $g\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, then by Proposition 3.1 $g(\tau)=c^{\wedge}(\tau)u_g(\tau)\ \forall\tau\in\mathbb{R}$, where $u_g\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. Then,

$$
\begin{aligned}
z(\tau)=\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)g(s)dW(s)&=\int_{-\infty}^{\tau}\mathcal{U}(\tau-s)c^{\wedge}(s)u_g(s)dW(s)\\
&=c^{\wedge}(\tau)\int_{-\infty}^{\tau}c^{\wedge}(s-\tau))\mathcal{U}(\tau-s)u_g(s)dW(s).
\end{aligned}
$$

Let $f(\tau)=\int_{-\infty}^{\tau}c^{\wedge}(s-\tau)\mathcal{U}(\tau-s)u_g(s)dW(s)$, for $\tau\in\mathbb{R}$. We have to show that $f\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. And by using the Ito's isometry property, we obtain

$$
\mathsf{E}\,\|f(\tau)\|^2=\mathsf{E}\left\|\int_{-\infty}^{\tau}c^{\wedge}(s-\tau))\mathcal{U}(\tau-s)u_g(s)dW(s)\right\|^2
$$

$$= \int_{-\infty}^{\tau} \|c^{\wedge}(s-\tau))\mathcal{U}(\tau-s)\|^2 \times \mathsf{E}\|u_g(s)\|^2 ds$$
$$\leq \int_{-\infty}^{\tau} \psi^2(\tau-s)\|u_g\|_{\infty}^2 ds$$
$$\leq \|u_g\|_{\infty}^2 \int_0^{\infty} \psi^2(\zeta)d\zeta \leq \|u_g\|_{\infty}^2 \|\psi\|_{L^2}^2 < \infty.$$

Let $\tau, \tau_0 \in \mathbb{R}$. Then, by using the Cauchy- Schwarz inequality, we get

$$\mathsf{E}\,\|f(\tau)-f(\tau_0)\|^2$$
$$= \mathsf{E}\left\|\int_{-\infty}^{\tau} c^{\wedge}(s-\tau))\mathcal{U}(\tau-s)u_g(s)dW(s) - \int_{-\infty}^{\tau_0} c^{\wedge}(s-\tau_0))\mathcal{U}(\tau_0-s)u_g(s)dW(s)\right\|^2$$
$$= \mathsf{E}\left\|\int_0^{\infty} c^{\wedge}(s))\mathcal{U}(s)u_g(\tau-s)dW(\tau+s) - \int_0^{\infty} c^{\wedge}(s))\mathcal{U}(s)u_g(\tau_0-s)dW(\tau_0+s)\right\|^2.$$

For $\zeta \in \mathbb{R}$, let $\widetilde{W}(s) = W(s+\zeta) - W(\zeta)$, $\forall s \in \mathbb{R}$. By using the fact that $\widetilde{W}$ is a Brownian motion and has the same distribution as W and by applying the Ito's isometry property and Fubini's theorem, we get that

$$\mathsf{E}\,\|f(\tau)-f(\tau_0)\|^2 \leq \mathsf{E}\left\|\int_0^{\infty} c^{\wedge}(-s)\mathcal{U}(s)\Big(u_g(\tau+s)-u_g(\tau_0+s)\Big)d\widetilde{W}(s)\right\|^2$$
$$\leq \int_0^{\infty} \|c^{\wedge}(-s)\mathcal{U}(s)\|^2 \mathsf{E}\|u_g(\tau+s)-u_g(\tau_0+s)\|^2 ds$$
$$\leq \int_0^{\infty} \psi^2(s)\mathsf{E}\|u_g(\tau+s)-u_g(\tau_0+s)\|^2 ds.$$

Since

$$\int_0^{\infty} \psi^2(s)\mathsf{E}\|u_g(\tau+s)-u_g(\tau_0+s)\|^2 ds \leq 4\|\psi\|_{L^2}^2 \|u_g\|_{\infty}^2 < \infty,$$

by using the fact that $u_g \in \mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$ and the dominated convergence theorem, it follows that

$$\lim_{\tau\to\tau_0} \int_0^{\infty} \psi^2(s)\mathsf{E}\|u_g(\tau+s)-u_g(t_0+s)\|^2 = 0.$$

Therefore

$$\lim_{\tau\to\tau_0} \mathsf{E}\,\|f(\tau)-f(t_0)\|^2 = 0.$$

We conclude that $f \in \mathcal{BC}\big(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\big)$. Hence by Proposition 3.1, we get that $z \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. Now let $q > 0$, then we have

$$\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\|c^{\wedge}(-\tau)[z(\tau+\omega)-cz(\tau)]\|^2 d\tau$$
$$= \frac{1}{2q}\int_{-q}^{q} \mathsf{E}\left\|c^{\wedge}(-\tau)\Big[\int_{-\infty}^{\tau+\omega} \mathcal{U}(\tau+\omega-s)g(s)dW(s) - c\int_{-\infty}^{\tau} \mathcal{U}(\tau-s)g(s)dW(s)\Big]\right\|^2 d\tau$$
$$= \frac{1}{2q}\int_{-q}^{q} \mathsf{E}\left\|c^{\wedge}(-\tau)\Big[\int_{-\infty}^{\tau} \mathcal{U}(\tau-s)g(s+\omega)dW(s+\omega) - c\int_{-\infty}^{\tau} \mathcal{U}(\tau-s)g(s)dW(s)\Big]\right\|^2 d\tau.$$

Let $\widetilde{W}(s) = W(s+\omega) - W(\omega)$. By using the fact that $\widetilde{W}$ is a Brownian motion and has the same distribution as W and by applying the Ito's isometry property and Fubinni's theorem, we get that

$$\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\|c^\wedge(-\tau)[z(\tau+\omega)-cz(\tau)]\|^2 d\tau \\
=&\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\left\|\int_{-\infty}^{\tau} c^\wedge(-\tau)\mathcal{U}(\tau-s)[g(s+\omega)-cg(s)]d\widetilde{W}(s)\right\|^2 d\tau \\
\leq&\frac{1}{2q}\int_{-q}^{q}\int_{-\infty}^{\tau} \mathsf{E}\left\|c^\wedge(-\tau)\mathcal{U}(\tau-s)[g(s+\omega)-cg(s)]\right\|^2 ds\, d\tau \\
\leq&\frac{1}{2q}\int_{-q}^{q}\int_{0}^{\infty} \mathsf{E}\left\|c^\wedge(-\tau)\mathcal{U}(s)[g(\tau-s+\omega)-cg(\tau-s)]\right\|^2 ds\, d\tau \\
\leq&\frac{1}{2q}\int_{-q}^{q}\int_{0}^{\infty} \mathsf{E}\left\|c^\wedge(-s)\mathcal{U}(s)c^\wedge(s-\tau)[g(\tau-s+\omega)-cg(\tau-s)]\right\|^2 ds\, d\tau \\
\leq&\frac{1}{2q}\int_{-q}^{q}\int_{0}^{\infty} \psi^2(s)\mathsf{E}\left\|c^\wedge(s-\tau)[f(\tau-s+\omega)-cg(\tau-s)]\right\|^2 ds\, d\tau. \\
\leq&\int_{0}^{\infty}\psi^2(s)\left[\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\left\|c^\wedge(s-\tau)[g(\tau-s+\omega)-cg(\tau-s)]\right\|^2 ds\right] d\tau.
\end{aligned}$$

By $g \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, Lemma3.3-(ii) and the dominated convergence theorem, we get that

$$\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\|c^\wedge(-\tau)[z(\tau+\omega)-cz(\tau)]\|^2 d\tau = 0.$$

Thus $z \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. □

3.4. Some superposition theorems in $\mathcal{PSAP}_{\omega,c}(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H}))$

Let's state and prove some superposition theorems. We define the set

$$\mathbb{L}(q) = \left\{h : \mathbb{R} \to \mathbb{R} \text{ is locally integrable and } \lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q} h(\tau)d\tau < \infty\right\}.$$

For given $F \in \mathcal{C}\big(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H})\big)$, we consider the following conditions:

(C1) (a) $\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)F(\tau,y)\|^2 < \infty$ uniformly for $y \in \mathbb{L}^2(\Omega,\mathbb{H})$.

(b) $\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega, cy) - cF(\tau,y)]\|^2 d\tau = 0$ uniformly for $y \in \mathbb{L}^2(\Omega,\mathbb{H})$.

(C2) There exists a constant $k_F > 0$ such that for all $y, \tilde{y} \in \mathbb{L}^2(\Omega,\mathbb{H})$ and $\tau \in \mathbb{R}$,

$$\mathsf{E}\|F(\tau,y) - F(\tau,\tilde{y})\|^2 \leq k_F \mathsf{E}\|y-\tilde{y}\|^2.$$

(C3) $F_\tau(y) := c^\wedge(-\tau)F(\tau, c^\wedge(\tau)y)$ is uniformly continuous for y in any bounded subset of $\mathbb{L}^2(\Omega,\mathbb{H})$ uniformly in $\tau \in \mathbb{R}$; that is , for any $\epsilon > 0$ and any bounded subset $Q \subset \mathbb{L}^2(\Omega,\mathbb{H})$, there exists $\delta > 0$ such that $x, y \in Q$ and $\mathsf{E}\|x-y\|^2 < \delta$ imply that

$$\mathsf{E}\|F_\tau(x) - F_\tau(y)\|^2 \leq \epsilon.$$

(C4) There exists a function $\mu(\cdot)\in\mathbb{L}(q)$ such that for any $\epsilon>0$, there is a constant $\kappa>0$ such that
$$\mathsf{E}\|F_\tau(x)-F_\tau(y)\|^2\le\mu(\tau)\epsilon$$
for all $x,y\in\mathbb{L}^2(\Omega,\mathbb{H})$ with $\mathsf{E}\|x-y\|^2<\kappa$ and $\tau\in\mathbb{R}$.

Theorem 3.3. *Suppose that* $F\in\mathcal{C}\big(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H})\big)$ *satisfies* **(C1)** *and* **(C2)**. *Then for each* $\varphi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, $\tau\mapsto F(\tau,\varphi(\tau))\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$.

Proof. From **(C1)**-(a), we have that $F(\cdot,\varphi(\cdot))\in\mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$. For each $\varphi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, we have that
$$\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2d\tau=0.$$

We have
$$\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega,\varphi(\tau+\omega))-cF(\tau,\varphi(\tau))]\|^2d\tau\\
=&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)\left[F(\tau+\omega,\varphi(\tau+\omega))-cF(\tau,\varphi(\tau))+cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)\right.\right.\\
&\qquad\qquad\left.\left.-cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)\right]\right\|^2d\tau\\
\le&\frac{1}{2q}\int_{-q}^{q}2\mathsf{E}\left\|c^\wedge(-\tau)\left[F(\tau+\omega,\varphi(\tau+\omega))-cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)\right]\right\|^2d\tau\\
&+\frac{1}{2q}\int_{-q}^{q}2\mathsf{E}\left\|c^\wedge(-\tau)\left[cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)-cF(\tau,\varphi(\tau))\right]\right\|^2d\tau\\
&:=2[J_1(q)+J_2(q)].
\end{aligned}$$

From **(C1)**-(b), we get $\lim\limits_{q\to\infty}J_1(q)=0$. For $J_2(q)$, we have
$$\begin{aligned}
0\le J_2(q)=&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)\left[cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)-cF(\tau,\varphi(\tau))\right]\right\|^2d\tau\\
=&\frac{1}{2q}\int_{-q}^{q}|c^\wedge(-\tau)|^2\,|c|^2\,\mathsf{E}\left\|F\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)-F(\tau,\varphi(\tau))\right\|^2d\tau\\
\le&\frac{1}{2q}\int_{-q}^{q}|c^\wedge(-\tau)|^2\,|c|^2\,k_F\,\mathsf{E}\left\|\frac{1}{c}\varphi(\tau+\omega)-\varphi(\tau)\right\|^2d\tau\\
\le&k_F\,\frac{1}{q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\right\|^2d\tau\to0\quad\text{as}\quad q\to\infty.
\end{aligned}$$

Then $\lim\limits_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega,\varphi(\tau+\omega))-cF(\tau,\varphi(\tau))]\|^2d\tau=0$. Hence $F(\cdot,\varphi(\cdot))\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. □

Theorem 3.4. *Suppose that* $F\in\mathcal{C}\big(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H})\big)$ *satisfies* **(C1)** *and* **(C3)**. *Then for each* $\varphi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, $F(\cdot,\varphi(\cdot))\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$.

Proof. From condition **(C1)**-(a), we have $F(\cdot,\varphi(\cdot)) \in \mathcal{BC}_{\omega,c}\big(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\big)$ and there exists $K>0$ such that $\sup_{\tau\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)F(\tau,y))\|^2 < K$ for all $y\in\mathbb{L}^2(\Omega,\mathbb{H})$. Now, we have

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega,\varphi(\tau+\omega))-cF(\tau,\varphi(\tau))]\|^2 d\tau\\
\leq&\frac{1}{2q}\int_{-q}^{q}2\mathsf{E}\left\|c^\wedge(-\tau)\left[F(\tau+\omega,\varphi(\tau+\omega))-cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)\right]\right\|^2 d\tau\\
&+\frac{1}{2q}\int_{-q}^{q}2\mathsf{E}\left\|c^\wedge(-\tau)\left[cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)-cF(\tau,\varphi(\tau))\right]\right\|^2 d\tau\\
&:=2[J_1(q)+J_2(q)].
\end{aligned}
$$

From **(C1)**-(b), we get $\lim_{q\to\infty} J_1(q)=0$. Let $Q:=\{c^\wedge(-\tau)\varphi(\tau)\,:\,\tau\in\mathbb{R}\}$. Since $\varphi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ then $Q\subset\mathbb{L}^2(\Omega,\mathbb{H})$. Let $\epsilon>0$. Then by **(C3)** there exists $\delta>0$ such that

$$\mathsf{E}\|F_\tau(c^\wedge(-\tau-\omega)\varphi(\tau+\omega))-F_\tau(c^\wedge(-\tau)\varphi(\tau))\|^2\leq\frac{\epsilon}{|c|^2}$$

if

$$\mathsf{E}\|c^\wedge(-\tau-\omega)\varphi(\tau+\omega)-c^\wedge(-\tau)\varphi(\tau)]\|^2=\mathsf{E}\|c^\wedge(-\tau-\omega)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2\leq\delta. \tag{3.1}$$

Note that relation (3.1) is equivalent to $\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega)-c\varphi(\tau)]\|^2\leq|c|^2\delta$.

Then, we have

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)\left[cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)-cF(\tau,\varphi(\tau))\right]\right\|^2 d\tau\\
=&|c|^2\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[F(\tau,c^\wedge(\tau)c^\wedge(-\tau-\omega)\varphi(\tau+\omega))-F(\tau,c^\wedge(-\tau)c^\wedge(\tau)\varphi(\tau))]\|^2 d\tau\\
=&|c|^2\frac{1}{2q}\int_{[-q,q]\setminus\mathcal{M}_{q,|c|^2\delta}(\varphi)}\mathsf{E}\left\|c^\wedge(-\tau)\left[F\left(\tau,c^\wedge(\tau)c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)\right.\right.\\
&\qquad\left.\left.-F(\tau,c^\wedge(\tau)c^\wedge(-\tau)\varphi(\tau))\right]\right\|^2 d\tau\\
&+|c|^2\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)}\mathsf{E}\left\|c^\wedge(-\tau)\left[F\left(\tau,c^\wedge(\tau)c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)\right.\right.\\
&\qquad\left.\left.-F(\tau,c^\wedge(\tau)c^\wedge(-\tau)\varphi(\tau))\right]\right\|^2 d\tau\\
=&|c|^2\frac{1}{2q}\int_{[-q,q]\setminus\mathcal{M}_{q,|c|^2\delta}(\varphi)}\mathsf{E}\left\|c^\wedge(-\tau)\left[F_\tau\left(c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)\right.\right.\\
&\qquad\left.\left.-F_\tau\left(c^\wedge(-\tau)\varphi(\tau)\right)\right]\right\|^2 d\tau\\
&+|c|^2\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)}\mathsf{E}\left\|c^\wedge(-\tau)\left[F_\tau\left(c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)-F_\tau\left(c^\wedge(-\tau)\varphi(\tau)\right)\right]\right\|^2 d\tau\\
\leq&\left(1-\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)}d\tau\right)\epsilon+4K|c|^2\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)}d\tau.
\end{aligned}
$$

Since $\varphi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ then we derive by Lemma 3.4 that $\lim\limits_{q\to\infty} J_2(q) = 0$. Thus, we obtain $\lim\limits_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega,\varphi(\tau+\omega)) - cF(\tau,\varphi(\tau))]\|^2 d\tau = 0$. Therefore, we get $\tau \mapsto F(\tau,\varphi(\tau)) \in \mathcal{SAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

□

Theorem 3.5. *Suppose that* $F \in \mathcal{C}\left(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}), \mathbb{L}^2(\Omega,\mathbb{H})\right)$ *satisfies* **(C1)** *and* **(C4)**. *Then for each* $\varphi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, $F(\cdot,\varphi(\cdot)) \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$.

Proof. From condition **(C1)**-(a), we know that $F(\cdot,\varphi(\cdot)) \in \mathcal{BC}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ and there exists $K > 0$ such that $\sup\limits_{\tau\in\mathbb{R}} \mathsf{E}\|c^\wedge(-\tau)F(\tau,y))\|^2 < K$ for all $y \in \mathbb{L}^2(\Omega,\mathbb{H})$. Now, we have

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega,\varphi(\tau+\omega)) - cF(\tau,\varphi(\tau))]\|^2 d\tau \\
\leq&\frac{1}{2q}\int_{-q}^{q} 2\mathsf{E}\left\|c^\wedge(-\tau)\left[F(\tau+\omega,\varphi(\tau+\omega)) - cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right)\right]\right\|^2 d\tau \\
&+\frac{1}{2q}\int_{-q}^{q} 2\mathsf{E}\left\|c^\wedge(-\tau)\left[cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right) - cF(\tau,\varphi(\tau))\right]\right\|^2 d\tau \\
&:= 2[J_1(q) + J_2(q)].
\end{aligned}
$$

From **(C1)**-(b), we get $\lim\limits_{q\to\infty} J_1(q) = 0$.

Let $\epsilon > 0$. Since for all $\tau \in \mathbb{R}$ $c^\wedge(-\tau)\varphi(\tau) \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, then by **(C3)** there exists $\delta > 0$ such that

$$
\mathsf{E}\|F_\tau(c^\wedge(-\tau-\omega)\varphi(\tau+\omega)) - F_\tau(c^\wedge(-\tau)\varphi(\tau))\|^2 \leq \frac{\mu(\tau)\epsilon}{|c|^2}
$$

if

$$
\mathsf{E}\|c^\wedge(-\tau-\omega)\varphi(\tau+\omega) - c^\wedge(-\tau)\varphi(\tau)]\|^2 = \mathsf{E}\|c^\wedge(-\tau-\omega)[\varphi(\tau+\omega) - c\varphi(\tau)]\|^2 \leq \delta. \tag{3.2}
$$

Note that relation (3.2) is equivalent to $\mathsf{E}\|c^\wedge(-\tau)[\varphi(\tau+\omega) - c\varphi(\tau)]\|^2 \leq |c|^2\delta$.

Then, we have

$$
\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\left\|c^\wedge(-\tau)\left[cF\left(\tau,\frac{1}{c}\varphi(\tau+\omega)\right) - cF(\tau,\varphi(\tau))\right]\right\|^2 d\tau \\
=&|c|^2\frac{1}{2q}\int_{-q}^{q} \mathsf{E}\left\|c^\wedge(-\tau)\left[F\left(\tau,c^\wedge(\tau)c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right) - F(\tau,c^\wedge(-\tau)c^\wedge(\tau)\varphi(\tau))\right]\right\|^2 d\tau \\
=&|c|^2\frac{1}{2q}\int_{[-q,q]\setminus\mathcal{M}_{q,|c|^2\delta}(\varphi)} \mathsf{E}\left\|c^\wedge(-\tau)\left[F\left(\tau,c^\wedge(\tau)c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)\right.\right. \\
&\qquad\qquad\left.\left. - F(\tau,c^\wedge(\tau)c^\wedge(-\tau)\varphi(\tau))\right]\right\|^2 d\tau \\
&+|c|^2\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)} \mathsf{E}\left\|c^\wedge(-\tau)\left[F\left(\tau,c^\wedge(\tau)c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)\right.\right. \\
&\qquad\qquad\left.\left. - F(\tau,c^\wedge(\tau)c^\wedge(-\tau)\varphi(\tau))\right]\right\|^2 d\tau
\end{aligned}
$$

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$$=|c|^2\frac{1}{2q}\int_{[-q,q]\setminus\mathcal{M}_{q,|c|^2\delta}(\varphi)}\mathsf{E}\left\|c^\wedge(-\tau)\left[F_\tau\left(c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)-F_\tau\left(c^\wedge(-\tau)\varphi(\tau)\right)\right]\right\|^2 d\tau$$

$$+|c|^2\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)}\mathsf{E}\left\|c^\wedge(-\tau)\left[F_\tau\left(c^\wedge(-\tau-\omega)\varphi(\tau+\omega)\right)-F_\tau\left(c^\wedge(-\tau)\varphi(\tau)\right)\right]\right\|^2 d\tau$$

$$\leq\epsilon\frac{1}{2q}\int_{-q}^{q}\mu(\tau)d\tau+4K|c|^2\frac{1}{2q}\int_{\mathcal{M}_{q,|c|^2\delta}(\varphi)}d\tau.$$

By the fact that $\mu\in\mathbb{L}(q)$ and Lemma 3.4-(ii), we derive that $\lim_{q\to\infty}J_2(q)=0$. Thus, we obtain $\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[F(\tau+\omega,\varphi(\tau+\omega))-cF(\tau,\varphi(\tau))]\|^2d\tau=0$. Therefore, we get $\tau\mapsto F(\tau,\varphi(\tau))\in\mathcal{SAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$.

□

Corollary 3.1. *Suppose that* $F\in\mathcal{C}\left(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H})\right)$ *satisfies* **(C1)** *and the following condition:*
$(\mathbf{C'4})$ *There exists a function* $\mu\in\mathbb{L}(q)$ *such that*

$$\mathsf{E}\|F_\tau(x)-F_\tau(y)\|^2\leq\mu(\tau)\mathsf{E}\|x-y\|^2.$$

for all $x,y\in\mathbb{L}^2(\Omega,\mathbb{H})$ *and* $\tau\in\mathbb{R}$. *Then for each* $\varphi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, $F(\cdot,\varphi(\cdot))\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$.

Proof. Since condition $(\mathbf{C'4})$ implies $(\mathbf{C4})$, the proof can be completed as in Theorem 3.5. □

4. A class of nonlinear stochastic fractional evolution equations

We discuss the existence and uniqueness of pseudo S-asymptotically (ω,c)-periodic mild solution for problems (1.1). We need to recall some facts about Weyl fractional integral and derivative of order $\alpha>0$, and α-resolvent operators (see [22]). First, suppose that $\mathbb{X}$ is a Banach space. For given function $h:\mathbb{R}\to\mathbb{X}$, the Weyl fractional integral of order $\alpha>0$ is defined by

$$\partial_\tau^{-\alpha}h(\tau):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\tau}(\tau-s)^{\alpha-1}h(s)ds,\qquad\tau\in\mathbb{R},$$

when this integral is convergent. The Weyl fractional derivative ∂_τ^α of order α is defined by

$$\partial_\tau^\alpha h(\tau)=\frac{d^n}{d\tau^n}\partial_\tau^{-(n-\alpha)}h(\tau),\quad\tau\in\mathbb{R},$$

where $n=[\alpha]+1$, and the notation $[\alpha]$ represents the integer part of α. Now, Let A be a closed and linear operator with domain $\mathcal{D}(A)$ defined on a Banach space $\mathbb{X}$, and $\alpha>0$. For a given kernel $b(\cdot)\in L^1_{loc}(\mathbb{R}_+)$, it is said that A is the generator of an α-resolvent family if there exists $\xi>0$ and a strongly continuous family $\mathcal{R}_\alpha:\mathbb{R}_+\to\mathcal{B}(\mathbb{X})$ such that

$$\left\{\frac{\lambda^\alpha}{1+\hat{b}(\lambda)}:\mathrm{Re}(\lambda)>\xi\right\}\subseteq\rho(A)$$

and for all $y \in \mathbb{X}$,

$$
\begin{aligned}
(\lambda^\alpha - (1+\hat{b}(\lambda))A)^{-1}y &= \frac{1}{1+\hat{b}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{b}(\lambda)} - A\right)^{-1} y \\
&= \int_0^\infty e^{-\lambda t}\mathcal{R}_\alpha(\tau) y d\tau, \quad \mathrm{Re}\lambda > \xi,
\end{aligned}
$$

where $\hat{b}$ denotes the Laplace transform of b. $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}$ is called the α-resolvent family generated by the operator A.

Motivated by Ponce [22], we present the concept of mild solutions for Eq.(1.1). For each $\tau \in \mathbb{R}$, $W(\tau)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_\tau)$ with $\mathcal{F}_\tau = \sigma\{W(u) - W(v) | u, v \leq \tau\}$.

Definition 4.1. An $\mathcal{F}_\tau$-progressively measurable process $\{\Phi(\tau)\}_{\tau\in\mathbb{R}}$ is called a mild solution of problem (1.1) if it satisfies the following stochastic integral equation

$$
\Phi(\tau) = \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) g(s, \Phi(s)) ds + \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) f(s, \Phi(s)) dW(s)
$$

for all $\tau \in \mathbb{R}$, where $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}$ is the resolvent family generated by the operator A.

Theorem 4.1. *Suppose that the operator A generates an α-resolvent operator $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0} \subset \mathcal{B}(\mathbb{H})$ such that for $\tau \geq 0$, $\|c^\wedge(-\tau)\mathcal{R}_\alpha(\tau)\| \leq \mathcal{K}_\alpha(\tau)$ where $\mathcal{K}_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Furthermore, assume that $g \in \mathcal{C}\big(\mathbb{R} \times \mathbb{L}^2(\Omega,\mathbb{H}), \mathbb{L}^2(\Omega,\mathbb{H})\big)$, $f \in \mathcal{C}\left(\mathbb{R} \times \mathbb{L}^2(\Omega,\mathbb{H}), \mathcal{L}(\mathbb{K}, \mathbb{H})\right)$ satisfies* **(C1)** *and there exist constants $L, L' > 0$ such that for any $\Phi_1, \Phi_2 \in \mathbb{L}^2(\Omega,\mathbb{H})$,*

$$
\begin{aligned}
&\mathsf{E}\|g(\tau, \Phi_1) - g(\tau, \Phi_2)\|^2 \leq L\, \mathsf{E}\|\Phi_1 - \Phi_2\|^2, \\
&\mathsf{E}\|f(\tau, \Phi_1) - f(\tau, \Phi_2)\|^2_{\mathcal{L}(\mathbb{K},\mathbb{H})} \leq L'\, \mathsf{E}\|\Phi_1 - \Phi_2\|^2,
\end{aligned}
$$

uniformly for all $\tau \in \mathbb{R}$.

Then equation (1.1) has a unique mild solution $\Phi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, provided

$$
2\left(\|\mathcal{K}_\alpha\|^2_{L^1} L + L'\|\mathcal{K}_\alpha\|^2_{L^2}\right) < 1.
$$

Proof. From Theorem 3.3, for each $\Phi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, the stochastic processes $s \mapsto f(s, \Phi(s))$ and $s \mapsto g(s, \Phi(s))$ belongs to $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. From Lemmas 3.1, 3.2 and 3.2-(a), we can define the operator

$$
\mathcal{S} : \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right) \to \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)
$$

by

$$
(\mathcal{S}\Phi)(\tau) = \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) g(s, \Phi(s)) ds + \int_{-\infty}^{\tau} \mathcal{R}_\alpha(\tau - s) f(s, \Phi(s)) dW(s). \quad (4.1)
$$

Let $\Phi_1, \Phi_2 \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ and $\tau \in \mathbb{R}$. Then using Cauchy- Schwarz's inequality and Ito's isometry property of stochastic integral, we have

$$
\mathsf{E}\|c^\wedge(-\tau)(\mathcal{S}\Phi_1)(\tau) - (\mathcal{S}\Phi_2)(\tau)\|^2
$$

$$
\begin{aligned}
&\leq 2\mathsf{E}\| \int_{-\infty}^{\tau} c^{\wedge}(-\tau)\mathcal{R}_\alpha(\tau-s)[g(s,\Phi_1(s))-g(s,\Phi_2(s))]ds\|^2 \\
&\quad + 2\mathsf{E}\| \int_{-\infty}^{\tau} c^{\wedge}(-\tau)\mathcal{R}_\alpha(\tau-s)[f(s,\Phi_1(s))-f(s,\Phi_2(s))]dW(s)\|^2 \\
&\leq 2\mathsf{E}\| \int_{-\infty}^{\tau} c^{\wedge}(-\tau+s)\mathcal{R}_\alpha(\tau-s)c^{\wedge}(-s)[g(s,\Phi_1(s))-g(s,\Phi_2(s))]ds\|^2 \\
&\quad + 2\mathsf{E}\| \int_{-\infty}^{\tau} c^{\wedge}(-\tau+s)\mathcal{R}_\alpha(\tau-s)c^{\wedge}(-s)[f(s,\Phi_1(s))-f(s,\Phi_2(s))]dW(s)\|^2 \\
&\leq 2\int_{-\infty}^{\tau}\mathcal{K}_\alpha(\tau-s)ds\left(\int_{-\infty}^{\tau}\mathcal{K}_\alpha(\tau-s)\mathsf{E}\|c^{\wedge}(-s)[g(s,\Phi_1(s))-g(s,\Phi_2(s))]\|^2ds\right) \\
&\quad + 2\left(\int_{-\infty}^{\tau}\mathcal{K}_\alpha^2(\tau-s)\mathsf{E}\|c^{\wedge}(-s)[f(s,\Phi_1(s))-f(s,\Phi_2(s))]\|^2\,ds\right) \\
&\leq 2L\left(\int_{-\infty}^{\tau}\mathcal{K}_\alpha(\tau-s)ds\right)^2\sup_{s\in\mathbb{R}}\mathsf{E}\|c^{\wedge}(-s)[\Phi_1(s)-\Phi_2(s)]\|^2 \\
&\quad + 2L'\left(\int_{-\infty}^{\tau}\mathcal{K}_\alpha^2(\tau-s)ds\right)\|\Phi_1-\Phi_2\|_{b,\omega,c}^2 \\
&\leq 2\left(\|\mathcal{K}_\alpha\|_{L^1}^2L+L'\|\mathcal{K}_\alpha\|_{L^2}^2\right)\|\Phi_1-\Phi_2\|_{b,\omega,c}^2.
\end{aligned}
$$

Therefore,

$$
\|\mathcal{S}\Phi_1-\mathcal{S}\Phi_2\|_{b,\omega,c}^2\leq 2\left(\|\mathcal{K}_\alpha\|_{L^1}^2L+L'\|\mathcal{K}_\alpha\|_{L^2}^2\right)\|\Phi_1-\Phi_2\|_{b,\omega,c}^2,
$$

which proves that $\mathcal{S}$ is a contraction owing to the condition

$$
2\left(\|\mathcal{K}_\alpha\|_{L^1}^2L+L'\|\mathcal{K}_\alpha\|_{L^2}^2\right)<1.
$$

Thus there exists a unique $\Phi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ such that $\mathcal{S}\Phi=\Phi$ via Banach fixed point theorem.

□

Theorem 4.2. *Suppose that the operator A generates an α-resolvent operator $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}\subset\mathcal{B}(\mathbb{H})$ such that $\|c^{\wedge}(-\tau)\mathcal{R}_\alpha(\tau)\|\leq\mathcal{K}_\alpha(\tau)$ where $\mathcal{K}_\alpha$ is a non-increasing function with $\mathcal{K}_0:=\sum_{n=0}^{\infty}\mathcal{K}_\alpha(n)<\infty$. Furthermore, assume that $g\in\mathcal{C}\big(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H})\big)$, $f\in\mathcal{C}\left(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H}),\mathcal{L}(\mathbb{K},\mathbb{H})\right)$ satisfies* **(C1)** *and there exist $\mu_1,\mu_2\in L^1(\mathbb{R},\mathbb{R}_+)$ such that*

$$
\begin{aligned}
&\mathsf{E}\|c^{\wedge}(-\tau)[g(\tau,c^{\wedge}(\tau)\Phi_1)-g(\tau,c^{\wedge}(\tau)\Phi_2)]\|^2\leq\mu_1(\tau)\,\mathsf{E}\|\Phi_1-\Phi_2\|^2,\\
&\mathsf{E}\|c^{\wedge}(-\tau)[f(\tau,c^{\wedge}(\tau)\Phi_1)-f(\tau,c^{\wedge}(\tau)\Phi_2)]\|_{\mathcal{L}(\mathbb{K},\mathbb{H})}^2\leq\mu_2(\tau)\,\mathsf{E}\|\Phi_1-\Phi_2\|^2,
\end{aligned}
$$

for $\Phi_1,\Phi_2\in\mathbb{L}^2(\Omega,\mathbb{H})$, for all $\tau\in\mathbb{R}$. Then equation (1.1) *has a unique mild solution $\Phi\in\mathcal{PSAP}_{\omega,c}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, provided*

$$
2\overline{\mu}\mathcal{K}_0\left(\|\mathcal{K}_\alpha\|_{L^1}+\mathcal{K}_0\right)<1,
$$

where $\overline{\mu} = \max\left\{ \sup_{\tau\in\mathbb{R}} \int_{\tau+1}^{\tau} \mu_1(s)ds,\ \sup_{\tau\in\mathbb{R}} \int_{\tau+1}^{\tau} \mu_2(s)ds \right\}$.

Proof. Since $\mathcal{K}_\alpha$ is a non-increasing function with $\sum_{n=0}^{\infty}\mathcal{K}_\alpha(n) < \infty$, then we deduce that $\mathcal{K}_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Then from Corollary 3.1, for each $\Phi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$, the stochastic processes $s \mapsto f(s,\Phi(s))$ and $s \mapsto g(s,\Phi(s))$ belongs to $\mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$. From Lemmas 3.1, 3.2 and 3.2-(a), we can define the operator

$$\mathcal{S} : \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right) \longrightarrow \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$$

as in (4.1). Let $\Phi_1, \Phi_2 \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ and $\tau \in \mathbb{R}$. Then using Cauchy-Schwarz's inequality and Ito's isometry property of stochastic integral, we have

$$\begin{aligned}
&\mathsf{E}\|c^\wedge(-\tau)(\mathcal{S}\Phi_1)(\tau) - (\mathcal{S}\Phi_2)(\tau)\|^2 \\
&\quad\leq 2\mathsf{E}\left\| \int_{-\infty}^{\tau} c^\wedge(-\tau)\mathcal{R}_\alpha(\tau-s)[g(s,\Phi_1(s)) - g(s,\Phi_2(s))]ds \right\|^2 \\
&\qquad + 2\mathsf{E}\left\| \int_{-\infty}^{\tau} c^\wedge(-\tau)\mathcal{R}_\alpha(\tau-s)[f(s,\Phi_1(s)) - f(s,\Phi_2(s))]dW(s) \right\|^2 \\
&\quad\leq 2\mathsf{E}\left\| \int_{-\infty}^{\tau} c^\wedge(-\tau+s)\mathcal{R}_\alpha(\tau-s)c^\wedge(-s)[g(s,\Phi_1(s)) - g(s,\Phi_2(s))]ds \right\|^2 \\
&\qquad + 2\mathsf{E}\left\| \int_{-\infty}^{\tau} c^\wedge(-\tau+s)\mathcal{R}_\alpha(\tau-s)c^\wedge(-s)[f(s,\Phi_1(s)) - f(s,\Phi_2(s))]dW(s) \right\|^2 \\
&\quad\leq 2\int_{-\infty}^{\tau}\mathcal{K}_\alpha(\tau-s)ds\left(\int_{-\infty}^{\tau}\mathcal{K}_\alpha(\tau-s)\mathsf{E}\|c^\wedge(-s)[g(s,\Phi_1(s)) - g(s,\Phi_2(s))]\|^2\, ds \right) \\
&\qquad + 2\left(\int_{-\infty}^{\tau}\mathcal{K}_\alpha^2(\tau-s)\mathsf{E}\|c^\wedge(-s)[f(s,\Phi_1(s)) - f(s,\Phi_2(s))]\|^2 ds \right) \\
&\quad\leq 2\|\mathcal{K}_\alpha\|_{L^1}\sum_{n=0}^{\infty}\left(\int_{\tau-(n+1)}^{\tau-n}\mu_1(s)\mathcal{K}_\alpha(\tau-s)\mathsf{E}\|c^\wedge(-s)[\Phi_1(s) - \Phi_2(s)]\|^2\, ds \right) \\
&\qquad + 2\sum_{n=0}^{\infty}\left(\int_{\tau-(n+1)}^{\tau-n}\mu_2(s)\mathcal{K}_\alpha^2(\tau-s)\mathsf{E}\|c^\wedge(-s)[\Phi_1(s) - \Phi_2(s)]\|^2 ds \right) \\
&\quad\leq 2\|\mathcal{K}_\alpha\|_{L^1}\sum_{n=0}^{\infty}\mathcal{K}_\alpha(n)\left(\int_{\tau-(n+1)}^{\tau-n}\mu_1(s)ds \right)\|\Phi_1 - \Phi_2\|_{b,\omega,c}^2 \\
&\qquad + 2\sum_{n=0}^{\infty}\mathcal{K}_\alpha^2(n)\left(\int_{\tau-(n+1)}^{\tau-n}\mu_2(s)ds \right)\|\Phi_1 - \Phi_2\|_{b,\omega,c}^2 \\
&\quad\leq 2\overline{\mu}\mathcal{K}_0\left(\|\mathcal{K}_\alpha\|_{L^1} + \mathcal{K}_0 \right)\|\Phi_1 - \Phi_2\|_{b,\omega,c}^2.
\end{aligned}$$

Therefore, we get

$$\|\mathcal{S}\Phi_1 - \mathcal{S}\Phi_2\|_{b,\omega,c}^2 \leq 2\overline{\mu}\mathcal{K}_0\left(\|\mathcal{K}_\alpha\|_{L^1} + \mathcal{K}_0 \right)\|\Phi_1 - \Phi_2\|_{b,\omega,c}^2.$$

Since $2\overline{\mu}\mathcal{K}_0\left(\|\mathcal{K}_\alpha\|_{L^1} + \mathcal{K}_0 \right) < 1$, then there exists a unique $\Phi \in \mathcal{PSAP}_{\omega,c}\left(\mathbb{R}, \mathbb{L}^2(\Omega,\mathbb{H})\right)$ such that $\mathcal{S}\Phi = \Phi$ via Banach fixed point theorem.

□

5. An illustrative example

Let $\mathbb{H} = L^2[0,\pi]$, $|c| \geq 1$, $\omega > 0$ $1 < \alpha < 2$, $\nu > 0$ and consider the following problem

$$\begin{cases} \partial_\tau^\alpha u(\tau,x) = & -\nu\, u(\tau,x) - \dfrac{\nu^2}{4}\displaystyle\int_{-\infty}^{\tau} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)} u(s,x)ds \\ & +g(\tau,u(\tau,x)) + f(\tau,u(\tau,x))\dfrac{\partial W(\tau)}{\partial t}, \quad (\tau,x)\in\mathbb{R}\times(0,\pi), \\ & u(\tau,0) = u(\tau,\pi) = 0, \end{cases} \tag{5.1}$$

where $W(\tau)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_\tau)$. The problem (5.1) can be written into the form (1.1) with $\Phi(\tau)(x) = u(\tau,x)$, $b(\tau) = \dfrac{\nu}{4}\dfrac{\tau^{\alpha-1}}{\Gamma(\alpha)}$ and $A = -\nu I$, I is the identity operator on the Hilbert space $\mathbb{H}$. It follows from [22, Example 4.17], that A generates a $\alpha-$resolvent family $\{\mathcal{R}_\alpha(\tau)\}_{\tau\geq 0}$ with its Laplace transform satisfying

$$\hat{\mathcal{R}}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + \nu/2)^2} = \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha+\nu/2)}\cdot\frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha+\nu/2)}$$

and

$$\mathcal{R}_\alpha(\tau) = (K*K)(\tau) \text{ where } K(\tau) = \tau^{\frac{\alpha}{2}-1}\mathcal{E}_{\alpha,\alpha/2}\Big(-\frac{\nu}{2}\tau^\alpha\Big)$$

and $\mathcal{E}_{\alpha,\alpha/2}(\cdot)$ is the Mittag-Leffter function (see [2]). From [21, Theorem 4.12], there exists a constant $C > 0$, depending only on α, such that, for $\tau \geq 0$

$$\|c^\wedge(-\tau)\mathcal{R}_\alpha(\tau)\| \leq \frac{|c|^\wedge(-\tau)C}{1+\nu\tau^\alpha} \leq \frac{C}{1+\nu\tau^\alpha} := \mathcal{K}_\alpha(\tau).$$

It follows that $\mathcal{K}_\alpha$ is a non-increasing function. Since $\alpha > 1$ then $\mathcal{K}_0 := \sum_{n=0}^{\infty}\mathcal{K}_\alpha(n) < \infty$. Simple calculations yield that :

$$\|\mathcal{K}_\alpha\|_{L^1} = \frac{C}{\alpha\nu^{1/\alpha}}\mathsf{B}\left(\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right) < \infty$$

and

$$\|\mathcal{K}_\alpha\|_{L^2}^2 = \int_0^\infty \frac{C^2}{(1+\nu\tau^\alpha)^2}\,d\tau = \frac{C^2}{\alpha\nu^{1/\alpha-1}}\mathsf{B}\left(\frac{1}{\alpha}, 2-\frac{1}{\alpha}\right) < \infty,$$

where $\mathsf{B}(\cdot,\cdot)$ denotes the Beta function.

5.1. Illustration of Theorem 4.1

To illustrate Theorem 4.1, let us take the forcing terms as follows:

$$g(\tau,u) = \psi(\tau)\cos(n(\tau)u) + \frac{1}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|),$$

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$$f(\tau,u)=\psi(\tau)e^{-\|n(\tau)u\|}+\frac{1}{1+\tau^2}\psi(\tau)\cos(\|a(\tau)u\|),$$

for $\tau\in\mathbb{R}$ and $u\in\mathbb{H}$, where $\psi\in\mathcal{P}_{\omega,c}(\mathbb{R},\mathbb{R})$ and $n,a\in\mathcal{P}_{\omega,\frac{1}{c}}(\mathbb{R},\mathbb{R})$. For $u\in\mathbb{H}$,

$$\begin{aligned}
&\sup_{\tau\in\mathbb{R}}\mathsf{E}\|c^\wedge(-\tau)g(\tau,u)\|^2\\
&\quad=\sup_{\tau\in\mathbb{R}}\mathsf{E}\|c^\wedge(-\tau)\left(\psi(\tau)\cos(n(\tau)u)+\frac{1}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|)\right)\|^2\\
&\quad\le 2\sup_{\tau\in\mathbb{R}}\mathsf{E}\|c^\wedge(-\tau)\psi(\tau)\cos(n(\tau)u)\|^2+2\sup_{\tau\in\mathbb{R}}\mathsf{E}\|\frac{c^\wedge(-\tau)}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|)\|^2\\
&\quad\le 2\sup_{\tau\in\mathbb{R}}\|c^\wedge(-\tau)\psi(\tau)\|^2\mathsf{E}\|\cos(n(\tau)u)\|^2+2\sup_{\tau\in\mathbb{R}}\|c^\wedge(-\tau)\psi(\tau)\|^2\mathsf{E}\|\sin(a(\tau)u)\|^2\\
&\quad\le 4\sup_{\tau\in\mathbb{R}}\|c^\wedge(-\tau)\psi(\tau)\|^2\le 4\|\psi\|^2_{\omega c}<\infty.
\end{aligned}$$

Similarly we have, $\sup_{\tau\in\mathbb{R}}\mathsf{E}\|c^\wedge(-\tau)f(\tau,u)\|^2<\infty$.

Since $\psi(\tau+\omega)\cos(n(\tau+\omega)cu)=\psi(\tau)\cos\left(\frac{1}{c}n(\tau)cu\right)=\psi(\tau)\cos(n(\tau)u)$,

then

$$\begin{aligned}
&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[g(\tau+\omega,cu)-cg(\tau,u)]\|^2d\tau\\
=&\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\left\|c^\wedge(-\tau)\left(\frac{1}{1+(\tau+\omega)^2}\psi(\tau+\omega)\sin(\|a(\tau+\omega)cu\|)\right.\right.\\
&\qquad\left.\left.-\frac{c}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|)\right)\right\|^2d\tau\\
\le&\frac{1}{2q}\int_{-q}^{q}\left(2\mathsf{E}\left\|c^\wedge(-\tau)\frac{c}{1+(\tau+\omega)^2}\psi(\tau)\sin(\|a(\tau)u\|)\right\|^2\right.\\
&\left.+2\mathsf{E}\left\|c^\wedge(-\tau)\frac{c}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|)\right\|^2\right)d\tau\\
\le&\frac{1}{2q}\int_{-q}^{q}\left[2\left(\frac{cc^\wedge(-\tau)\psi(\tau)}{1+(\tau+\omega)^2}\right)^2+2\left(\frac{cc^\wedge(-\tau)\psi(\tau)}{1+\tau^2}\right)^2\right]d\tau\\
\le&2c^2\|\psi\|^2_{\omega c}\left(\frac{1}{2q}\int_{-q}^{q}\left[\left(\frac{1}{1+(\tau+\omega)^2}\right)^2+\left(\frac{1}{1+\tau^2}\right)^2\right]d\tau\right).
\end{aligned}$$

Since

$$\frac{1}{2q}\int_{-q}^{q}\left|\frac{1}{1+\tau^2}\right|^2d\tau\le\frac{1}{2q}\int_{-q}^{q}\frac{d\tau}{1+\tau^2}=\frac{\arctan(q)}{q}\to 0\text{ as }q\to\infty,$$

and

$$\frac{1}{2q}\int_{-q}^{q}\left|\frac{1}{1+(\tau+\omega)^2}\right|^2d\tau\le\frac{q+\omega}{q}\left(\frac{1}{2(q+\omega)}\int_{-q-\omega}^{q+\omega}\left|\frac{1}{1+\tau^2}\right|^2d\tau\right)\to 0\text{ as }q\to\infty,$$

then it follows that

$$\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[g(\tau+\omega,cu)-cg(\tau,u)]\|^2d\tau=0.$$

Similarly, we have

$$\lim_{q\to\infty}\frac{1}{2q}\int_{-q}^{q}\mathsf{E}\|c^\wedge(-\tau)[f(\tau+\omega,cu)-cf(\tau,u)]\|^2d\tau=0.$$

Therefore f and g satisfy **(C1)**. Remark that $\psi(\tau+\omega)n(\tau+\omega)=c\psi(\tau)\frac{1}{c}n(\tau)=\psi(\tau)n(\tau)$ and $\psi(\tau+\omega)a(\tau+\omega)=c\psi(\tau)\frac{1}{c}a(\tau)=\psi(\tau)a(\tau)$. Hence the functions ψn and ψa are periodic. Then it follows that $(\psi n),(\psi a)\in BC(\mathbb{R},\mathbb{R})$. Let $u,v\in\mathbb{H}$,

$$\begin{aligned}\mathsf{E}\|g(\tau,u)-g(\tau,v)\|^2&\le 2\mathsf{E}\|\psi(\tau)\cos(n(\tau)u)-\psi(\tau)\cos(n(\tau)v)\|^2\\&\quad+2\mathsf{E}\left\|\frac{1}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|)-\frac{1}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)v\|)\right\|^2\\&\le 2\|\psi n\|_\infty^2\mathsf{E}\|u-v\|^2+2\|\psi a\|_\infty^2\mathsf{E}\|u-v\|^2\\&\le 2(\|\psi n\|_\infty^2+\|\psi a\|_\infty^2)\mathsf{E}\|u-v\|^2.\end{aligned}$$

The same arguments performed above give us

$$\mathsf{E}\|f(\tau,u)-f(\tau,v)\|^2\le 2(\|\psi n\|_\infty^2+\|\psi a\|_\infty^2)\mathsf{E}\|u-v\|^2.$$

Therefore, by Theorem 4.1, the problem (5.1) has a unique square-mean pseudo-S-asymptotically (ω,c)-periodic mild solution on $\mathbb{R}$ provided that $(\|\psi n\|_\infty^2+\|\psi a\|_\infty^2)$ is small enough.

5.2. Illustration of Theorem 4.2

Now, for Theorem 4.2, we take f and g as follows

$$g(\tau,u)=\frac{1}{1+\tau^2}\psi(\tau)\sin(\|a(\tau)u\|),\text{ and } f(\tau,u)=\frac{1}{1+\tau^2}p(\tau)\cos(\|b(\tau)u\|),$$

for $\tau\in\mathbb{R}$ and $u\in\mathbb{H}$, where $\psi,p\in\mathcal{P}_{\omega,c}(\mathbb{R},\mathbb{R})$ and $a,b\in\mathcal{P}_{\omega,\frac{1}{c}}(\mathbb{R},\mathbb{R})$. Similarly, the same arguments performed above show that f and g satisfy **(C1)**. Let $u,v\in\mathbb{H}$ and $\tau\in\mathbb{R}$. Then

$$\begin{aligned}&\mathsf{E}\|c^\wedge(-\tau)\,[g(\tau,c^\wedge(\tau)u)-g(\tau,c^\wedge(\tau)v)]\,\|^2\\&\le\mathsf{E}\left\|\frac{c^\wedge(-\tau)}{1+\tau^2}\psi(\tau)\sin(\|c^\wedge(\tau)a(\tau)u\|)-\frac{c^\wedge(-\tau)}{1+\tau^2}\psi(\tau)\sin(\|c^\wedge(\tau)a(\tau)v\|)\right\|^2\\&\le\left(\frac{|c|^\wedge(-\tau)}{1+\tau^2}|c|^\wedge(\tau)|\psi(\tau)a(\tau)|\right)^2\mathsf{E}\|u-v\|^2\\&\le\|\psi a\|_\infty^2\left(\frac{1}{1+\tau^2}\right)^2\mathsf{E}\|u-v\|^2.\end{aligned}$$

Let $\mu_1(\tau)=\|\psi a\|_\infty^2\left(\frac{1}{1+\tau^2}\right)^2$ for $\tau\in\mathbb{R}$. We have

$$\int_{\mathbb{R}}|\mu_1(\tau)|d\tau=2\|\psi a\|_\infty^2\int_0^\infty\left(\frac{1}{1+\tau^2}\right)^2d\tau$$

$$\leq 2\|\psi a\|_\infty^2 \int_0^\infty \frac{d\tau}{1+\tau^2} = \pi\|\psi a\|_\infty^2 < \infty.$$

Then $\mu_1 \in L^1(\mathbb{R}, \mathbb{R}_+)$. Similarly, we get

$$\mathsf{E}\|c^\wedge(-\tau) f(\tau, c^\wedge(\tau)u) - f(\tau, c^\wedge(\tau)v)\|^2 \leq \mu_2(\tau)\mathsf{E}\|u - v\|^2,$$

where $\mu_2(\tau) = \|\psi b\|_\infty^2 \left(\frac{1}{1+\tau^2}\right)^2$ belongs in $L^1(\mathbb{R}, \mathbb{R}_+)$. Therefore, by Theorem 4.2, the problem (5.1) has a unique square-mean pseudo-S-asymptotically (ω, c)-periodic mild solution on $\mathbb{R}$ provided that $\|\psi a\|_\infty + \|\psi b\|_\infty$ is small enough.

Statements and Declarations

Competing interests
The authors declare no competing interest.
Data Availability
Not Applicable.

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