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A Study on Demonstrating the Consistency of the Formulation of the Parameters of 5 or Less Independent Variables of Multiple Linear Regression with Data Examples

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Abstract In this study, it is aimed to demonstrate the consistency of the previously given formulations of the parameters of 5 or less independent variables of multiple linear regression with data examples. All the values in the parameter formulas of 5 or less independent variables with respect to one dependent variable were found to be the same as all the parameter values obtained from the related equation system. In other words, the values found with the parameter formulas and the parameter values found with the computer program were found to be the same.

Keywords Linear regression, multiple linear regression, formulas of parameters, 5 or less independent variables

MSC(2010) 62J05, 62J12

1. Introduction

In mathematics, statistics and many sciences, regression is one of the important topics. In this study, we will first work with the example given for simple linear regression. After that we will work with the example given for multiple linear regression, MLR.

Regression analysis is an important statistical tool for analyzing the relationships between dependent, and independent variables. The main goal of regression analysis is to determine and estimate parameters of a function that describes the best fit for given data sets. There are many linear types of regression analysis models such as simple and multiple regression models. Regression analysis is the widely used statistical tool for understanding relationships among variables. It is used when there is a continuous dependent variable which can be predicted by independent variables [1]

Seber defined linear regression analysis as a common technique of estimating the relationship between any two random variables, the explanatory variable X, and the dependent variable Y [11]. The simple relationship among dependent and explanatory variables can be defined as follows:

$$Y = \mathrm{f}(X_1, X_2, \ldots, X_N) + \varepsilon,$$

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where a random error representing the discrepancy in the approximation is assumed to be ε. It accounts for the failure of the model to fit the data exactly. The function $f(X_1, X_2, \ldots X_N)$ describes the relationship between the dependent variable Y, and the explanatory variables $X_1, X_2, \ldots X_N$.

When the relationship is linear, it may be represented mathematically using a straight line equation. The regression coefficient describes the change in Y that is associated with a unit change in X. This line is frequently computed using the least square procedure [5].

Linear regression is one of the fundamental techniques in the statistical analysis of data. We assume a straight-line model for a response variable Y as a function of one or more predictor (or explanatory) variables X [4]. In this study, first, we look at exactly one predictor variable and then we will look at two or more predictor variables.

Multiple linear regression is as follows when the error is omitted

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_3 + \ldots + \hat{\beta}_N X_N,$$

where $\hat{Y}$ is the equation of the regression and $\hat{\beta}_i$ ($i = 0, 1, 2, \ldots, N$) are unknown parameters of the regression, X_i ($i = 1, 2, \ldots, N$) are the independent variables of the regression.

The unknown parameters, $\hat{\beta}$, are estimated by using the method of least squares in multiple linear regression. The estimates of the $\hat{\beta}$ coefficients are the values that minimize the sum of squared errors for the sample. The obtained formula for this is given in this study on matrix notation.

Equations like this can easily be handled by any computer program that performs ordinary multiple regression. But in this study to get the parameters of multiple linear regression without using a computer program, the general formula of the parameters will be given. After that the parameter values obtained by the formula will be compared with the parameter values obtained by the linear equation system.

Multiple regression analysis is one of the most widely used statistical procedures. Multiple linear regression using the least square procedure is extensively used in astronomy to model observational data, analyze simulated data, and compare empirical data with theoretical models [3]. Its popularity is fostered by its applicability to varied types of data and problems, ease of interpretation, robustness to violations of the underlying assumptions, and widespread availability [7].

If a simple linear regression model with one predictor variable, X_1, is started, then a second predictor variable, X_2, is added, Error sum of squares (SSE) will decrease (or stay the same) while Total sum of squares (SST) remains constant, and thus R-squared (R^2) will increase (or stay the same). In other words, R^2 always increases (or stays the same) as more predictors are added to a multiple linear regression model, even if the predictors added are unrelated to the response variable. Thus, by itself, R^2 cannot be used to help us identify which predictors should be included or excluded in a model. But an alternative measure, adjusted R^2, does not necessarily increase as more predictors are added, and can be used to help us identify which predictors should be included or excluded in a model. Due to the malfunctioning of R^2, the researchers preferred to use adjusted R^2 [2].

In fact, the adjusted R^2 statistic does not change by adding variables to the model. In addition, the adjusted R^2 will often decrease by adding excessive parameters. This is the best way to add unnecessary variables to the model without changing the R^2 significantly [15].

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The regression sum of squares always increases, and the error sum of squares decreases due to adding more variables to the regression model. Adding an external variable to the model continues until one decides that the result will feature good accuracy. Actually, the effectiveness of the model will decrease by adding further insignificant variables because this increases the mean square error [10, 12].

Using the least squares method in a computer program, Ye and Liu in their study [14] found the parameters of the regression model with 3 independent and one dependent variable for a sample data set. By using least square method, this study was conducted to test the consistency of parameter formulas obtained by Korkmaz in his study [6] for multiple linear models with 5 or less independent variables.

2. Materials and methods

An important objective of regression analysis is to estimate the unknown parameters in the regression model. This process is also called fitting the model to the data. There are several parameter estimation techniques. One of these techniques is the method of least squares [8,13]. In this study, for linear and multiple linear regression, the method of least squares is used.

In his study [6], Korkmaz presented the formulas of the parameters of the multiple linear regression with 4 or less independent variables under the relevant tables. Since the denominators and numerators of the parameters for 5 independent variables are very long, Korkmaz did not summarize the formula for the parameters of 5 independent variables in a relevant table in his study [6]. That is, Korkmaz presented only the relevant formula table, which is in a certain order, for the parameters of 5 independent variables in his study [6].

In this study, a sample data table is used for 1 independent variable and 1 dependent variable. Then, sample data tables are arranged by keeping one dependent variable constant, increasing the number of independent variables up to five one by one, and preserving the previous independent variable values.

3. Results

3.1. One independent variable with one dependent variable

If we use one independent variable and one dependent variable, then we use the following equation (3.1) for linear regression,

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X. \tag{3.1}$$

To minimize SSE, we use the method of least squares by means of the derivatives of the parameters,

$$SSE = \sum_{i=1}^{N} (Y_i - \hat{Y})^2 = \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X)^2,$$

$$\frac{\partial SSE}{\partial \hat{\beta}_0} = -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0,$$

$$\frac{\partial SSE}{\partial \hat{\beta}_1} = -2\sum_{i=1}^{N}(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)X_i = 0,$$

$$\sum_{i=1}^{N} Y_i = \hat{\beta}_0 N + \hat{\beta}_1 \sum_{i=1}^{N} X_i, \tag{3.2}$$

$$\sum_{i=1}^{N} X_i Y_i = \hat{\beta}_0 \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2. \tag{3.3}$$

Equations (3.2) and (3.3) are called normal equations. Matrix display of this system with two parameters can be written as the following equation (3.4):

$$\begin{bmatrix} \sum_{i=1}^{N} Y_i \\ \sum_{i=1}^{N} X_i Y_i \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^{N} X_i \\ \sum_{i=1}^{N} X_i & \sum_{i=1}^{N} X_i^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}. \tag{3.4}$$

If both sides of the equation (3.2) are divided by N, we get

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1, \tag{3.5}$$

where

$$\bar{X} = \frac{\sum_{i=1}^{N} X_i}{N} \quad \text{and} \quad \bar{Y} = \frac{\sum_{i=1}^{N} Y_i}{N}.$$

If this $\hat{\beta}_0$ value in equation (3.5) is substituted in the equation (3.3), we can get

$$\sum_{i=1}^{N} X_i Y_i = (\bar{Y} - \hat{\beta}_1 \bar{X}\ ) \sum_{i=1}^{N} X_i + \hat{\beta}_1 \sum_{i=1}^{N} X_i^2,$$
$$\sum_{i=1}^{N} X_i Y_i - \bar{Y} \sum_{i=1}^{N} X_i = \hat{\beta}_1 (\sum_{i=1}^{N} X_i^2 - \bar{X} \sum_{i=1}^{N} X_i),$$

and then we can get

$$\hat{\beta}_1 = \frac{S_{XY}}{S_{XX}}, \tag{3.6}$$

where

$$S_{XY} = \sum_{i=1}^{N} X_i Y_i - \frac{\left(\sum_{i=1}^{N} X_i\right)\left(\sum_{i=1}^{N} Y_i\right)}{N} \quad \text{and} \quad S_{XX} = \sum_{i=1}^{N} X_i^2 - \frac{\left(\sum_{i=1}^{N} X_i\right)^2}{N}.$$

The values of 1 independent parameter against one dependent parameter are given in Table 1.

By using the values in Table 1, since S_{XX}=14,5 and S_{XY}= -7, the relevant parameter values of equations (3.6) and (3.5) are found as follows respectively:

$$\hat{\beta}_1 = -0.4828,$$

$$\hat{\beta}_0 = \quad 7.5724.$$

Table 1. The values of 1 independent parameter against one dependent parameter

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**∑**
X	6	4	2	5	4	3	5	6	5	5	**45**
Y	2	5	6	5	7	6	6	6	5	6	**54**
X*X	36	16	4	25	16	9	25	36	25	25	**217**
X*Y	12	20	12	25	28	18	30	36	25	30	**236**
Y*Y	4	25	36	25	49	36	36	36	25	36	**308**

The system of equation (3.4) can be written in the following:

$$\begin{bmatrix} 10 & 45 \\ 45 & 217 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} 54 \\ 236 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} 7.5724 \\ -0.4828 \end{bmatrix}.$$

The results obtained with both the formula method and the system of equation are exactly the same.

3.2. Two independent variables with one dependent variable

If we use two independent variables and one dependent variable, then we use the following equation (3.7) for multiple regression.

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2. \tag{3.7}$$

To minimize SSE, we use the method of least squares by means of the derivatives of the parameters.

$$SSE = \sum_{i=1}^{N} (Y_i - \hat{Y})^2 = \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X)^2,$$

$$\frac{\partial SSE}{\partial \hat{\beta}_0} = -2 \textstyle\sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) = 0,$$

$$\frac{\partial SSE}{\partial \hat{\beta}_1} = -2 \textstyle\sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) X_{1i} = 0,$$

$$\frac{\partial SSE}{\partial \hat{\beta}_2} = -2 \textstyle\sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}) X_{2i} = 0,$$

$$\sum_{i=1}^{N} Y_i = \hat{\beta}_0 N + \hat{\beta}_1 \sum_{i=1}^{N} X_{1i} + \hat{\beta}_2 \sum_{i=1}^{N} X_{2i}, \tag{3.8}$$

$$\sum_{i=1}^{N} X_{1i} Y_i = \hat{\beta}_0 \sum_{i=1}^{N} X_{1i} + \hat{\beta}_1 \sum_{i=1}^{N} X_{1i}^2 + \hat{\beta}_2 \sum_{i=1}^{N} X_{1i} X_{2i}, \tag{3.9}$$

$$\sum_{i=1}^{N} X_{2i}Y_i = \hat{\beta}_0 \sum_{i=1}^{N} X_{2i} + \hat{\beta}_1 \sum_{i=1}^{N} X_{1i}X_{2i} + \hat{\beta}_2 \sum_{i=1}^{N} X_{2i}^2. \tag{3.10}$$

Equations (3.8), (3.9) and (3.10) are called normal equations. Matrix display of this system with 3 parameters can be written as the following equation (3.11):

$$\begin{bmatrix} \sum_{i=1}^{N} Y_i \\ \sum_{i=1}^{N} X_{1i}Y_i, \\ \sum_{i=1}^{N} X_{2i}Y_i \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^{N} X_{1i} & \sum_{i=1}^{N} X_{2i} \\ \sum_{i=1}^{N} X_{1i} & \sum_{i=1}^{N} X_{1i}^2 & \sum_{i=1}^{N} X_{1i}X_{2i} \\ \sum_{i=1}^{N} X_{2i} & \sum_{i=1}^{N} X_{1i}X_{2i} & \sum_{i=1}^{N} X_{2i}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}. \tag{3.11}$$

If both sides of equation (3.8) are divided by N,

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}_1 - \hat{\beta}_2\bar{X}_2 \tag{3.12}$$

is found. If this $\hat{\beta}_0$ value is substituted in equation (3.9),

$$\sum_{i=1}^{N} X_{1i}Y_i = \bar{Y}\sum_{i=1}^{N} X_{1i} - \hat{\beta}_1\bar{X}_1\sum_{i=1}^{N} X_{1i} - \hat{\beta}_2\bar{X}_2\sum_{i=1}^{N} X_{1i} + \hat{\beta}_1\sum_{i=1}^{N} X_{1i}{}^2 + \hat{\beta}_2\sum_{i=1}^{N} X_{1i}X_{2i},$$

$$\sum_{i=1}^{N} X_{1i}Y_i - \bar{Y}\sum_{i=1}^{N} X_{1i} = \hat{\beta}_1(\sum_{i=1}^{N} X_{1i}{}^2 - \bar{X}_1\sum_{i=1}^{N} X_{1i}) + \hat{\beta}_2(\sum_{i=1}^{N} X_{1i}X_{2i} - \bar{X}_2\sum_{i=1}^{N} X_{1i})$$

can be obtained. And then

$$S_{X_1Y} = \hat{\beta}_1 S_{X_1X_1} + \hat{\beta}_2 S_{X_1X_2} \tag{3.13}$$

is obtained. And similarly, if this $\hat{\beta}_0$ value is substituted in equation (3.10),

$$\sum_{i=1}^{N} X_{2i}Y_i = \bar{Y}\sum_{i=1}^{N} X_{2i} - \hat{\beta}_1\bar{X}_1\sum_{i=1}^{N} X_{2i} - \hat{\beta}_2\bar{X}_2\sum_{i=1}^{N} X_{2i} + \hat{\beta}_1\sum_{i=1}^{N} X_{1i}X_{2i} + \hat{\beta}_2\sum_{i=1}^{N} X_{2i}{}^2,$$

$$\sum_{i=1}^{N} X_{2i}Y_i - \bar{Y}\sum_{i=1}^{N} X_{2i} = \hat{\beta}_1(\sum_{i=1}^{N} X_{1i}X_{2i} - \bar{X}_1\sum_{i=1}^{N} X_{2i}) + \hat{\beta}_2(\sum_{i=1}^{N} X_{2i}{}^2 - \bar{X}_2\sum_{i=1}^{N} X_{2i})$$

can be obtained. And then

$$S_{X_2Y} = \hat{\beta}_1 S_{X_1X_2} + \hat{\beta}_2 S_{X_2X_2} \tag{3.14}$$

is obtained. These equations (3.13) and (3.14) can be solved by using adding or substituting method and the matrix form of these is the following equation (3.15),

$$\begin{bmatrix} S_{X_1Y} \\ S_{X_2Y} \end{bmatrix} = \begin{bmatrix} S_{X_1X_1} & S_{X_1X_2} \\ S_{X_1X_2} & S_{X_2X_2} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}. \tag{3.15}$$

In his study [6], Korkmaz gave the formulas for the parameters of the 2 independent variables under each relevant Table as follows:

$$\hat{\beta}_1 = \frac{S_{X_1Y}S_{X_2X_2} - S_{X_2Y}S_{X_1X_2}}{S_{X_1X_1}S_{X_2X_2} - (S_{X_1X_2})^2} \tag{3.16}$$

and

$$\hat{\beta}_2 = \frac{S_{X_2Y}S_{X_1X_1} - S_{X_1Y}S_{X_1X_2}}{S_{X_1X_1}S_{X_2X_2} - (S_{X_1X_2})^2}, \tag{3.17}$$

in addition to the constant parameter (equation (3.12))

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}_1 - \hat{\beta}_2\bar{X}_2.$$

The values of 2 independent parameter against one dependent parameter are given in Table 2.

Table 2. The values of 2 independent parameter against one dependent parameter

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**∑**
X_1	6	4	2	5	4	3	5	6	5	5	**45**
X_2	2	4	5	4	2	6	3	2	6	5	**39**
Y	2	5	6	5	7	6	6	6	5	6	**54**
X_1*X_1	36	16	4	25	16	9	25	36	25	25	**217**
X_1*X_2	12	16	10	20	8	18	15	12	30	25	**166**
X_1*Y	12	20	12	25	28	18	30	36	25	30	**236**
X_2*X_2	4	16	25	16	4	36	9	4	36	25	**175**
X_2*Y	4	20	30	20	14	36	18	12	30	30	**214**
Y*Y	4	25	36	25	49	36	36	36	25	36	**308**

The system of equation (3.11) can be written in the following:

$$\begin{bmatrix} 10 & 45 & 39 \\ 45 & 217 & 166 \\ 39 & 166 & 175 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} 54 \\ 236 \\ 214 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} 8.0596 \\ -0.5294 \\ -0.0711 \end{bmatrix}.$$

By using the values in Table 2, the system of equation (3.15) can be written in the following:

$$\begin{bmatrix} -7 \\ 3,4 \end{bmatrix} = \begin{bmatrix} 14,5 & -9,5 \\ -9,5 & 22,9 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} -0.5294 \\ -0.0711 \end{bmatrix}$$

and after getting the values of $\hat{\beta}_1$ and $\hat{\beta}_2$, by using equation (3.12), we can find the value of $\hat{\beta}_0$ below:

$$\hat{\beta}_0 = 8.0596$$

By using the values in Table 2, the relevant parameter values of equations (3.16), (3.17) and (3.12) are found as follows respectively:

$$\hat{\beta}_1 = -0.5294,$$
$$\hat{\beta}_2 = -0.0711,$$
$$\hat{\beta}_0 = \quad 8.0596.$$

The results obtained with the formula method and the results of the systems of equations (3.11) and (3.15) and equation (3.12) are exactly the same.

3.3. Three independent variables with one dependent variable

If we use three independent variables and one dependent variable, then we use the following equation (3.18) for multiple regression,

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_3. \tag{3.18}$$

To minimize SSE below, after using the method of least squares by means of the derivatives of the parameters, we can get the normal equations of the system,

$$SSE = \sum_{i=1}^{N} (Y_i - \hat{Y})^2 = \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2 - \hat{\beta}_3 X_3)^2.$$

Matrix display of this system with 4 parameters can be written as follows:

$$\begin{bmatrix} \sum_{i=1}^N Y_i \\ \sum_{i=1}^N X_{1i} Y_i \\ \sum_{i=1}^N X_{2i} Y_i \\ \sum_{i=1}^N X_{3i} Y_i \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^N X_{1i} & \sum_{i=1}^N X_{2i} & \sum_{i=1}^N X_{3i} \\ \sum_{i=1}^N X_{1i} & \sum_{i=1}^N X_{1i}^2 & \sum_{i=1}^N X_{1i} X_{2i} & \sum_{i=1}^N X_{1i} X_{3i} \\ \sum_{i=1}^N X_{2i} & \sum_{i=1}^N X_{1i} X_{2i} & \sum_{i=1}^N X_{2i}^2 & \sum_{i=1}^N X_{3i} X_{2i} \\ \sum_{i=1}^N X_{3i} & \sum_{i=1}^N X_{1i} X_{3i} & \sum_{i=1}^N X_{3i} X_{2i} & \sum_{i=1}^N X_{3i}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix}. \tag{3.19}$$

For 3 independent variables, after performing similar operations to those for 2 independent variables, we have the following equation (3.20):

$$\begin{bmatrix} S_{X_1Y} \\ S_{X_2Y} \\ S_{X_3Y} \end{bmatrix} = \begin{bmatrix} S_{X_1X_1} & S_{X_1X_2} & S_{X_1X_3} \\ S_{X_2X_1} & S_{X_2X_2} & S_{X_2X_3} \\ S_{X_3X_1} & S_{X_3X_2} & S_{X_3X_3} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} \tag{3.20}$$

and

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3.$$

In his study [6], Korkmaz gave the formulas for the parameters of the 3 independent variables under each relevant Table as follows:

$$\hat{\beta}_1 = \frac{\left(S_{X_2X_2} S_{X_3X_3} - \left(S_{X_2X_3}\right)^2\right) S_{YX_1} + \left(S_{X_1X_3} S_{X_2X_3} - S_{X_1X_2} S_{X_3X_3}\right) S_{YX_2} + \left(S_{X_1X_2} S_{X_3X_2} - S_{X_1X_3} S_{X_2X_2}\right) S_{YX_3}}{\left(S_{X_2X_2} S_{X_3X_3} - \left(S_{X_2X_3}\right)^2\right) S_{X_1X_1} + \left(S_{X_1X_3} S_{X_2X_3} - S_{X_1X_2} S_{X_3X_3}\right) S_{X_1X_2} + \left(S_{X_1X_2} S_{X_3X_2} - S_{X_1X_3} S_{X_2X_2}\right) S_{X_1X_3}} \tag{3.21}$$

Table 3. The values of 3 independent parameter against one dependent parameter

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**∑**
X_1	6	4	2	5	4	3	5	6	5	5	**45**
X_2	2	4	5	4	2	6	3	2	6	5	**39**
X_3	1	3	3,91	2,64	3,09	1,55	2	2,18	1,64	2,18	**23,19**
Y	2	5	6	5	7	6	6	6	5	6	**54**
X_1*X_1	36	16	4	25	16	9	25	36	25	25	**217**
X_1*X_2	12	16	10	20	8	18	15	12	30	25	**166**
X_1*X_3	6	12	7,82	13,2	12,36	4,65	10	13,08	8,2	10,9	**98,21**
X_1*Y	12	20	12	25	28	18	30	36	25	30	**236**
X_2*X_2	4	16	25	16	4	36	9	4	36	25	**175**
X_2*X_3	2	12	19,55	10,56	6,18	9,3	6	4,36	9,84	10,9	**90,69**
X_2*Y	4	20	30	20	14	36	18	12	30	30	**214**
X_3*X_3	1	9	15,2881	6,9696	9,5481	2,4025	4	4,7524	2,6896	4,7524	**60,4027**
X_3*Y	2	15	23,46	13,2	21,63	9,3	12	13,08	8,2	13,08	**130,95**
Y*Y	4	25	36	25	49	36	36	36	25	36	**308**

$$\hat{\beta}_2 = \frac{\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right)S_{YX_1} + \left(S_{X_1X_1}S_{X_3X_3} - \left(S_{X_1X_3}\right)^2\right)S_{YX_2} + (S_{X_2X_1}S_{X_3X_1} - S_{X_2X_3}S_{X_1X_1})S_{YX_3}}{\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right)S_{X_2X_1} + \left(S_{X_1X_1}S_{X_3X_3} - \left(S_{X_1X_3}\right)^2\right)S_{X_2X_2} + (S_{X_2X_1}S_{X_3X_1} - S_{X_2X_3}S_{X_1X_1})S_{X_2X_3}} \tag{3.22}$$

and

$$\hat{\beta}_3 = \frac{\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right)S_{YX_1} + \left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right)S_{YX_2} + \left(S_{X_1X_1}S_{X_2X_2} - \left(S_{X_1X_2}\right)^2\right)S_{YX_3}}{\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right)S_{X_3X_1} + \left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right)S_{X_3X_2} + \left(S_{X_1X_1}S_{X_2X_2} - \left(S_{X_1X_2}\right)^2\right)S_{YX_3}} \tag{3.23}$$

in addition to the constant parameter

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}_1 - \hat{\beta}_2\bar{X}_2 - \hat{\beta}_3\bar{X}_3. \tag{3.24}$$

The values of 3 independent parameter against one dependent parameter are given in Table 3.

The system of equation (3.19) can be written in the following equation system:

$$\begin{bmatrix} 10 & 45 & 39 & 23.19 \\ 45 & 217 & 166 & 98.21 \\ 39 & 166 & 175 & 90.69 \\ 23.19 & 98.21 & 90.69 & 60.4027 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 54 \\ 236 \\ 214 \\ 130.95 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} 3.5173 \\ -0.0795 \\ 0.1069 \\ 0.7862 \end{bmatrix}.$$

By using the values in Table 3, the system of equation (3.20) can be written in the following:

$$\begin{bmatrix} -7 \\ 3.4 \\ 5.724 \end{bmatrix} = \begin{bmatrix} 14.5 & -9.5 & -6.145 \\ -9.5 & 22.9 & 0.249 \\ -6.145 & 0.249 & 6.62509 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} -0.0795 \\ 0.1069 \\ 0.7862 \end{bmatrix}$$

and after getting the values of $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$, by using equation (3.24), we can find the value of $\hat{\beta}_0$ below.

$$\hat{\beta}_0 = 3.5173.$$

By using the values in Table 3, the relevant parameter values of equations (3.21), (3.22), (3.23) and (3.24) are found as follows respectively:

$$\begin{aligned} \hat{\beta}_1 &= -0.0795, \\ \hat{\beta}_2 &= \quad 0.1069, \\ \hat{\beta}_3 &= \quad 0.7862, \\ \hat{\beta}_0 &= \quad 3.5173. \end{aligned}$$

The results obtained with the formula method and the results of the systems of equations (3.19) and (3.20) and equation (3.24) are exactly the same.

3.4. Four independent variables with one dependent variable

If we use four independent variables and one dependent variable, then we use the following equation (3.25) for multiple regression,

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_3 + \hat{\beta}_4 X_4. \tag{3.25}$$

To minimize SSE below, after using the method of least squares by means of the derivatives of the parameters, we can get the normal equations of the system,

$$SSE = \sum_{i=1}^{N} (Y_i - \hat{Y})^2 = \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2 - \hat{\beta}_3 X_3 - \hat{\beta}_4 X_4)^2.$$

Matrix display of this system with 5 parameters can be written as the following equation (3.26):

$$\begin{bmatrix} \sum_{i=1}^N Y_i \\ \sum_{i=1}^N X_{1i} Y_i \\ \sum_{i=1}^N X_{2i} Y_i \\ \sum_{i=1}^N X_{3i} Y_i \\ \sum_{i=1}^N X_{4i} Y_i \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^N X_{1i} & \sum_{i=1}^N X_{2i} & \sum_{i=1}^N X_{3i} & \sum_{i=1}^N X_{4i} \\ \sum_{i=1}^N X_{1i} & \sum_{i=1}^N X_{1i}^2 & \sum_{i=1}^N X_{1i} X_{2i} & \sum_{i=1}^N X_{1i} X_{3i} & \sum_{i=1}^N X_{1i} X_{4i} \\ \sum_{i=1}^N X_{2i} & \sum_{i=1}^N X_{1i} X_{2i} & \sum_{i=1}^N X_{2i}^2 & \sum_{i=1}^N X_{2i} X_{3i} & \sum_{i=1}^N X_{2i} X_{4i} \\ \sum_{i=1}^N X_{3i} & \sum_{i=1}^N X_{1i} X_{3i} & \sum_{i=1}^N X_{2i} X_{3i} & \sum_{i=1}^N X_{3i}^2 & \sum_{i=1}^N X_{3i} X_{4i} \\ \sum_{i=1}^N X_{4i} & \sum_{i=1}^N X_{1i} X_{4i} & \sum_{i=1}^N X_{2i} X_{4i} & \sum_{i=1}^N X_{3i} X_{4i} & \sum_{i=1}^N X_{4i}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{bmatrix}. \tag{3.26}$$

For 4 independent variables, after similar operations to those done before, we have:

$$\begin{bmatrix} S_{X_1Y} \\ S_{X_2Y} \\ S_{X_3Y} \\ S_{X_4Y} \end{bmatrix} = \begin{bmatrix} S_{X_1X_1} & S_{X_1X_2} & S_{X_1X_3} & S_{X_1X_4} \\ S_{X_2X_1} & S_{X_2X_2} & S_{X_2X_3} & S_{X_2X_4} \\ S_{X_3X_1} & S_{X_3X_2} & S_{X_3X_3} & S_{X_3X_4} \\ S_{X_4X_1} & S_{X_4X_2} & S_{X_4X_3} & S_{X_4X_4} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{bmatrix} \tag{3.27}$$

in addition to the constant parameter

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}_1 - \hat{\beta}_2\bar{X}_2 - \hat{\beta}_3\bar{X}_3 - \hat{\beta}_4\bar{X}_4. \tag{3.28}$$

In his study [6], Korkmaz gave the formulas for the parameters of the 4 independent variables under each relevant Table as follows:

$$\hat{\beta}_1 = \frac{S_{YX_1}E_1 + S_{YX_2}F_1 + S_{YX_3}G_1 + S_{YX_4}H_1}{S_{X_1X_1}E_1 + S_{X_1X_2}F_1 + S_{X_1X_3}G_1 + S_{X_1X_4}H_1}, \tag{3.29}$$

where

$$E_1 = S_{X_2X_2}\left(S_{X_3X_3}S_{X_4X_4} - S_{X_3X_4}S_{X_3X_4}\right) + S_{X_2X_3}\left(S_{X_2X_4}S_{X_3X_4} - S_{X_2X_3}S_{X_4X_4}\right) + S_{X_2X_4}\left(S_{X_2X_3}S_{X_4X_3} - S_{X_2X_4}S_{X_3X_3}\right),$$

$$F_1 = S_{X_1X_2}\left(S_{X_3X_4}S_{X_3X_4} - S_{X_3X_3}S_{X_4X_4}\right) + S_{X_1X_3}\left(S_{X_2X_3}S_{X_4X_4} - S_{X_2X_4}S_{X_3X_4}\right) + S_{X_1X_4}\left(S_{X_2X_4}S_{X_3X_3} - S_{X_2X_3}S_{X_4X_3}\right),$$

$$G_1 = S_{X_1X_2}\left(S_{X_3X_2}S_{X_4X_4} - S_{X_3X_4}S_{X_2X_4}\right) + S_{X_1X_3}\left(S_{X_2X_4}S_{X_2X_4} - S_{X_2X_2}S_{X_4X_4}\right) + S_{X_1X_4}\left(S_{X_3X_4}S_{X_2X_2} - S_{X_3X_2}S_{X_4X_2}\right),$$

$$H_1 = S_{X_1X_2}\left(S_{X_4X_2}S_{X_3X_3} - S_{X_4X_3}S_{X_2X_3}\right) + S_{X_1X_3}\left(S_{X_4X_3}S_{X_2X_2} - S_{X_4X_2}S_{X_3X_2}\right) + S_{X_1X_4}\left(S_{X_2X_3}S_{X_2X_3} - S_{X_2X_2}S_{X_3X_3}\right).$$

$$\hat{\beta}_2 = \frac{S_{YX_1}E_2 + S_{YX_2}F_2 + S_{YX_3}G_2 + S_{YX_4}H_2}{S_{X_2X_1}E_2 + S_{X_2X_2}F_2 + S_{X_2X_3}G_2 + S_{X_2X_4}H_2}, \tag{3.30}$$

where

$$E_2 = S_{X_2X_1}\left(S_{X_3X_4}S_{X_3X_4} - S_{X_3X_3}S_{X_4X_4}\right) + S_{X_2X_3}\left(S_{X_1X_3}S_{X_4X_4} - S_{X_1X_4}S_{X_3X_4}\right) + S_{X_2X_4}\left(S_{X_1X_4}S_{X_3X_3} - S_{X_1X_3}S_{X_4X_3}\right),$$

$$F_2 = S_{X_3X_1}\left(S_{X_3X_4}S_{X_1X_4} - S_{X_3X_1}S_{X_4X_4}\right) + S_{X_3X_3}\left(S_{X_1X_1}S_{X_4X_4} - S_{X_1X_4}S_{X_1X_4}\right) + S_{X_3X_4}\left(S_{X_3X_1}S_{X_4X_1} - S_{X_3X_4}S_{X_1X_1}\right),$$

$$G_2 = S_{X_2X_1}\left(S_{X_3X_1}S_{X_4X_4} - S_{X_3X_4}S_{X_1X_4}\right) + S_{X_2X_3}\left(S_{X_1X_4}S_{X_1X_4} - S_{X_1X_1}S_{X_4X_4}\right) + S_{X_2X_4}\left(S_{X_3X_4}S_{X_1X_1} - S_{X_3X_1}S_{X_4X_1}\right),$$

$$H_2 = S_{X_2X_1}\left(S_{X_4X_1}S_{X_3X_3} - S_{X_4X_3}S_{X_1X_3}\right) + S_{X_2X_3}\left(S_{X_4X_3}S_{X_1X_1} - S_{X_4X_1}S_{X_3X_1}\right) + S_{X_2X_4}\left(S_{X_1X_3}S_{X_1X_3} - S_{X_1X_1}S_{X_3X_3}\right).$$

$$\hat{\beta}_3 = \frac{S_{YX_1}E_3 + S_{YX_2}F_3 + S_{YX_3}G_3 + S_{YX_4}H_3}{S_{X_3X_1}E_3 + S_{X_3X_2}F_3 + S_{X_3X_3}G_3 + S_{X_3X_4}H_3} \tag{3.31}$$

where

$$E_3 = S_{X_3X_1}\left(S_{X_2X_4}S_{X_2X_4} - S_{X_2X_2}S_{X_4X_4}\right) + S_{X_3X_2}\left(S_{X_1X_2}S_{X_4X_4} - S_{X_1X_4}S_{X_2X_4}\right) + S_{X_3X_4}\left(S_{X_1X_4}S_{X_2X_2} - S_{X_1X_2}S_{X_4X_2}\right),$$

$$F_3 = S_{X_3X_1}\left(S_{X_2X_1}S_{X_4X_4} - S_{X_2X_4}S_{X_1X_4}\right) + S_{X_3X_2}\left(S_{X_1X_4}S_{X_1X_4} - S_{X_1X_1}S_{X_4X_4}\right) + S_{X_3X_4}\left(S_{X_2X_4}S_{X_1X_1} - S_{X_2X_1}S_{X_4X_1}\right),$$

$$G_3 = S_{X_4X_1}\left(S_{X_4X_2}S_{X_1X_2} - S_{X_4X_1}S_{X_2X_2}\right) + S_{X_4X_2}\left(S_{X_4X_1}S_{X_2X_1} - S_{X_4X_2}S_{X_1X_1}\right) + S_{X_4X_4}\left(S_{X_1X_1}S_{X_2X_2} - S_{X_1X_2}S_{X_1X_2}\right),$$

$$H_3 = S_{X_3X_1}\left(S_{X_4X_1}S_{X_2X_2} - S_{X_4X_2}S_{X_1X_2}\right) + S_{X_3X_2}\left(S_{X_4X_2}S_{X_1X_1} - S_{X_4X_1}S_{X_2X_1}\right) + S_{X_3X_4}\left(S_{X_1X_2}S_{X_1X_2} - S_{X_1X_1}S_{X_2X_2}\right).$$

$$\hat{\beta}_4 = \frac{S_{YX_1}E_4 + S_{YX_2}F_4 + S_{YX_3}G_4 + S_{YX_4}H_4}{S_{X_4X_1}E_4 + S_{X_4X_2}F_4 + S_{X_4X_3}G_4 + S_{X_4X_4}H_4} \tag{3.32}$$

where

$$E_4 = S_{x_4x_1}\left(S_{x_2x_3}S_{x_2x_3} - S_{x_2x_2}S_{x_3x_3}\right) + S_{x_4x_2}\left(S_{x_1x_2}S_{x_3x_3} - S_{x_1x_3}S_{x_2x_3}\right) + S_{x_4x_3}\left(S_{x_1x_3}S_{x_2x_2} - S_{x_1x_2}S_{x_3x_2}\right),$$

$$F_4 = S_{x_4x_1}\left(S_{x_2x_1}S_{x_3x_3} - S_{x_2x_3}S_{x_1x_3}\right) + S_{x_4x_2}\left(S_{x_1x_3}S_{x_1x_3} - S_{x_1x_1}S_{x_3x_3}\right) + S_{x_4x_3}\left(S_{x_2x_3}S_{x_1x_1} - S_{x_2x_1}S_{x_3x_1}\right),$$

$$G_4 = S_{x_4x_1}\left(S_{x_3x_1}S_{x_2x_2} - S_{x_3x_2}S_{x_1x_2}\right) + S_{x_4x_2}\left(S_{x_3x_2}S_{x_1x_1} - S_{x_3x_1}S_{x_2x_1}\right) + S_{x_4x_3}\left(S_{x_1x_2}S_{x_1x_2} - S_{x_1x_1}S_{x_2x_2}\right),$$

$$H_4 = S_{x_3x_1}\left(S_{x_3x_2}S_{x_1x_2} - S_{x_3x_1}S_{x_2x_2}\right) + S_{x_3x_2}\left(S_{x_3x_1}S_{x_2x_1} - S_{x_3x_2}S_{x_1x_1}\right) + S_{x_3x_3}\left(S_{x_1x_1}S_{x_2x_2} - S_{x_1x_2}S_{x_1x_2}\right).$$

The values of 4 independent parameters against one dependent parameter are given in Table 4. The system of equation (3.26) can be written in the following equation

Table 4. The values of 4 independent parameter against one dependent parameter

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	∑
$\mathbf{X_1}$	6	4	2	5	4	3	5	6	5	5	**45**
$\mathbf{X_2}$	2	4	5	4	2	6	3	2	6	5	**39**
$\mathbf{X_3}$	1	3	3,91	2,64	3,09	1,55	2	2,18	1,64	2,18	**23,19**
$\mathbf{X_4}$	10	20	30	40	50	60	70	80	90	100	**550**
Y	2	5	6	5	7	6	6	6	5	6	**54**
$\mathbf{X_1{*}X_1}$	36	16	4	25	16	9	25	36	25	25	**217**
$\mathbf{X_1{*}X_2}$	12	16	10	20	8	18	15	12	30	25	**166**
$\mathbf{X_1{*}X_3}$	6	12	7,82	13,2	12,36	4,65	10	13,08	8,2	10,9	**98,21**
$\mathbf{X_1{*}X_4}$	60	80	60	200	200	180	350	480	450	500	**2560**
$\mathbf{X_1{*}Y}$	12	20	12	25	28	18	30	36	25	30	**236**
$\mathbf{X_2{*}X_2}$	4	16	25	16	4	36	9	4	36	25	**175**
$\mathbf{X_2{*}X_3}$	2	12	19,55	10,56	6,18	9,3	6	4,36	9,84	10,9	**90,69**
$\mathbf{X_2{*}X_4}$	20	80	150	160	100	360	210	160	540	500	**2280**
$\mathbf{X_2{*}Y}$	4	20	30	20	14	36	18	12	30	30	**214**
$\mathbf{X_3{*}X_3}$	1	9	15,2881	6,9696	9,5481	2,4025	4	4,7524	2,6896	4,7524	**60,4027**
$\mathbf{X_3{*}X_4}$	10	60	117,3	105,6	154,5	93	140	174,4	147,6	218	**1220,4**
$\mathbf{X_3{*}Y}$	2	15	23,46	13,2	21,63	9,3	12	13,08	8,2	13,08	**130,95**
$\mathbf{X_4{*}X_4}$	100	400	900	1600	2500	3600	4900	6400	8100	10000	**38500**
$\mathbf{X_4{*}Y}$	20	100	180	200	350	360	420	480	450	600	**3160**
Y*Y	4	25	36	25	49	36	36	36	25	36	**308**

system:

$$\begin{bmatrix} 10 & 45 & 39 & 23.19 & 550 \\ 45 & 217 & 166 & 98.21 & 2560 \\ 39 & 166 & 175 & 90.69 & 2280 \\ 23.19 & 98.21 & 90.69 & 60.4027 & 1220.40 \\ 550 & 2560 & 2280 & 1220.40 & 38500 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{bmatrix} = \begin{bmatrix} 54 \\ 236 \\ 214 \\ 130.95 \\ 3160 \end{bmatrix}$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} 7.4453 \\ -0.8217 \\ -0.4444 \\ 0.4665 \\ 0.0419 \end{bmatrix}.$$

By using the values in Table 4, the system of equation (3.27) can be written in the following:

$$\begin{bmatrix} -7 \\ 3.4 \\ 5.724 \\ 190 \end{bmatrix} = \begin{bmatrix} 14.5 & -9.5 & -6.145 & 85 \\ -9.5 & 22.9 & 0.249 & 135 \\ -6.145 & 0.249 & 6.62509 & -55.05 \\ 85 & 135 & -55.05 & 8250 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} -0.8217 \\ -0.4444 \\ 0.4665 \\ 0.0419 \end{bmatrix}$$

and after getting the values of $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ and $\hat{\beta}_4$, by using equation (3.28), we can find the value of $\hat{\beta}_0$ below,

$$\hat{\beta}_0 = 7.4453.$$

By using the values in Table 4, the relevant parameter values of equations (3.28), (3.29), (3.30), (3.31) and (3.32) are found as follows respectively:

$$\hat{\beta}_0 = 7.4453,$$

$$\hat{\beta}_1 = -0.8217,$$

$$\hat{\beta}_2 = -0.4444,$$

$$\hat{\beta}_3 = \quad 0.4665,$$

$$\hat{\beta}_4 = \quad 0.0419.$$

The results obtained with the formula method and the results of the systems of equations (3.26) and (3.27) and equation (3.28) are exactly the same.

3.5. Five independent variables with one dependent variable

If we use five independent variables and one dependent variable, then we use the following equation (3.33) for multiple regression,

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_3 + \hat{\beta}_4 X_4 + \hat{\beta}_5 X_5. \tag{3.33}$$

To minimize SSE below, after using the method of least squares by means of the derivatives of the parameters, we can get the normal equations of the system,

$$SSE = \sum_{i=1}^{N} (Y_i - \hat{Y})^2 = \sum_{i=1}^{N} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_1 - \hat{\beta}_2 X_2 - \hat{\beta}_3 X_3 - \hat{\beta}_4 X_4 - \hat{\beta}_5 X_5)^2.$$

Matrix display of this system with 5 parameters can be written as the following equation (3.34):

$$\begin{bmatrix} \sum_{i=1}^N Y_i \\ \sum_{i=1}^N X_{1i}Y_i \\ \sum_{i=1}^N X_{2i}Y_i \\ \sum_{i=1}^N X_{3i}Y_i \\ \sum_{i=1}^N X_{4i}Y_i \\ \sum_{i=1}^N X_{5i}Y_i \end{bmatrix} = \begin{bmatrix} N & \sum_{i=1}^N X_{1i} & \sum_{i=1}^N X_{2i} & \sum_{i=1}^N X_{3i} & \sum_{i=1}^N X_{4i} & \sum_{i=1}^N X_{5i} \\ \sum_{i=1}^N X_{1i} & \sum_{i=1}^N X_{1i}^2 & \sum_{i=1}^N X_{1i}X_{2i} & \sum_{i=1}^N X_{1i}X_{3i} & \sum_{i=1}^N X_{1i}X_{4i} & \sum_{i=1}^N X_{1i}X_{5i} \\ \sum_{i=1}^N X_{2i} & \sum_{i=1}^N X_{2i}X_{1i} & \sum_{i=1}^N X_{2i}^2 & \sum_{i=1}^N X_{2i}X_{3i} & \sum_{i=1}^N X_{2i}X_{4i} & \sum_{i=1}^N X_{2i}X_{5i} \\ \sum_{i=1}^N X_{3i} & \sum_{i=1}^N X_{3i}X_{1i} & \sum_{i=1}^N X_{3i}X_{2i} & \sum_{i=1}^N X_{3i}^2 & \sum_{i=1}^N X_{3i}X_{4i} & \sum_{i=1}^N X_{3i}X_{5i} \\ \sum_{i=1}^N X_{4i} & \sum_{i=1}^N X_{4i}X_{1i} & \sum_{i=1}^N X_{4i}X_{2i} & \sum_{i=1}^N X_{4i}X_{3i} & \sum_{i=1}^N X_{4i}^2 & \sum_{i=1}^N X_{4i}X_{5i} \\ \sum_{i=1}^N X_{5i} & \sum_{i=1}^N X_{5i}X_{1i} & \sum_{i=1}^N X_{5i}X_{2i} & \sum_{i=1}^N X_{5i}X_{3i} & \sum_{i=1}^N X_{5i}X_{4i} & \sum_{i=1}^N X_{5i}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \end{bmatrix}. \tag{3.34}$$

For 5 independent variables, after similar operations to those done before, we have:

$$\begin{bmatrix} S_{X_1Y} \\ S_{X_2Y} \\ S_{X_3Y} \\ S_{X_4Y} \\ S_{X_5Y} \end{bmatrix} = \begin{bmatrix} S_{X_1X_1} & S_{X_1X_2} & S_{X_1X_3} & S_{X_1X_4} & S_{X_1X_5} \\ S_{X_2X_1} & S_{X_2X_2} & S_{X_2X_3} & S_{X_2X_4} & S_{X_2X_5} \\ S_{X_3X_1} & S_{X_3X_2} & S_{X_3X_3} & S_{X_3X_4} & S_{X_3X_5} \\ S_{X_4X_1} & S_{X_4X_2} & S_{X_4X_3} & S_{X_4X_4} & S_{X_4X_5} \\ S_{X_5X_1} & S_{X_5X_2} & S_{X_5X_3} & S_{X_5X_4} & S_{X_5X_5} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \end{bmatrix} \tag{3.35}$$

in addition to the constant parameter

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}_1 - \hat{\beta}_2 \bar{X}_2 - \hat{\beta}_3 \bar{X}_3 - \hat{\beta}_4 \bar{X}_4 - \hat{\beta}_5 \bar{X}_5. \tag{3.36}$$

Korkmaz in [6] is summarized the formula for the parameters of 5 independent variables in relevant Tables in his study. Since the denominator and numerator of the parameters for 5 independent variables are very long, the formulas of the parameters in the Table, which is in a certain order, are not given under the relevant Table in his work.

In this study, using this Table in a certain order, some appropriate abbreviations are made, and the parameter formulas are written as follows:

$$\hat{\beta}_1 = \frac{S_{YX_1}K_1 + S_{YX_2}L_1 + S_{YX_3}M_1 + S_{YX_4}N_1 + S_{YX_5}O_1}{S_{X_1X_1}K_1 + S_{X_1X_2}L_1 + S_{X_1X_3}M_1 + S_{X_1X_4}N_1 + S_{X_1X_5}O_1} \tag{3.37}$$

where

$$K_1 = A_{11}S_{X_2X_2} + B_{11}S_{X_2X_3} + C_{11}S_{X_2X_4} + D_{11}S_{X_2X_5},$$

$$L_1 = A_{12}S_{X_1X_2} + B_{12}S_{X_1X_3} + C_{12}S_{X_1X_4} + D_{12}S_{X_1X_5},$$
$$M_1 = A_{13}S_{X_1X_2} + B_{13}S_{X_1X_3} + C_{13}S_{X_1X_4} + D_{13}S_{X_1X_5},$$
$$N_1 = A_{14}S_{X_1X_2} + B_{14}S_{X_1X_3} + C_{14}S_{X_1X_4} + D_{14}S_{X_1X_5},$$
$$O_1 = A_{15}S_{X_1X_2} + B_{15}S_{X_1X_3} + C_{15}S_{X_1X_4} + D_{15}S_{X_1X_5},$$
$$A_{11} = S_{X_5X_3}\left(S_{X_5X_4}S_{X_3X_4} - S_{X_5X_3}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_5X_3}S_{X_4X_3} - S_{X_5X_4}S_{X_3X_3}\right) + S_{X_5X_5}\left(S_{X_3X_3}S_{X_4X_4} - S_{X_3X_4}S_{X_3X_4}\right),$$
$$B_{11} = S_{X_5X_2}\left(S_{X_3X_5}S_{X_4X_4} - S_{X_3X_4}S_{X_5X_4}\right) + S_{X_5X_4}\left(S_{X_2X_3}S_{X_4X_5} - S_{X_2X_4}S_{X_3X_5}\right) + S_{X_5X_5}\left(S_{X_2X_4}S_{X_3X_4} - S_{X_2X_3}S_{X_4X_4}\right),$$
$$C_{11} = S_{X_5X_2}\left(S_{X_5X_4}S_{X_3X_3} - S_{X_5X_3}S_{X_4X_3}\right) + S_{X_5X_3}\left(S_{X_2X_4}S_{X_3X_5} - S_{X_2X_3}S_{X_4X_5}\right) + S_{X_5X_5}\left(S_{X_2X_3}S_{X_4X_3} - S_{X_2X_4}S_{X_3X_3}\right),$$
$$D_{11} = S_{X_5X_2}\left(S_{X_3X_4}S_{X_3X_4} - S_{X_3X_3}S_{X_4X_4}\right) + S_{X_5X_3}\left(S_{X_2X_3}S_{X_4X_4} - S_{X_2X_4}S_{X_3X_4}\right) + S_{X_5X_4}\left(S_{X_2X_4}S_{X_3X_3} - S_{X_2X_3}S_{X_4X_3}\right),$$

$$A_{12} = S_{X_5X_3}\left(S_{X_5X_3}S_{X_4X_4} - S_{X_5X_4}S_{X_3X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_3X_3} - S_{X_5X_3}S_{X_4X_3}\right) + S_{X_5X_5}\left(S_{X_3X_4}S_{X_3X_4} - S_{X_3X_3}S_{X_4X_4}\right),$$
$$B_{12} = S_{X_5X_2}\left(S_{X_3X_4}S_{X_5X_4} - S_{X_3X_5}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_2X_4}S_{X_3X_5} - S_{X_2X_3}S_{X_4X_5}\right) + S_{X_5X_5}\left(S_{X_2X_3}S_{X_4X_4} - S_{X_2X_4}S_{X_3X_4}\right),$$
$$C_{12} = S_{X_5X_2}\left(S_{X_3X_4}S_{X_3X_5} - S_{X_3X_3}S_{X_4X_5}\right) + S_{X_5X_3}\left(S_{X_2X_3}S_{X_4X_5} - S_{X_2X_4}S_{X_3X_5}\right) + S_{X_5X_5}\left(S_{X_2X_4}S_{X_3X_3} - S_{X_2X_3}S_{X_4X_3}\right),$$
$$D_{12} = S_{X_5X_2}\left(S_{X_3X_3}S_{X_4X_4} - S_{X_3X_4}S_{X_3X_4}\right) + S_{X_5X_3}\left(S_{X_2X_4}S_{X_3X_4} - S_{X_2X_3}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_2X_3}S_{X_4X_3} - S_{X_2X_4}S_{X_3X_3}\right),$$

$$A_{13} = S_{X_5X_3}\left(S_{X_2X_4}S_{X_5X_4} - S_{X_2X_5}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_3X_4}S_{X_2X_5} - S_{X_3X_2}S_{X_4X_5}\right) + S_{X_5X_5}\left(S_{X_3X_2}S_{X_4X_4} - S_{X_3X_4}S_{X_2X_4}\right),$$
$$B_{13} = S_{X_5X_2}\left(S_{X_5X_2}S_{X_4X_4} - S_{X_5X_4}S_{X_2X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_2X_2} - S_{X_5X_2}S_{X_4X_2}\right) + S_{X_5X_5}\left(S_{X_2X_4}S_{X_2X_4} - S_{X_2X_2}S_{X_4X_4}\right),$$
$$C_{13} = S_{X_5X_2}\left(S_{X_3X_2}S_{X_4X_5} - S_{X_3X_4}S_{X_2X_5}\right) + S_{X_5X_3}\left(S_{X_2X_4}S_{X_2X_5} - S_{X_2X_2}S_{X_4X_5}\right) + S_{X_5X_5}\left(S_{X_3X_4}S_{X_2X_2} - S_{X_3X_2}S_{X_4X_2}\right),$$
$$D_{13} = S_{X_5X_2}\left(S_{X_3X_4}S_{X_2X_4} - S_{X_3X_2}S_{X_4X_4}\right) + S_{X_5X_3}\left(S_{X_2X_2}S_{X_4X_4} - S_{X_2X_4}S_{X_2X_4}\right) + S_{X_5X_4}\left(S_{X_3X_2}S_{X_4X_2} - S_{X_3X_4}S_{X_2X_2}\right),$$

$$A_{14} = S_{X_5X_3}\left(S_{X_4X_3}S_{X_2X_5} - S_{X_4X_2}S_{X_3X_5}\right) + S_{X_5X_4}\left(S_{X_2X_3}S_{X_5X_3} - S_{X_2X_5}S_{X_3X_3}\right) + S_{X_5X_5}\left(S_{X_4X_2}S_{X_3X_3} - S_{X_4X_3}S_{X_2X_3}\right),$$
$$B_{14} = S_{X_5X_2}\left(S_{X_4X_2}S_{X_3X_5} - S_{X_4X_3}S_{X_2X_5}\right) + S_{X_5X_4}\left(S_{X_3X_2}S_{X_5X_2} - S_{X_3X_5}S_{X_2X_2}\right) + S_{X_5X_5}\left(S_{X_4X_3}S_{X_2X_2} - S_{X_4X_2}S_{X_3X_2}\right),$$
$$C_{14} = S_{X_5X_2}\left(S_{X_5X_2}S_{X_3X_3} - S_{X_5X_3}S_{X_2X_3}\right) + S_{X_5X_3}\left(S_{X_5X_3}S_{X_2X_2} - S_{X_5X_2}S_{X_3X_2}\right) + S_{X_5X_5}\left(S_{X_2X_3}S_{X_2X_3} - S_{X_2X_2}S_{X_3X_3}\right),$$
$$D_{14} = S_{X_5X_2}\left(S_{X_2X_3}S_{X_4X_3} - S_{X_2X_4}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_3X_2}S_{X_4X_2} - S_{X_3X_4}S_{X_2X_2}\right) + S_{X_5X_4}\left(S_{X_2X_2}S_{X_3X_3} - S_{X_2X_3}S_{X_2X_3}\right),$$

$$A_{15} = S_{X_5X_2}\left(S_{X_3X_3}S_{X_4X_4} - S_{X_3X_4}S_{X_3X_4}\right) + S_{X_5X_3}\left(S_{X_2X_4}S_{X_3X_4} - S_{X_2X_3}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_2X_3}S_{X_4X_3} - S_{X_2X_4}S_{X_3X_3}\right),$$
$$B_{15} = S_{X_5X_2}\left(S_{X_2X_4}S_{X_3X_4} - S_{X_2X_3}S_{X_4X_4}\right) + S_{X_5X_3}\left(S_{X_2X_2}S_{X_4X_4} - S_{X_2X_4}S_{X_2X_4}\right) + S_{X_5X_4}\left(S_{X_3X_2}S_{X_4X_2} - S_{X_3X_4}S_{X_2X_2}\right),$$
$$C_{15} = S_{X_5X_2}\left(S_{X_4X_3}S_{X_2X_3} - S_{X_4X_2}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_4X_2}S_{X_3X_2} - S_{X_4X_3}S_{X_2X_2}\right) + S_{X_5X_4}\left(S_{X_2X_2}S_{X_3X_3} - S_{X_2X_3}S_{X_2X_3}\right),$$
$$D_{15} = S_{X_2X_2}\left(S_{X_3X_4}S_{X_3X_4} - S_{X_3X_3}S_{X_4X_4}\right) + S_{X_2X_3}\left(S_{X_2X_3}S_{X_4X_4} - S_{X_2X_4}S_{X_3X_4}\right) + S_{X_2X_4}\left(S_{X_2X_4}S_{X_3X_3} - S_{X_2X_3}S_{X_4X_3}\right).$$

$$\hat{\beta}_2 = \frac{S_{YX_1}K_2 + S_{YX_2}L_2 + S_{YX_3}M_2 + S_{YX_4}N_2 + S_{YX_5}O_2}{S_{X_2X_1}K_2 + S_{X_2X_2}L_2 + S_{X_2X_3}M_2 + S_{X_2X_4}N_2 + S_{X_2X_5}O_2} \tag{3.38}$$

where

$$K_2 = A_{21}S_{X_2X_1} + B_{21}S_{X_2X_3} + C_{21}S_{X_2X_4} + D_{21}S_{X_2X_5},$$
$$L_2 = A_{22}S_{X_3X_1} + B_{22}S_{X_3X_3} + C_{22}S_{X_3X_4} + D_{22}S_{X_3X_5},$$
$$M_2 = A_{23}S_{X_2X_1} + B_{23}S_{X_2X_3} + C_{23}S_{X_2X_4} + D_{23}S_{X_2X_5},$$
$$N_2 = A_{24}S_{X_2X_1} + B_{24}S_{X_2X_3} + C_{24}S_{X_2X_4} + D_{24}S_{X_2X_5},$$

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$$O_2 = A_{25}S_{X_2X_1} + B_{25}S_{X_2X_3} + C_{25}S_{X_2X_4} + D_{25}S_{X_2X_5},$$

$$A_{21} = S_{X_5X_3}\left(S_{X_5X_3}S_{X_4X_4} - S_{X_5X_4}S_{X_3X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_3X_3} - S_{X_5X_3}S_{X_4X_3}\right) + S_{X_5X_5}\left(S_{X_3X_4}S_{X_3X_4} - S_{X_3X_3}S_{X_4X_4}\right),$$

$$B_{21} = S_{X_5X_1}\left(S_{X_3X_4}S_{X_5X_4} - S_{X_3X_5}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_1X_4}S_{X_3X_5} - S_{X_1X_3}S_{X_4X_5}\right) + S_{X_5X_5}\left(S_{X_1X_3}S_{X_4X_4} - S_{X_1X_4}S_{X_3X_4}\right),$$

$$C_{21} = S_{X_5X_1}\left(S_{X_4X_3}S_{X_5X_3} - S_{X_4X_5}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_1X_3}S_{X_4X_5} - S_{X_1X_4}S_{X_3X_5}\right) + S_{X_5X_5}\left(S_{X_1X_4}S_{X_3X_3} - S_{X_1X_3}S_{X_4X_3}\right),$$

$$D_{21} = S_{X_5X_1}\left(S_{X_3X_3}S_{X_4X_4} - S_{X_3X_4}S_{X_3X_4}\right) + S_{X_5X_3}\left(S_{X_1X_4}S_{X_3X_4} - S_{X_1X_3}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_1X_3}S_{X_4X_3} - S_{X_1X_4}S_{X_3X_3}\right),$$

$$A_{22} = S_{X_5X_3}\left(S_{X_5X_1}S_{X_4X_4} - S_{X_5X_4}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_1X_3} - S_{X_5X_1}S_{X_4X_3}\right) + S_{X_5X_5}\left(S_{X_4X_3}S_{X_4X_1} - S_{X_4X_4}S_{X_3X_1}\right),$$

$$B_{22} = S_{X_5X_1}\left(S_{X_5X_4}S_{X_1X_4} - S_{X_5X_1}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_5X_1}S_{X_4X_1} - S_{X_5X_4}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_4X_4}S_{X_1X_1} - S_{X_4X_1}S_{X_4X_1}\right),$$

$$C_{22} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_4X_3} - S_{X_5X_4}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_5X_4}S_{X_1X_1} - S_{X_5X_1}S_{X_4X_1}\right) + S_{X_5X_5}\left(S_{X_3X_1}S_{X_4X_1} - S_{X_3X_4}S_{X_1X_1}\right),$$

$$D_{22} = S_{X_5X_1}\left(S_{X_3X_1}S_{X_4X_4} - S_{X_3X_4}S_{X_1X_4}\right) + S_{X_5X_3}\left(S_{X_4X_1}S_{X_4X_1} - S_{X_4X_4}S_{X_1X_1}\right) + S_{X_5X_4}\left(S_{X_4X_3}S_{X_1X_1} - S_{X_4X_1}S_{X_3X_1}\right),$$

$$A_{23} = S_{X_5X_3}\left(S_{X_1X_4}S_{X_5X_4} - S_{X_1X_5}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_3X_4}S_{X_1X_5} - S_{X_3X_1}S_{X_4X_5}\right) + S_{X_5X_5}\left(S_{X_3X_1}S_{X_4X_4} - S_{X_3X_4}S_{X_1X_4}\right),$$

$$B_{23} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_4X_4} - S_{X_5X_4}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_1X_1} - S_{X_5X_1}S_{X_4X_1}\right) + S_{X_5X_5}\left(S_{X_1X_4}S_{X_1X_4} - S_{X_1X_1}S_{X_4X_4}\right),$$

$$C_{23} = S_{X_5X_1}\left(S_{X_3X_1}S_{X_4X_5} - S_{X_3X_4}S_{X_1X_5}\right) + S_{X_5X_3}\left(S_{X_4X_1}S_{X_5X_1} - S_{X_4X_5}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_3X_4}S_{X_1X_1} - S_{X_3X_1}S_{X_4X_1}\right),$$

$$D_{23} = S_{X_5X_1}\left(S_{X_4X_3}S_{X_4X_1} - S_{X_4X_4}S_{X_3X_1}\right) + S_{X_5X_3}\left(S_{X_4X_4}S_{X_1X_1} - S_{X_4X_1}S_{X_4X_1}\right) + S_{X_5X_4}\left(S_{X_3X_1}S_{X_4X_1} - S_{X_3X_4}S_{X_1X_1}\right),$$

$$A_{24} = S_{X_5X_3}\left(S_{X_4X_3}S_{X_1X_5} - S_{X_4X_1}S_{X_3X_5}\right) + S_{X_5X_4}\left(S_{X_5X_3}S_{X_1X_3} - S_{X_5X_1}S_{X_3X_3}\right) + S_{X_5X_5}\left(S_{X_4X_1}S_{X_3X_3} - S_{X_4X_3}S_{X_1X_3}\right),$$

$$B_{24} = S_{X_5X_1}\left(S_{X_4X_1}S_{X_3X_5} - S_{X_4X_3}S_{X_1X_5}\right) + S_{X_5X_4}\left(S_{X_3X_1}S_{X_5X_1} - S_{X_3X_5}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_4X_3}S_{X_1X_1} - S_{X_4X_1}S_{X_3X_1}\right),$$

$$C_{24} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_3X_3} - S_{X_5X_3}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_5X_3}S_{X_1X_1} - S_{X_5X_1}S_{X_3X_1}\right) + S_{X_5X_5}\left(S_{X_1X_3}S_{X_1X_3} - S_{X_1X_1}S_{X_3X_3}\right),$$

$$D_{24} = S_{X_5X_1}\left(S_{X_4X_3}S_{X_1X_3} - S_{X_4X_1}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_4X_1}S_{X_3X_2} - S_{X_4X_3}S_{X_1X_1}\right) + S_{X_5X_4}\left(S_{X_1X_1}S_{X_3X_3} - S_{X_1X_3}S_{X_1X_3}\right),$$

$$A_{25} = S_{X_5X_1}\left(S_{X_3X_3}S_{X_4X_4} - S_{X_3X_4}S_{X_3X_4}\right) + S_{X_5X_3}\left(S_{X_1X_4}S_{X_3X_4} - S_{X_1X_3}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_1X_3}S_{X_4X_3} - S_{X_1X_4}S_{X_3X_3}\right),$$

$$B_{25} = S_{X_5X_1}\left(S_{X_3X_4}S_{X_1X_4} - S_{X_3X_1}S_{X_4X_4}\right) + S_{X_5X_3}\left(S_{X_1X_1}S_{X_4X_4} - S_{X_1X_4}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_3X_1}S_{X_4X_1} - S_{X_3X_4}S_{X_1X_1}\right),$$

$$C_{25} = S_{X_5X_1}\left(S_{X_3X_4}S_{X_3X_1} - S_{X_3X_3}S_{X_4X_1}\right) + S_{X_5X_3}\left(S_{X_1X_4}S_{X_1X_3} - S_{X_1X_1}S_{X_4X_3}\right) + S_{X_5X_4}\left(S_{X_1X_1}S_{X_3X_3} - S_{X_1X_3}S_{X_1X_3}\right),$$

$$D_{25} = S_{X_4X_1}\left(S_{X_4X_1}S_{X_3X_3} - S_{X_4X_3}S_{X_1X_3}\right) + S_{X_4X_3}\left(S_{X_4X_3}S_{X_1X_1} - S_{X_4X_1}S_{X_3X_1}\right) + S_{X_4X_4}\left(S_{X_1X_3}S_{X_1X_3} - S_{X_1X_1}S_{X_3X_3}\right).$$

$$\hat{\beta}_3 = \frac{S_{YX_1}K_3 + S_{YX_2}L_3 + S_{YX_3}M_3 + S_{YX_4}N_3 + S_{YX_5}O_3}{S_{X_3X_1}K_3 + S_{X_3X_2}L_3 + S_{X_3X_3}M_3 + S_{X_3X_4}N_3 + S_{X_3X_5}O_3} \tag{3.39}$$

where

$$K_3 = A_{31}S_{X_3X_1} + B_{31}S_{X_3X_2} + C_{31}S_{X_3X_4} + D_{31}S_{X_3X_5},$$

$$L_3 = A_{32}S_{X_3X_1} + B_{32}S_{X_3X_2} + C_{32}S_{X_3X_4} + D_{32}S_{X_3X_5},$$

$$M_3 = A_{33}S_{X_4X_1} + B_{33}S_{X_4X_2} + C_{33}S_{X_4X_4} + D_{33}S_{X_4X_5},$$

$$N_3 = A_{34}S_{X_3X_1} + B_{34}S_{X_3X_2} + C_{34}S_{X_3X_4} + D_{34}S_{X_3X_5},$$

$$O_3 = A_{35}S_{X_3X_1} + B_{35}S_{X_3X_2} + C_{35}S_{X_3X_4} + D_{35}S_{X_3X_5},$$

$$A_{31} = S_{X_5X_2}\left(S_{X_5X_2}S_{X_4X_4} - S_{X_5X_4}S_{X_2X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_2X_2} - S_{X_5X_2}S_{X_4X_2}\right) + S_{X_5X_5}\left(S_{X_4X_2}S_{X_4X_2} - S_{X_4X_4}S_{X_2X_2}\right),$$

$$B_{31} = S_{X_5X_1}\left(S_{X_5X_4}S_{X_2X_4} - S_{X_5X_2}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_5X_2}S_{X_4X_1} - S_{X_5X_4}S_{X_2X_1}\right) + S_{X_5X_5}\left(S_{X_2X_1}S_{X_4X_4} - S_{X_2X_4}S_{X_1X_4}\right),$$

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$$C_{31} = S_{X_5X_1}\left(S_{X_5X_2}S_{X_4X_2} - S_{X_5X_4}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_5X_4}S_{X_2X_1} - S_{X_5X_2}S_{X_4X_1}\right) + S_{X_5X_5}\left(S_{X_4X_1}S_{X_2X_2} - S_{X_4X_2}S_{X_1X_2}\right),$$

$$D_{31} = S_{X_5X_1}\left(S_{X_4X_4}S_{X_2X_2} - S_{X_4X_2}S_{X_4X_2}\right) + S_{X_5X_2}\left(S_{X_4X_2}S_{X_4X_1} - S_{X_4X_4}S_{X_2X_1}\right) + S_{X_5X_4}\left(S_{X_4X_2}S_{X_1X_2} - S_{X_4X_1}S_{X_2X_2}\right),$$

$$A_{32} = S_{X_5X_2}\left(S_{X_5X_4}S_{X_1X_4} - S_{X_5X_1}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_5X_1}S_{X_4X_2} - S_{X_5X_4}S_{X_1X_2}\right) + S_{X_5X_5}\left(S_{X_2X_1}S_{X_4X_4} - S_{X_2X_4}S_{X_1X_4}\right),$$

$$B_{32} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_4X_4} - S_{X_5X_4}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_5X_4}S_{X_1X_1} - S_{X_5X_1}S_{X_4X_1}\right) + S_{X_5X_5}\left(S_{X_4X_1}S_{X_4X_1} - S_{X_4X_4}S_{X_1X_1}\right),$$

$$C_{32} = S_{X_5X_1}\left(S_{X_5X_4}S_{X_1X_2} - S_{X_5X_1}S_{X_4X_2}\right) + S_{X_5X_2}\left(S_{X_5X_1}S_{X_4X_1} - S_{X_5X_4}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_4X_2}S_{X_1X_1} - S_{X_4X_1}S_{X_2X_1}\right),$$

$$D_{32} = S_{X_5X_1}\left(S_{X_2X_4}S_{X_1X_4} - S_{X_2X_1}S_{X_4X_4}\right) + S_{X_5X_2}\left(S_{X_1X_1}S_{X_4X_4} - S_{X_1X_4}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_4X_1}S_{X_2X_1} - S_{X_4X_2}S_{X_1X_1}\right),$$

$$A_{33} = S_{X_5X_2}\left(S_{X_5X_2}S_{X_1X_4} - S_{X_5X_1}S_{X_2X_4}\right) + S_{X_5X_4}\left(S_{X_5X_1}S_{X_2X_2} - S_{X_5X_2}S_{X_1X_2}\right) + S_{X_5X_5}\left(S_{X_4X_2}S_{X_1X_2} - S_{X_4X_1}S_{X_2X_2}\right),$$

$$B_{33} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_2X_4} - S_{X_5X_2}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_5X_2}S_{X_1X_1} - S_{X_5X_1}S_{X_2X_1}\right) + S_{X_5X_5}\left(S_{X_4X_1}S_{X_2X_1} - S_{X_4X_2}S_{X_1X_1}\right),$$

$$C_{33} = S_{X_5X_1}\left(S_{X_5X_2}S_{X_1X_2} - S_{X_5X_1}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_5X_1}S_{X_2X_1} - S_{X_5X_2}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_1X_1}S_{X_2X_2} - S_{X_1X_2}S_{X_1X_2}\right),$$

$$D_{33} = S_{X_5X_1}\left(S_{X_4X_1}S_{X_2X_2} - S_{X_4X_2}S_{X_1X_2}\right) + S_{X_5X_2}\left(S_{X_4X_2}S_{X_1X_1} - S_{X_4X_1}S_{X_2X_1}\right) + S_{X_5X_4}\left(S_{X_2X_1}S_{X_2X_1} - S_{X_2X_2}S_{X_1X_1}\right),$$

$$A_{34} = S_{X_5X_2}\left(S_{X_5X_1}S_{X_2X_4} - S_{X_5X_2}S_{X_1X_4}\right) + S_{X_5X_4}\left(S_{X_5X_2}S_{X_1X_2} - S_{X_5X_1}S_{X_2X_2}\right) + S_{X_5X_5}\left(S_{X_4X_1}S_{X_2X_2} - S_{X_4X_2}S_{X_1X_2}\right),$$

$$B_{34} = S_{X_5X_1}\left(S_{X_5X_2}S_{X_1X_4} - S_{X_5X_1}S_{X_2X_4}\right) + S_{X_5X_4}\left(S_{X_5X_1}S_{X_2X_1} - S_{X_5X_2}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_4X_2}S_{X_1X_1} - S_{X_4X_1}S_{X_2X_1}\right),$$

$$C_{34} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_2X_2} - S_{X_5X_2}S_{X_1X_2}\right) + S_{X_5X_2}\left(S_{X_5X_2}S_{X_1X_1} - S_{X_5X_1}S_{X_2X_1}\right) + S_{X_5X_5}\left(S_{X_2X_1}S_{X_2X_1} - S_{X_2X_2}S_{X_1X_1}\right),$$

$$D_{34} = S_{X_5X_1}\left(S_{X_4X_2}S_{X_1X_2} - S_{X_4X_1}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_4X_1}S_{X_2X_1} - S_{X_4X_2}S_{X_1X_1}\right) + S_{X_5X_4}\left(S_{X_2X_2}S_{X_1X_1} - S_{X_2X_1}S_{X_2X_1}\right),$$

$$A_{35} = S_{X_5X_1}\left(S_{X_2X_2}S_{X_4X_4} - S_{X_2X_4}S_{X_2X_4}\right) + S_{X_5X_2}\left(S_{X_2X_4}S_{X_1X_4} - S_{X_2X_1}S_{X_4X_4}\right) + S_{X_5X_4}\left(S_{X_4X_2}S_{X_1X_2} - S_{X_4X_1}S_{X_2X_2}\right),$$

$$B_{35} = S_{X_5X_1}\left(S_{X_4X_2}S_{X_4X_1} - S_{X_4X_4}S_{X_2X_1}\right) + S_{X_5X_2}\left(S_{X_4X_4}S_{X_1X_1} - S_{X_4X_1}S_{X_4X_1}\right) + S_{X_5X_4}\left(S_{X_4X_1}S_{X_2X_1} - S_{X_4X_2}S_{X_1X_1}\right),$$

$$C_{35} = S_{X_5X_1}\left(S_{X_4X_2}S_{X_1X_2} - S_{X_4X_1}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_4X_1}S_{X_2X_1} - S_{X_4X_2}S_{X_1X_1}\right) + S_{X_5X_4}\left(S_{X_2X_2}S_{X_1X_1} - S_{X_2X_1}S_{X_2X_1}\right),$$

$$D_{35} = S_{X_4X_1}\left(S_{X_4X_1}S_{X_2X_2} - S_{X_4X_2}S_{X_1X_2}\right) + S_{X_4X_2}\left(S_{X_4X_2}S_{X_1X_1} - S_{X_4X_1}S_{X_2X_1}\right) + S_{X_4X_4}\left(S_{X_2X_1}S_{X_2X_1} - S_{X_2X_2}S_{X_1X_1}\right).$$

$$\hat{\beta}_4 = \frac{S_{YX_1}K_4 + S_{YX_2}L_4 + S_{YX_3}M_4 + S_{YX_4}N_4 + S_{YX_5}O_4}{S_{X_4X_1}K_4 + S_{X_4X_2}L_4 + S_{X_4X_3}M_4 + S_{X_4X_4}N_4 + S_{X_4X_5}O_4} \qquad (3.40)$$

where

$$K_4 = A_{41}S_{X_4X_1} + B_{41}S_{X_4X_2} + C_{41}S_{X_4X_3} + D_{41}S_{X_4X_5},$$
$$L_4 = A_{42}S_{X_4X_1} + B_{42}S_{X_4X_2} + C_{42}S_{X_4X_3} + D_{42}S_{X_4X_5},$$
$$M_4 = A_{43}S_{X_4X_1} + B_{43}S_{X_4X_2} + C_{43}S_{X_4X_3} + D_{43}S_{X_4X_5},$$
$$N_4 = A_{44}S_{X_5X_1} + B_{44}S_{X_5X_2} + C_{44}S_{X_5X_3} + D_{44}S_{X_5X_5},$$
$$O_4 = A_{45}S_{X_4X_1} + B_{45}S_{X_4X_2} + C_{45}S_{X_4X_3} + D_{45}S_{X_4X_5},$$

$$A_{41} = S_{X_5X_2}\left(S_{X_5X_2}S_{X_3X_3} - S_{X_5X_3}S_{X_2X_3}\right) + S_{X_5X_3}\left(S_{X_5X_3}S_{X_2X_2} - S_{X_5X_2}S_{X_3X_2}\right) + S_{X_5X_5}\left(S_{X_3X_2}S_{X_3X_2} - S_{X_3X_3}S_{X_2X_2}\right),$$

$$B_{41} = S_{X_5X_1}\left(S_{X_5X_3}S_{X_2X_3} - S_{X_5X_2}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_5X_2}S_{X_3X_1} - S_{X_5X_3}S_{X_2X_1}\right) + S_{X_5X_5}\left(S_{X_2X_1}S_{X_3X_3} - S_{X_2X_3}S_{X_1X_3}\right),$$

$$C_{41} = S_{X_5X_1}\left(S_{X_5X_2}S_{X_3X_2} - S_{X_5X_3}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_5X_3}S_{X_2X_1} - S_{X_5X_2}S_{X_3X_1}\right) + S_{X_5X_5}\left(S_{X_3X_1}S_{X_2X_2} - S_{X_3X_2}S_{X_1X_2}\right),$$

$$D_{41} = S_{X_5X_1}\left(S_{X_3X_3}S_{X_2X_2} - S_{X_3X_2}S_{X_3X_2}\right) + S_{X_5X_2}\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right),$$

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$$A_{42} = S_{X_5X_2}\left(S_{X_5X_3}S_{X_1X_3} - S_{X_5X_1}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_5X_1}S_{X_3X_2} - S_{X_5X_3}S_{X_1X_2}\right) + S_{X_5X_5}\left(S_{X_2X_1}S_{X_3X_3} - S_{X_2X_3}S_{X_1X_3}\right),$$

$$B_{42} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_3X_3} - S_{X_5X_3}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_5X_3}S_{X_1X_1} - S_{X_5X_1}S_{X_3X_1}\right) + S_{X_5X_5}\left(S_{X_3X_1}S_{X_3X_1} - S_{X_3X_3}S_{X_1X_1}\right),$$

$$C_{42} = S_{X_5X_1}\left(S_{X_5X_3}S_{X_1X_2} - S_{X_5X_1}S_{X_3X_2}\right) + S_{X_5X_2}\left(S_{X_5X_1}S_{X_3X_1} - S_{X_5X_3}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_3X_2}S_{X_1X_1} - S_{X_3X_1}S_{X_2X_1}\right),$$

$$D_{42} = S_{X_5X_1}\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right) + S_{X_5X_2}\left(S_{X_1X_1}S_{X_3X_3} - S_{X_1X_3}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right),$$

$$A_{43} = S_{X_5X_2}\left(S_{X_5X_1}S_{X_2X_3} - S_{X_5X_2}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_5X_2}S_{X_1X_2} - S_{X_5X_1}S_{X_2X_2}\right) + S_{X_5X_5}\left(S_{X_3X_1}S_{X_2X_2} - S_{X_3X_2}S_{X_1X_2}\right),$$

$$B_{43} = S_{X_5X_1}\left(S_{X_5X_2}S_{X_1X_3} - S_{X_5X_1}S_{X_2X_3}\right) + S_{X_5X_3}\left(S_{X_5X_1}S_{X_2X_1} - S_{X_5X_2}S_{X_1X_1}\right) + S_{X_5X_5}\left(S_{X_3X_2}S_{X_1X_1} - S_{X_3X_1}S_{X_2X_1}\right),$$

$$C_{43} = S_{X_5X_1}\left(S_{X_5X_1}S_{X_2X_2} - S_{X_5X_2}S_{X_1X_2}\right) + S_{X_5X_2}\left(S_{X_5X_2}S_{X_1X_1} - S_{X_5X_1}S_{X_2X_1}\right) + S_{X_5X_5}\left(S_{X_2X_1}S_{X_2X_1} - S_{X_2X_2}S_{X_1X_1}\right),$$

$$D_{43} = S_{X_5X_1}\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right) + S_{X_5X_3}\left(S_{X_1X_1}S_{X_2X_2} - S_{X_1X_2}S_{X_1X_2}\right),$$

$$A_{44} = S_{X_5X_1}\left(S_{X_3X_2}S_{X_3X_2} - S_{X_3X_3}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_2X_1}S_{X_3X_3} - S_{X_2X_3}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_3X_1}S_{X_2X_2} - S_{X_3X_2}S_{X_1X_2}\right),$$

$$B_{44} = S_{X_5X_1}\left(S_{X_2X_1}S_{X_3X_3} - S_{X_2X_3}S_{X_1X_3}\right) + S_{X_5X_2}\left(S_{X_3X_1}S_{X_3X_1} - S_{X_3X_3}S_{X_1X_1}\right) + S_{X_5X_3}\left(S_{X_3X_2}S_{X_1X_1} - S_{X_3X_1}S_{X_2X_1}\right),$$

$$C_{44} = S_{X_5X_1}\left(S_{X_3X_1}S_{X_2X_2} - S_{X_3X_2}S_{X_1X_2}\right) + S_{X_5X_2}\left(S_{X_3X_2}S_{X_1X_1} - S_{X_3X_1}S_{X_2X_1}\right) + S_{X_5X_3}\left(S_{X_2X_1}S_{X_2X_1} - S_{X_2X_2}S_{X_1X_1}\right),$$

$$D_{44} = S_{X_3X_1}\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right) + S_{X_3X_2}\left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right) + S_{X_3X_3}\left(S_{X_1X_1}S_{X_2X_2} - S_{X_1X_2}S_{X_1X_2}\right),$$

$$A_{45} = S_{X_5X_1}\left(S_{X_2X_2}S_{X_3X_3} - S_{X_2X_3}S_{X_2X_3}\right) + S_{X_5X_2}\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right) + S_{X_5X_3}\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right),$$

$$B_{45} = S_{X_5X_1}\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right) + S_{X_5X_2}\left(S_{X_1X_1}S_{X_3X_3} - S_{X_1X_3}S_{X_1X_3}\right) + S_{X_5X_3}\left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right),$$

$$C_{45} = S_{X_5X_1}\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right) + S_{X_5X_2}\left(S_{X_3X_1}S_{X_2X_1} - S_{X_3X_2}S_{X_1X_1}\right) + S_{X_5X_3}\left(S_{X_1X_1}S_{X_2X_2} - S_{X_1X_2}S_{X_1X_2}\right),$$

$$D_{45} = S_{X_3X_1}\left(S_{X_3X_1}S_{X_2X_2} - S_{X_3X_2}S_{X_1X_2}\right) + S_{X_3X_2}\left(S_{X_3X_2}S_{X_1X_1} - S_{X_3X_1}S_{X_2X_1}\right) + S_{X_3X_3}\left(S_{X_2X_1}S_{X_2X_1} - S_{X_2X_2}S_{X_1X_1}\right).$$

$$\hat{\beta}_5 = \frac{S_{YX_1}K_5 + S_{YX_2}L_5 + S_{YX_3}M_5 + S_{YX_4}N_5 + S_{YX_5}O_5}{S_{X_5X_1}K_5 + S_{X_5X_2}L_5 + S_{X_5X_3}M_5 + S_{X_5X_4}N_5 + S_{X_5X_5}O_5} \tag{3.41}$$

where

$$K_5 = A_{51}S_{X_5X_1} + B_{51}S_{X_5X_2} + C_{51}S_{X_5X_3} + D_{51}S_{X_5X_4},$$
$$L_5 = A_{52}S_{X_5X_1} + B_{52}S_{X_5X_2} + C_{52}S_{X_5X_3} + D_{52}S_{X_5X_4},$$
$$M_5 = A_{53}S_{X_5X_1} + B_{53}S_{X_5X_2} + C_{53}S_{X_5X_3} + D_{53}S_{X_5X_4},$$
$$N_5 = A_{54}S_{X_5X_1} + B_{54}S_{X_5X_2} + C_{54}S_{X_5X_3} + D_{54}S_{X_5X_4},$$
$$O_5 = A_{55}S_{X_4X_1} + B_{55}S_{X_4X_2} + C_{55}S_{X_4X_3} + D_{55}S_{X_4X_4},$$

$$A_{51} = S_{X_4X_2}\left(S_{X_4X_2}S_{X_3X_3} - S_{X_4X_3}S_{X_2X_3}\right) + S_{X_4X_3}\left(S_{X_4X_3}S_{X_2X_2} - S_{X_4X_2}S_{X_3X_2}\right) + S_{X_4X_4}\left(S_{X_3X_2}S_{X_3X_2} - S_{X_3X_3}S_{X_2X_2}\right),$$

$$B_{51} = S_{X_4X_1}\left(S_{4X_3}S_{X_2X_3} - S_{X_4X_2}S_{X_3X_3}\right) + S_{X_4X_3}\left(S_{X_4X_2}S_{X_3X_1} - S_{X_4X_3}S_{X_2X_1}\right) + S_{X_4X_4}\left(S_{X_2X_1}S_{X_3X_3} - S_{X_2X_3}S_{X_1X_3}\right),$$

$$C_{51} = S_{X_4X_1}\left(S_{X_4X_2}S_{X_3X_2} - S_{X_4X_3}S_{X_2X_2}\right) + S_{X_4X_2}\left(S_{X_4X_3}S_{X_2X_1} - S_{X_4X_2}S_{X_3X_1}\right) + S_{X_4X_4}\left(S_{X_3X_1}S_{X_2X_2} - S_{X_3X_2}S_{X_1X_2}\right),$$

$$D_{51} = S_{X_4X_1}\left(S_{X_2X_2}S_{X_3X_3} - S_{X_2X_3}S_{X_2X_3}\right) + S_{X_4X_2}\left(S_{X_2X_3}S_{X_1X_3} - S_{X_2X_1}S_{X_3X_3}\right) + S_{X_4X_3}\left(S_{X_3X_2}S_{X_1X_2} - S_{X_3X_1}S_{X_2X_2}\right),$$

$$A_{52} = S_{X_4X_2}\left(S_{X_4X_3}S_{X_1X_3} - S_{X_4X_1}S_{X_3X_3}\right) + S_{X_4X_3}\left(S_{X_4X_1}S_{X_3X_2} - S_{X_4X_3}S_{X_1X_2}\right) + S_{X_4X_4}\left(S_{X_2X_1}S_{X_3X_3} - S_{X_2X_3}S_{X_1X_3}\right),$$

$$B_{52} = S_{X_4X_1}\left(S_{X_4X_1}S_{X_3X_3} - S_{X_4X_3}S_{X_1X_3}\right) + S_{X_4X_3}\left(S_{X_4X_3}S_{X_1X_1} - S_{X_4X_1}S_{X_3X_1}\right) + S_{X_4X_4}\left(S_{X_3X_1}S_{X_3X_1} - S_{X_3X_3}S_{X_1X_1}\right),$$

$$C_{52} = S_{x_4x_1}\left(S_{x_4x_3}S_{x_1x_2} - S_{x_4x_1}S_{x_3x_2}\right) + S_{x_4x_2}\left(S_{x_4x_1}S_{x_3x_1} - S_{x_4x_3}S_{x_1x_1}\right) + S_{x_4x_4}\left(S_{x_3x_2}S_{x_1x_1} - S_{x_3x_1}S_{x_2x_1}\right),$$
$$D_{52} = S_{x_4x_1}\left(S_{x_2x_3}S_{x_1x_3} - S_{x_2x_1}S_{x_3x_3}\right) + S_{x_4x_2}\left(S_{x_1x_1}S_{x_3x_3} - S_{x_1x_3}S_{x_1x_3}\right) + S_{x_4x_3}\left(S_{x_3x_1}S_{x_2x_1} - S_{x_3x_2}S_{x_1x_1}\right),$$

$$A_{53} = S_{x_4x_2}\left(S_{x_4x_1}S_{x_2x_3} - S_{x_4x_2}S_{x_1x_3}\right) + S_{x_4x_3}\left(S_{x_4x_2}S_{x_1x_2} - S_{x_4x_1}S_{x_2x_2}\right) + S_{x_4x_4}\left(S_{x_3x_1}S_{x_2x_2} - S_{x_3x_2}S_{x_1x_2}\right),$$
$$B_{53} = S_{x_4x_1}\left(S_{x_4x_2}S_{x_1x_3} - S_{x_4x_1}S_{x_2x_3}\right) + S_{x_4x_3}\left(S_{x_4x_1}S_{x_2x_1} - S_{x_4x_2}S_{x_1x_1}\right) + S_{x_4x_4}\left(S_{x_3x_2}S_{x_1x_1} - S_{x_3x_1}S_{x_2x_1}\right),$$
$$C_{53} = S_{x_4x_1}\left(S_{x_4x_1}S_{x_2x_2} - S_{x_4x_2}S_{x_1x_2}\right) + S_{x_4x_2}\left(S_{x_4x_2}S_{x_1x_1} - S_{x_4x_1}S_{x_2x_1}\right) + S_{x_4x_4}\left(S_{x_2x_1}S_{x_2x_1} - S_{x_2x_2}S_{x_1x_1}\right),$$
$$D_{53} = S_{x_4x_1}\left(S_{x_3x_2}S_{x_1x_2} - S_{x_3x_1}S_{x_2x_2}\right) + S_{x_4x_2}\left(S_{x_3x_1}S_{x_2x_1} - S_{x_3x_2}S_{x_1x_1}\right) + S_{x_4x_3}\left(S_{x_1x_1}S_{x_2x_2} - S_{x_1x_2}S_{x_1x_2}\right),$$

$$A_{54} = S_{x_4x_1}\left(S_{x_2x_2}S_{x_3x_3} - S_{x_2x_3}S_{x_2x_3}\right) + S_{x_4x_2}\left(S_{x_2x_3}S_{x_1x_3} - S_{x_2x_1}S_{x_3x_3}\right) + S_{x_4x_3}\left(S_{x_3x_2}S_{x_1x_2} - S_{x_3x_1}S_{x_2x_2}\right),$$
$$B_{54} = S_{x_4x_1}\left(S_{x_2x_3}S_{x_1x_3} - S_{x_2x_1}S_{x_3x_3}\right) + S_{x_4x_2}\left(S_{x_1x_1}S_{x_3x_3} - S_{x_1x_3}S_{x_1x_3}\right) + S_{x_4x_3}\left(S_{x_3x_1}S_{x_2x_1} - S_{x_3x_2}S_{x_1x_1}\right),$$
$$C_{54} = S_{x_4x_1}\left(S_{x_3x_2}S_{x_1x_2} - S_{x_3x_1}S_{x_2x_2}\right) + S_{x_4x_2}\left(S_{x_3x_1}S_{x_2x_1} - S_{x_3x_2}S_{x_1x_1}\right) + S_{x_4x_3}\left(S_{x_1x_1}S_{x_2x_2} - S_{x_1x_2}S_{x_1x_2}\right),$$
$$D_{54} = S_{x_3x_1}\left(S_{x_3x_1}S_{x_2x_2} - S_{x_3x_2}S_{x_1x_2}\right) + S_{x_3x_2}\left(S_{x_3x_2}S_{x_1x_1} - S_{x_3x_1}S_{x_2x_1}\right) + S_{x_3x_3}\left(S_{x_2x_1}S_{x_2x_1} - S_{x_2x_2}S_{x_1x_1}\right),$$

$$A_{55} = S_{x_4x_1}\left(S_{x_3x_2}S_{x_3x_2} - S_{x_3x_3}S_{x_2x_2}\right) + S_{x_4x_2}\left(S_{x_2x_1}S_{x_3x_3} - S_{x_2x_3}S_{x_1x_3}\right) + S_{x_4x_3}\left(S_{x_3x_1}S_{x_2x_2} - S_{x_3x_2}S_{x_1x_2}\right),$$
$$B_{55} = S_{x_4x_1}\left(S_{x_2x_1}S_{x_3x_3} - S_{x_2x_3}S_{x_1x_3}\right) + S_{x_4x_2}\left(S_{x_3x_1}S_{x_3x_1} - S_{x_3x_3}S_{x_1x_1}\right) + S_{x_4x_3}\left(S_{x_3x_2}S_{x_1x_1} - S_{x_3x_1}S_{x_2x_1}\right),$$
$$C_{55} = S_{x_4x_1}\left(S_{x_3x_1}S_{x_2x_2} - S_{x_3x_2}S_{x_1x_2}\right) + S_{x_4x_2}\left(S_{x_3x_2}S_{x_1x_1} - S_{x_3x_1}S_{x_2x_1}\right) + S_{x_4x_3}\left(S_{x_2x_1}S_{x_2x_1} - S_{x_2x_2}S_{x_1x_1}\right),$$
$$D_{55} = S_{x_3x_1}\left(S_{x_3x_2}S_{x_1x_2} - S_{x_3x_1}S_{x_2x_2}\right) + S_{x_3x_2}\left(S_{x_3x_1}S_{x_2x_1} - S_{x_3x_2}S_{x_1x_1}\right) + S_{x_3x_3}\left(S_{x_1x_1}S_{x_2x_2} - S_{x_1x_2}S_{x_1x_2}\right).$$

The values of 5 independent parameter against one dependent parameter are given in Table 5. The system of equation (3.34) can be written in the following equation system:

$$\begin{bmatrix} 10 & 45 & 39 & 23.19 & 550 & 3513 \\ 45 & 217 & 166 & 98.21 & 2560 & 16270 \\ 39 & 166 & 175 & 90.69 & 2280 & 14546 \\ 23.19 & 98.21 & 90.69 & 60.4027 & 1220.40 & 7865.43 \\ 550 & 2560 & 2280 & 1220.40 & 38500 & 240300 \\ 3513 & 16270 & 14546 & 7865.43 & 240300 & 1508819 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \end{bmatrix} = \begin{bmatrix} 54 \\ 236 \\ 214 \\ 130.95 \\ 3160 \\ 20001 \end{bmatrix}.$$

By solving the system above, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} 6.8037 \\ -0.7156 \\ -0.3576 \\ 0.6012 \\ 0.1010 \\ -0.0106 \end{bmatrix}.$$

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Table 5. The values of 5 independent parameter against one dependent parameter

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**∑**
$\mathbf{X_1}$	6	4	2	5	4	3	5	6	5	5	**45**
$\mathbf{X_2}$	2	4	5	4	2	6	3	2	6	5	**39**
$\mathbf{X_3}$	1	3	3,91	2,64	3,09	1,55	2	2,18	1,64	2,18	**23,19**
$\mathbf{X_4}$	10	20	30	40	50	60	70	80	90	100	**550**
$\mathbf{X_5}$	110	154	225	256	330	352	412	458	557	659	**3513**
$\mathbf{Y}$	2	5	6	5	7	6	6	6	5	6	**54**
$\mathbf{X_1*X_1}$	36	16	4	25	16	9	25	36	25	25	**217**
$\mathbf{X_1*X_2}$	12	16	10	20	8	18	15	12	30	25	**166**
$\mathbf{X_1*X_3}$	6	12	7,82	13,2	12,36	4,65	10	13,08	8,2	10,9	**98,21**
$\mathbf{X_1*X_4}$	60	80	60	200	200	180	350	480	450	500	**2560**
$\mathbf{X_1*X_5}$	660	616	450	1280	1320	1056	2060	2748	2785	3295	**16270**
$\mathbf{X_1*Y}$	12	20	12	25	28	18	30	36	25	30	**236**
$\mathbf{X_2*X_2}$	4	16	25	16	4	36	9	4	36	25	**175**
$\mathbf{X_2*X_3}$	2	12	19,55	10,56	6,18	9,3	6	4,36	9,84	10,9	**90,69**
$\mathbf{X_2*X_4}$	20	80	150	160	100	360	210	160	540	500	**2280**
$\mathbf{X_2*X_5}$	220	616	1125	1024	660	2112	1236	916	3342	3295	**14546**
$\mathbf{X_2*Y}$	4	20	30	20	14	36	18	12	30	30	**214**
$\mathbf{X_3*X_3}$	1	9	15,2881	6,9696	9,5481	2,4025	4	4,7524	2,6896	4,7524	**60,4027**
$\mathbf{X_3*X_4}$	10	60	117,3	105,6	154,5	93	140	174,4	147,6	218	**1220,4**
$\mathbf{X_3*X_5}$	110	462	879,75	675,84	1019,7	545,6	824	998,44	913,48	1436,62	**7865,43**
$\mathbf{X_3*Y}$	2	15	23,46	13,2	21,63	9,3	12	13,08	8,2	13,08	**130,95**
$\mathbf{X_4*X_4}$	100	400	900	1600	2500	3600	4900	6400	8100	10000	**38500**
$\mathbf{X_4*X_5}$	1100	3080	6750	10240	16500	21120	28840	36640	50130	65900	**240300**
$\mathbf{X_4*Y}$	20	100	180	200	350	360	420	480	450	600	**3160**
$\mathbf{X_5*X_5}$	12100	23716	50625	65536	108900	123904	169744	209764	310249	434281	**1508819**
$\mathbf{X_5*Y}$	220	770	1350	1280	2310	2112	2472	2748	2785	3954	**20001**
$\mathbf{Y*Y}$	4	25	36	25	49	36	36	36	25	36	**308**

By using the values in Table 5, the system of equation (3.35) can be written in the following:

$$\begin{bmatrix} -7 \\ 3.4 \\ 5.724 \\ 190 \\ 1030.8 \end{bmatrix} = \begin{bmatrix} 14.5 & -9.5 & -6.145 & 85 & 461.5 \\ -9.5 & 22.9 & 0.249 & 135 & 845.3 \\ -6.145 & 0.249 & 6.6251 & -55.05 & -281.217 \\ 85 & 135 & -55.05 & 8250 & 47085 \\ 461.5 & 845.3 & -281.217 & 47085 & 2.7470021x10^5 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \\ \hat{\beta}_4 \\ \hat{\beta}_5 \end{bmatrix}.$$

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By solving the system, we can find unknown parameters as follows:

$$\hat{\beta} = \begin{bmatrix} -0.7156 \\ -0.3576 \\ 0.6012 \\ 0.1010 \\ -0.0106 \end{bmatrix}$$

and after getting the values of $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4$ and $\hat{\beta}_5$, by using equation (3.36), we can find the value of $\hat{\beta}_0$ below,

$$\hat{\beta}_0 = 6.8037.$$

By using the values in Table 5, the relevant parameter values of equations (3.36), (3.37), (3.38), (3.39), (3.40) and (3.41) are found as follows respectively:

$$\begin{aligned} \hat{\beta}_0 &= 6.8037, \\ \hat{\beta}_1 &= -0.7156, \\ \hat{\beta}_2 &= -0.3576, \\ \hat{\beta}_3 &= 0.6012, \\ \hat{\beta}_4 &= 0.1010, \\ \hat{\beta}_5 &= -0.0106. \end{aligned}$$

The results obtained with the formula method and the results of the systems of equations (3.34) and (3.35) and equation (3.36) are exactly the same.

4. Discussion

As can be seen, this study was conducted to test the consistency of the parameter formulas obtained by Korkmaz in [6] for multiple linear models with 5 or less independent variables. Actually, in [9], Pires et al. gave the criteria to select statistically valid regression parameters using multiple linear regression models. Because of this reference, the researcher should be cautious when selecting statistically valid regression parameters of multiple linear regression. Therefore, the number of parameters of multiple linear regression will not be as many as the researchers want. In other words, the researchers may not be able to increase the number of independent variables too much.

In [6], the aim of Korkmaz's work was to get the general formula of the parameters of multiple linear regression without using a computer program. In that study [6], it was stated that the parameters of the multiple linear regression can be easily found using this formula. Thus, it was emphasized in that study [6] that researchers could estimate the regression equation.

5. Conclusions

Since the denominator and numerator of the parameters for 4 and smaller independent variables are not very long, the formulas of the parameters in the tables, which are in a certain order, are summarized under the relevant tables [6]. Since the denominator and numerator of the parameters for 5 independent variables are very long, the formulas of the parameters in a table, which is in a certain order, are not summarized under the relevant table [6]. In this study, the formulas of the parameters with 5 independent variables in a certain order are summarized, just like the formulas of the parameters with 4 or less independent variables in a certain order, by using the tables that Korkmaz used in his study [6].

While the parameter formulas were given for 5 or less independent variables, the tables were not given again in this study because they were given by Korkmaz in [6] and they take up a lot of space.

Through this study the researcher completely sees the general formulas of the parameter values with five or less independent variables that he or she wants to find.

In this study, by using the data examples, all the values in the parameter formulas were found to be the same as all the parameter values found with the related equation system. That is, the parameter values found are in a consistency.

Since different methods may have rounding errors in the same parameter values due to different calculations among the methods, 4 digits after the dot were used in the parameter values in this study. Although the methods are different, parameter values were always the same in models with 5 or less independent variables.

In this study, it has been shown that the parameter formula values of the multiple linear regression model with 5 or less independent variables are completely consistent with the parameter values obtained by other classical methods.

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