Journal of Nonlinear Modeling and Analysis
Volume 7, Number 1, February 2025, 229–240
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.229

On Strong Form Of $\beta^* - I-$ Open Sets via Ideals Topological Spaces

Radhwan Mohammed Aqeel[1,†] and Fawzia Abdullah Ahmed[2]

Abstract The concept of *strong* $\beta^* - I-$ open sets in ideal topological spaces is investigated and some of their properties are obtained, also we study these in relation to some other types of sets. Furthermore, by using the new notion, we define the strong $\beta^* - I-$ interior and strong $\beta^* - I-$ closure operators.

Keywords Local functions, ideal topological spaces, strong $\beta^* - I-$ open sets, strong $\beta^* - I-$ closed sets

MSC(2010) 54A05,54A010,18F05,54B99.

1. Introduction

Kuratowski established the fundamental concept of ideal topological spaces [23]. Later vaidyanathaswamy [30] studied the concept in point set topology. Hamlett and Janković [20] discovered a new topology τ^* that is finer than τ and uses previous ones. They also created the concept of ideal topological spaces and introduced a new Kuratowski closure operator cl^*. The contributions of Hamlett and Janković in ideal topological spaces started the generalization of several important features in general topology via topological ideal [21]. They also established the use of topological ideals in the extension of topological notions by introducing the concept of $I-$ open sets [22]. Hatir and Noiri introduced the ideas of $\alpha - I-$ open, $semi - I-$ open and $\beta - I-$ open sets in ideal topological spaces [18]. Hatir and Keskin introduced the idea of strong $\beta - I-$ open sets [16]. Ekici introduced the concepts of $\beta^* - I-$ open sets [10]. Aqeel and Bin Kuddah ([6], [5]) presented the concepts of $S.S^* - I-$ open sets and $S.P^* - I-$ open sets. Recently, the ideal topological spaces have evolved through paactical research that has studied many new concepts, including [7,13,27]. We define strong $\beta^* - I-$ open sets and strong $\beta^* - I-$ closed sets in this article. Several traits and qualities are investigated.

2. Preliminaries

In this section, we summarize the definitions and results that are needed in the sequel. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, then $cl(A)$ and $int(A)$ denote the closure and interior

†The corresponding author.
Email address: raqeel1976@yahoo.com(R.M.Aqeel),
fawziaahmed89@yahoo.com(F.A.Ahmed)
[1]Department of Mathematics, University of Aden, 374, Aden, Yemen
[2]Department of Mathematics, University of Aden, 5, Aden, Yemen

of A in (X, τ), respectively. An ideal I in a topological space (X, τ) is a nonempty collection of subsets of X that satisfies the following two conditions [23]:

(i) If $A \in I$ and $B \subset A$, then $B \in I$.

(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Let (X, τ) be a topological space, and I be an ideal on X. An ideal topological space is a topological space (X, τ) with an ideal I on X and it is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I$ for each neighborhood U of $x\}$ is called the local function of A with respect to I and τ [30]. It is obvious that $(.)^* : p(X) \to p(X)$ is a set operator. Throughout this paper, we use A^* instead of $A^*(I, \tau)$. $cl^*(A)$ and $int^*(A)$ denote the closure and interior of A in (X, τ^*) respectively. In [20], Note $cl^*(A) = A \cup A^*$ defines a Kuratowski operator for a topology τ^*, finer than τ. We start with recalling some lemmas and definitions that are necessary for this study in the sequel.

Among the results published in [1,2,4–6,10,12,14–16,18,19,22,24,25,28,29] we mention the following results in the form of Definition 2.1.

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is called:

(i) pre $-$ open if $A \subset int(cl(A))$;

(ii) strong $pre^* - I-$ open($S.P^* - I-$ open) if $A \subset int^*(cl^*(A))$;

(iii) $\alpha - I-$ open if $A \subset int(cl^*(int(A)))$;

(iv) strong $\alpha^* - I-$ open($S.\alpha^* - I-$ open) if $A \subset int^*(cl^*(int^*(A)))$;

(v) $semi^* - I-$ open if $A \subset cl(int^*(A))$;

(vi) strong $semi^* - I-$ open($S.S^* - I-$ open) if $A \subset cl^*(int^*(A))$;

(vii) $\beta-$ open if $A \subset cl(int(cl(A)))$;

(viii) $\beta - I-$ open if $A \subset cl(int(cl^*(A)))$;

(ix) $\beta^* - I-$ open if $A \subset cl(int^*(cl(A)))$;

(x) strong $\beta - I-$ open($S.\beta - I-$ open) if $A \subset cl^*(int(cl^*(A)))$;

(xi) $b - I-$ open if $A \subset cl^*(int(A)) \cup int(cl^*(A))$;

(xii) weakly semi $-I-$ open if $A \subset cl^*(int(cl(A)))$;

(xiii) $*-$ dense in itself if $A \subset A^*$;

(xiv) $*-$ perfect if $A = A^*$;

(xv) $I-$ open if $A \subset int(A^*)$;

(xvi) Almost strong $-I-$ open if $A \subset cl^*(int(A^*))$;

(xvii) Regular $-$ open if $A = int(cl(A))$;

(xviii) $t-$ set if $int(A) = int(cl(A))$;

(xix) $t - I-$ set if $int(A) = int(cl^*(A))$;

(xx) $\delta - I-$ open if $int(cl^*(A)) \subset cl^*(int(A))$.

The complement of pre $-$ open ((resp. strong $pre^* - I-$ open, $\alpha - I-$ open,...)) sets is called pre $-$ closed ((resp. strong $pre^* - I-$ closed, $\alpha - I-$ closed,...)) sets.

Definition 2.2. [3] A subset A of an ideal topological space (X, τ, I) is called $I - R-$ closed if $A = cl^*(int(A))$.

Definition 2.3. [9] In an ideal topological space (X, τ, I), I is said to be codence if $\tau \cap I = \phi$.

Lemma 2.1. *[20] Let (X, τ, I) be an ideal topological space, where I is codence. Then the following hold:*

(i) $cl(A) = cl^*(A)$, *for every* $* -$ *open set* A,

(ii) $int(A) = int^*(A)$, *for every* $* -$ *closed set* A.

Lemma 2.2. *[17] Let A be a subset of an ideal topological space (X, τ, I) and U be an open set. Then $U \cap cl^*(A) \subset cl^*(U \cap A)$.*

Lemma 2.3. *[8] Let (X, τ, I) be an ideal topological space and A be a $* - dense$ $in\ itself$ subset of X. Then $A^* = cl(A^*) = cl(A) = cl^*(A)$.*

Corollary 2.1. *[26] For each $A \subset (X, \tau, I)$ we have*

(i) $(\cup cl^*(A_\alpha) : \alpha \in \Delta) \subset cl^*(\cup A_\alpha : \alpha \in \Delta)$.

(ii) $cl^*(\cap A_\alpha : \alpha \in \Delta)) \subset (\cap cl^*(A_\alpha) : \alpha \in \Delta)$.

Lemma 2.4. *[11] An ideal topological space (X, τ, I) is I-extremally disconnected if and only if $cl^*(int(A)) \subset int(cl^*(A))$, for every subset A of X.*

Theorem 2.1. *[26] For two subsets, A and B of a space (X, τ, I), the following properties hold:*

(i) *If* $A \subset B$, *then* $cl^*(A) \subset cl^*(B)$;

(ii) $cl^*(cl^*(A)) = cl^*(A)$;

(iii) $cl^*(A \cap B) \subset cl^*(A) \cap cl^*(B)$;

(iv) $cl^*(A \cup B) = cl^*(A) \cup cl^*(B)$;

(v) $A \subset cl^*(A) \subset cl(A)$.

Lemma 2.5. *[31] Let A and B be subsets of (X, τ, I) and let $int^*(A)$ denote the interior of A with respect to τ^*. The following properties hold:*

(i) *If* $A \subset B$, *then* $int^*(A) \subset int^*(B)$;

(ii) *If A is an open in (X, τ, I), then* $A = int(A)$ *and* $A = int^*(A)$;

(iii) $int(A) \subset int^*(A) \subset A$;

(iv) $int^*(A \cap B) = int^*(A) \cap int^*(B)$;

(v) $int^*(A) \cup int^*(B) \subset int^*(A \cup B)$.

3. Strong β^*-I- open sets and strong β^*-I- closed sets

Based on the Definition 2.1 [(ii), (iii), (iv), (vi), (ix), (x)], in this section, we will define the new type of sets: strong $\beta^* - I-$ open, strong $\beta^* - I-$ closed and deduce their characteristics and relationships with other sets.

Definition 3.1. Given a space (X, τ, I) and $A \subset X$, A is said to be a strong β^*-I- open set (briefly $S.\beta^*-I-$ open) if $A \subset cl^*(int^*(cl^*(A)))$. We denote that all $S.\beta^*-I-$ open sets by $S.\beta^*IO(X)$ that is $S.\beta^*IO(X) = \{A \subset X : A \subset cl^*(int^*(cl^*(A)))\}$.

Definition 3.2. A set $F \subset (X, \tau, I)$ is called strong β^*-I- closed (briefly $S.\beta^*-I-$ closed) if its complement is an $S.\beta^* - I-$ open set. We denote that all $S.\beta^* - I-$ closed by $S.\beta^*IC(X)$.

The following diagram holds for any subset A of a space (X, τ, I).

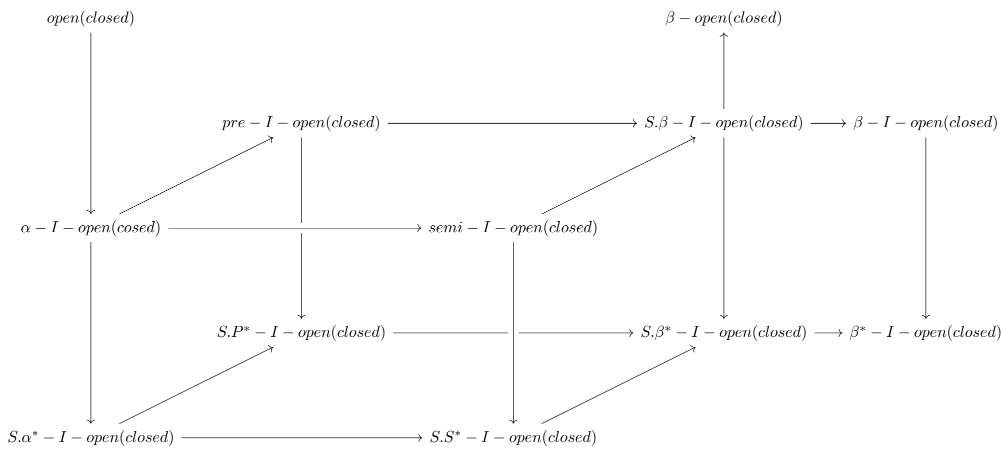

Figure 1.

Remark 3.1. The reverse of the implication in Figure 1 is not true in general as shown in the following examples.

Example 3.1. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then

(i) $A = \{b\} \in S.\beta^*IO(X)$, but A is not $S.\alpha^*-I-$ open, A is not $S.S^*-I-$ open and $A \notin \tau$.

(ii) $A = \{a, d\}$ is $\beta^* - I-$ open while $A \notin S.\beta^*IO(X)$.

Example 3.2. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}\}$. Then $A = \{b\} \in S.\beta^*IO(X)$, but it is not $S.\beta - I - open$.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\phi, \{b\}\}$. Then $A = \{b, c\} \in S.\beta^*IO(X)$ while A is not $S.P^* - I-$ open.

Remark 3.2. It clear that, $S.\beta^*-I-open$ sets and $\beta-I-open$ sets are independent concepts (Example 3.4).

Example 3.4. From Example 3.2 we get

(i) $A = \{a, c\}$ is $\beta - I-$ open while $A \notin S.\beta^*IO(X)$.

DOI https://doi.org/10.12150/jnma.2025.229 | Generated on 2025-04-09 07:30:37

(ii) $A = \{b, c\} \in S.\beta^*IO(X)$ while A is not $\beta - I-$ open.

Theorem 3.1. *Let (X, τ, I) be an ideal topological space. Then $A \in S.\beta^*IO(X)$ if and only if $cl^*(A) = cl^*(int^*(cl^*(A)))$.*

Proof. Letting $A \in S.\beta^*IO(X)$, then $A \subset cl^*(int^*(cl^*(A)))$. We obtain

$$cl^*(A) \subset cl^*(cl^*(int^*(cl^*(A)))) = cl^*(int^*(cl^*(A))).$$

Also, it is obvious that $cl^*(int^*(cl^*(A))) \subset cl^*(A)$. Thus $cl^* = cl^*(int^*(cl^*(A)))$.

Conversely, let $cl^*(A) = cl^*(int^*(cl^*(A)))$. Since (cl^*) is a closure oprator and $A \subset cl^*(A)$, $\forall A \subset X$, by using hypothesis we have $A \subset cl^*(int^*(cl^*(A)))$. Hence $A \in S.\beta^*IO(X)$. □

Remark 3.3. $S.\beta^* - I-$ closed sets and $\beta-$ closed sets are independent notions. We show that with the following examples.

Example 3.5. From Example 3.1 if $A = \{b, c\}$, then A is a $\beta-$ closed set but $A \notin S.\beta^*IC(X)$.

Example 3.6. From Example 3.2 if $A = \{a\}$, then A is not $\beta-$ closed set while $A \in S.\beta^*IC(X)$.

Theorem 3.2. *Let (X, τ, I) be an ideal topological space and $A, B \subset X$. Then $B \in S.\beta^*IO(X)$ if there exists $A \in S.\beta^*IO(X)$ such that $A \subset B \subset cl^*(A)$.*

Proof. It is obtained that $cl^*(A) = cl^*(B)$ by taking the $*-$ closure of $A \subset B \subset cl^*(A)$. Suppose that $A \in S.\beta^*IO(X)$. Then $A \subset cl^*(int^*(cl^*(A)))$ and we obtain that

$$B \subset cl^*(A) \subset cl^*(cl^*(int^*(cl^*(A)))) = cl^*(int^*(cl^*(A))) = cl^*(int^*(cl^*(B))).$$

Therefore $B \subset cl^*(int^*(cl^*(B)))$ and this shows that $B \in S.\beta^*IO(X)$. □

Corollary 3.1. *Let $B \subset (X, \tau, I)$. Then $B \in S.\beta^*IO(X)$ if there exists a $*-$ open set A such that $A \subset B \subset cl^*(A)$.*

Proof. This follows directly from Theorem 3.2. □

Theorem 3.3. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then the following properties are equivalent:*

(i) $A \in S.\beta^*IO(X)$;

(ii) *There exists $S.P^* - I-$ open B in X such that $B \subset cl^*(A) \subset cl^*(B)$;*

(iii) $cl^*(A) = cl^*(int^*(cl^*(A)))$.

Proof. $(i) \Rightarrow (ii)$ Let $A \in S.\beta^*IO(X)$. Then $A \subset cl^*(int^*(cl^*(A)))$. We set $B = int^*(cl^*(A))$, which is $*-$ open, then we have $B \subset int^*(cl^*(B))$. Therefore, we obtain B is $S.P^* - I-$ open and $B \subset cl^*(A) \subset cl^*(B)$.

$(ii) \Rightarrow (iii)$ Assume that there exists $S.P^* - I-$ open B in X such that $B \subset cl^*(A) \subset cl^*(B)$. Then $B \subset int^*(cl^*(B))$. This implies that

$$\begin{aligned} cl^*(B) &\subset cl^*(int^*(cl^*(B))) \\ &= cl^*(int^*(cl^*(cl^*(A)))) \\ &= cl^*(int^*(cl^*(A))) \\ &\subset cl^*(A) \\ &\subset cl^*(B). \end{aligned}$$

DOI https://doi.org/10.12150/jnma.2025.229 | Generated on 2025-04-09 07:30:37

Therefore, we obtain $cl^*(B) = cl^*(int^*(cl^*(A)))$.

On the other hand, since $B \subset cl^*(A) \subset cl^*(B)$, $cl^*(A) = cl^*(int^*(cl^*(A)))$.
$(iii) \Rightarrow (i)$ This proof is given in Theorem 3.1. □

Theorem 3.4. *Let $A \subset (X, \tau, I)$. Then $A \in S.\beta^*IC(X)$ if and only if*
$$int^*(cl^*(int^*(A))) \subset A.$$

Proof. Let $A \in S.\beta^*IC(X)$. Then $X - A \in S.\beta^*IO(X)$ and hence
$$X - A \subset cl^*(int^*(cl^*(X - A))) = X - int^*(cl^*(int^*(A))).$$
Therefore, $int^*(cl^*(int^*(A)) \subset A$.

Conversely, let $int^*(cl^*(int^*(A))) \subset A$. Then $X - A \subset cl^*(int^*(cl^*(X - A)))$ and hence $X - A \in S.\beta^*IO(X)$. Therefore, $A \in S.\beta^*IC(X)$. □

Theorem 3.5. *Let (X, τ, I) be an ideal topological space and I be codense. Then $A \in S.\beta^*IC(X)$ if and only if $int^*(cl(int^*(A))) \subset A$.*

Proof. Let $A \in S.\beta^*IC(X)$. Then $A \supset int^*(cl^*(int^*(A))) = int^*(cl(int^*(A)))$. Conversely, let $A \subset X$ such that $A \supset int^*(cl(int^*(A))) \supset int^*(cl^*(int^*(A)))$. Hence $A \in S.\beta^*IC(X)$. □

Theorem 3.6. *Let (X, τ, I) be an ideal topological space. Then $B \in S.\beta^*IC(X)$ if there exists $A \in S.\beta^*IC(X)$ such that $int^*(A) \subset B \subset A$.*

Proof. If $A \in S.\beta^*IC(X)$ such that $int^*(A) \subset B \subset A$, then $A \supset int^*(cl^*(int^*(A)))$ and $int^*(A) = int^*(B)$. Hence,
$B \supset int^*(A) \supset int^*(int^*(cl^*(int^*(A)))) = int^*(cl^*(int^*(A))) = int^*(cl^*(int^*(B)))$,
which shows that $B \in S.\beta^*IC(X)$. □

Corollary 3.2. *Let $B \subset (X, \tau, I)$. Then $B \in S.\beta^*IC(X)$ if there exists a $*-$ closed set A such that $int^*(A) \subset B \subset A$.*

Proof. The proof can be obtained from Theorem 3.6. □

Corollary 3.3. *Let (X, τ, I) be an ideal topological space.Then $cl^*(A) \in S.\beta^*IO(X)$ if $A \in S.\beta^*IO(X)$.*

Proof. Let $A \in S.\beta^*IO(X)$. Then by Theorem 3.1
$$cl^*(A) = cl^*(int^*(cl^*(A))).$$
Therefore, $cl^*(A) \subset cl^*(int^*(cl^*(cl^*(A))))$.

This implies $cl^*(A) \in S.\beta^*IO(X)$. □

Theorem 3.7. *Let (X, τ, I) be an ideal topological space. If $A \cup (X - A^*) \in S.\beta^*IC(X)$, then $A^* - A \in S.\beta^*IO(X)$.*

Proof. Suppose $A \cup (X - A^*) \in S.\beta^*IC(X)$. Since $X - (A^* - A) = A \cup (X - A^*)$, then $A^* - A \in S.\beta^*IO(X)$. □

Corollary 3.4. *Let (X, τ, I) be an ideal topological space. Then $A \in S.\beta^*IO(X)$ if A is an $I-$ open set .*

Proof. Let A be an $I-$ open set, then
$A \subset int(A^*) \subset int(A^* \cup A) = int(cl^*(A)) \subset int^*(cl^*(A)) \subset cl^*(int^*(cl^*(A)))$.
Hence $A \in S.\beta^*IO(X)$. □

Theorem 3.8. *Let (X, τ, I) be an ideal topological space where I is codense and A is $*-$ dense in itself. Then $A \in S.\beta^*IO(X)$ if and only if A is an almost strong $-I-$ open set.*

DOI https://doi.org/10.12150/jnma.2025.229 | Generated on 2025-04-09 07:30:37

Proof. Let $A \in S.\beta^*IO(X)$ and $*-$ dense in itself. Then
$$A \subset cl^*(int^*(cl^*(A))) = cl^*(int(cl^*(A)) = cl^*(int(A^*)).$$
This implies that A is almost strong $-I-$ open.

Conversely, let A be an almost strong $-I-$ open set. Then
$$A \subset cl^*(int(A^*)) = cl^*(int(cl^*(A)) \subset cl^*(int^*(cl^*(A))).$$
This implies that $A \in S.\beta^*IO(X)$. □

Theorem 3.9. *Let (X, τ, I) be an ideal topological space. Then if A is $b-I-$ open in X, then $A \in S.\beta^*IO(X)$.*

Proof. Let A be $b-I-$ open in X. Then
$$A \subset cl^*(int(A)) \cup int(cl^*(A)) \subset cl^*(int(cl^*(A))) \subset cl^*(int^*(cl^*(A))).$$
Thus, $A \in S.\beta^*IO(X)$. □

Theorem 3.10. *Let (X, τ, I) be a space, $A, B \subset X$. Then*

(i) If $U_\gamma \in S\beta^*IO(X)$ (resp. $U_\gamma \in S\beta^*IC(X)$), $\forall \gamma \in \Delta$, then $\bigcup\{U_\gamma : \gamma \in \Delta\}$ $\in S\beta^*IO(X)$ (resp. $\bigcap\{U_\gamma : \gamma \in \Delta\} \in S\beta^*IC(X)$).

(ii) If $A \in S\beta^*IO(X)$ (resp. $A \in S\beta^*IC(X)$) and $B \in \tau$ (resp. B is closed), then $A \cap B \in S\beta^*IO(X)$ (resp. $A \cup B \in S\beta^*IC(X)$).

(iii) If $A \in S.\beta^*IC(X)$ and B is a $t-I-$ set, then $A \cap B$ is a $\beta^* - I-$ closed set.

Proof.

(i) We only need to prove the case of $U_\gamma \in S\beta^*IO(X)$.
Since $U_\gamma \in S\beta^*IO(X)$, we have $U_\gamma \subset cl^*(int^*(cl^*(U_\gamma)))$, for each $\gamma \in \Delta$. Then we obtain
$$\begin{aligned}\textstyle\bigcup_{\gamma\in\Delta} U_\gamma &\subset \textstyle\bigcup_{\gamma\in\Delta} cl^*(int^*(cl^*(U_\gamma)))\\ &\subset cl^*(\textstyle\bigcup_{\gamma\in\Delta} int^*(cl^*(U_\gamma)))\\ &\subset cl^*(int^*(\textstyle\bigcup_{\gamma\in\Delta} cl^*(U_\gamma))\\ &\subset cl^*(int^*(cl^*(\textstyle\bigcup_{\gamma\in\Delta} U_\gamma))).\end{aligned}$$
Hence $\bigcup_{\gamma\in\Delta} U_\gamma \in S\beta^*IO(X)$. With res. $U \in S\beta^*IC(X))$ the proof is similar.

(ii) We only need to prove the case of $A \in S\beta^*IO(X)$.
Let $A \in S\beta^*IO(X)$ and $B \in \tau$. Then $A \subset cl^*(int^*(cl^*(A)))$. Thus, we obtain
$$\begin{aligned}A \cap B &\subset cl^*(int^*(cl^*(A))) \cap B\\ &\subset cl^*(int^*(cl^*(A)) \cap B)\\ &= cl^*(int^*(cl^*(A)) \cap int^*(B))\\ &= cl^*(int^*(cl^*(A) \cap B))\\ &\subset cl^*(int^*(cl^*(A \cap B))).\end{aligned}$$
Hence $A \cap B \in S.\beta^*IO(X)$. With res. $A \in S\beta^*IC(X)$ the proof is similar.

(iii) Let $A \in S.\beta^*IC(X)$ and B is a $t-I-$ set. Then $A \supset int^*(cl^*(int^*(A)))$ and

$int(B) = int(cl^*(B))$. Now

$$\begin{aligned} A \cap B &\supset int^*(cl^*(int^*(A))) \cap B \\ &\supset int^*(cl^*(int^*(A))) \cap int(B) \\ &= int^*(cl^*(int^*(A))) \cap int(cl^*(B)). \\ &\supset int(cl^*(int(A))) \cap int(cl^*(int(B))). \\ &= int(cl^*(int(A)) \cap cl^*(int(B))). \\ &\supset int(cl^*(int(A) \cap int(B))). \\ &= int(cl^*(int(A \cap B))). \end{aligned}$$

Hence $A \cap B$ is $\beta^* - I-$ closed.

□

Remark 3.4. The intersection of two $S.\beta^* - I-$ open sets need not be an $S.\beta^* - I - open$ set and the union of two $S.\beta^* - I-$ closed sets need not be an $S.\beta^* - I-$ closed set and as is illustrated by the following example.

Example 3.7. Letting $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{a, b, d\}\}$ and $I = \{\phi, \{a\}\}$, then

(i) $A = \{a, b, c\} \in S.\beta^*IO(X)$ and $B = \{a, c, d\} \in S.\beta^*IO(X)$, but $A \cap B = \{a, c\} \notin S.\beta^*IO(X)$.

(ii) $A = \{d\} \in S.\beta^*IC(X)$ and $B = \{b, c\} \in S.\beta^*IC(X)$, but $A \cup B = \{b, c, d\} \notin S.\beta^*IC(X)$.

Theorem 3.11. *Let (X, τ, I) be an ideal topological space, where I is codense and A is a $*-$ dense set. Then the following statements are equivalent:*

(i) $A \in S.\beta^*IO(X)$,

(ii) A is an almost strong $-I-$ open set,

(iii) A is a $\beta^* - I-$ open set.

Proof. $(i) \Rightarrow (ii)$ Letting $A \in S.\beta^*IO(X)$, then
$$A \subset cl^*(int^*(cl^*(A))) = cl^*(int(cl^*(A))) = cl^*(int(A^*)).$$
Hence A is almost strong $-I-$ open.
$(ii) \Rightarrow (iii)$ Letting A be an almost strong $-I-$ open set, then
$$A \subset cl^*(int(A^*)) = cl^*(int(cl(A))) \subset cl(int^*(cl(A))).$$
Hence A is $\beta^* - I-$ open.
$(iii) \Rightarrow (i)$ Letting A be $\beta^* - I-$ open, then $A \subset cl(int^*(cl(A))) = cl^*(int^*(cl^*(A)))$.
Hence $A \in S.\beta^*IO(X)$. □

Theorem 3.12. *Let (X, τ, I) be an ideal topological space, where I is codense. Then for any set $A \subset X$ the followings hold:*

(i) If A is a semi$^* - I-$ open, then $A \in S.\beta^*IO(X)$.

(ii) $A \in S.\beta^*IO(X)$ if it is regular $-$ closed.

(iii) If $A \in S.\beta^*IO(X)$, then it is $\beta-$ open.

DOI https://doi.org/10.12150/jnma.2025.229 | Generated on 2025-04-09 07:30:37

Proof.

(i) Let A is $semi^* - I-$ open. Then
$$A \subset cl(int^*(A)) = cl^*(int^*(A)) \subset cl^*(int^*(cl^*(A))).$$
Hence $A \in S.\beta^* IO(X)$.

(ii) Let A be a regular $-$ closed set. Then
$$A = cl(int(A)) \subset cl(int^*(cl^*(A))) = cl^*(int^*(cl^*(A))).$$
Hence $A \in S.\beta^* IO(X)$.

(iii) Let $A \in S.\beta^* IO(X)$. Then
$$A \subset cl^*(int^*(cl^*(A))) = cl^*(int(cl^*(A))) \subset cl(int(cl(A))).$$
Hence A is $\beta-$ open.

□

Remark 3.5. The converse of Theorem 3.12 is not true in general as shown in the following example.

Example 3.8. From Example 3.7 we obtain

(i) $A = \{b\} \in S.\beta^* IO(X)$ while it is not $semi^* - I-$ open.

(ii) $A = \{a, b, c\} \in S.\beta^* IO(X)$, however A is not regular $-$ closed.

(iii) $A = \{a, c\} \notin S.\beta^* IO(X)$, but A is $\beta-$ open.

Corollary 3.5. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then $A \in S.\beta^* IO(X)$ if A is a pre$-$ open and $t-$set .*

Proof. Let A be pre $-$ open. Then $A \subset int(cl(A))$. Since A is a $t-$ set, then $int(cl(A)) = int(A)$. Now $A \subset int(cl(A)) = int(A) \subset cl^*(int^*(cl^*(A)))$.
Hence $A \in S.\beta^* IO(X)$. □

Theorem 3.13. *Let (X, τ, I) be an ideal topological space where I is codense, $A \subset X$ be a $*-$ closed set and (X, τ, I) be an $I-$ extremally disconnected. Then A is an $S.\alpha^* - I-$ open set if $A \in S.\beta^* IO(X)$.*

Proof. Letting $A \in S.\beta^* IO(X)$. Then
$A \subset cl^*(int^*(cl^*(A))) = cl^*(int(A)) \subset int(cl^*(A)) = int(A) \subset int^*(cl^*(int^*(A)))$,
which shows A is $S.\alpha^* - I-$ open. □

Theorem 3.14. *Let (X, τ, I) be an ideal topological space and $A \subset X$ be a $\delta - I-$ set where I is codense. Then A is an $S.S^* - I-$ open set if $A \in S.\beta^* IO(X)$.*

Proof. Let $A \in S.\beta^* IO(X)$ and A is a $\delta - I-$ set. Then

$$\begin{aligned} A &\subset cl^*(int^*(cl^*(A))) \\ &= cl^*(int(cl^*(A))) \\ &\subset cl^*(cl^*(int(A))) \\ &= cl^*(int(A)) \\ &\subset cl^*(int^*(A)). \end{aligned}$$

Hence A is $S.S^* - I-$ open. □

DOI https://doi.org/10.12150/jnma.2025.229 | Generated on 2025-04-09 07:30:37

Theorem 3.15. *Let (X, τ, I) be an ideal topological space and $A \subset X$ be a $*-$ closed (resp. $*-$ open) set. Then A is an $S.S^* - I-$ open (resp. $S.S^* - I-$ closed) set if $A \in S.\beta^* IO(X)$ (resp. $A \in S.\beta^* IC(X)$).*

Proof. We only prove the case where A is $*-$ closed.
Let $A \in S.\beta^* IO(X)$ and $*-$ closed. Then $A \subset cl^*(int^*(cl^*(A))) = cl^*(int^*(A))$. Hence A is $S.S^* - I-$ open. With res. $*-$ open the proof is similar. □

Theorem 3.16. *Let (X, τ, I) be an ideal topological space and $A \subset X$. If A is an $I - R-$ closed set, then $A \in S.\beta^* IO(X)$.*

Proof. Let A be $I - R-$ closed. Then $A = cl^*(int(A)) \subset cl^*(int^*(cl^*(A)))$. Hence $A \in S.\beta^* IO(X)$. □

Remark 3.6. The converse of Theorem 3.16 is not true in general as shown in the following example.

Example 3.9. From Example 3.1 we obtain $A = \{a, b\} \in S.\beta^* IO(X)$, but it is not $I - R-$ closed.

Definition 3.3. The strong $\beta^* - I-$ interior of a subset A of a space (X, τ, I) is defined to be the union of all $S.\beta^* - I-$ open sets of X contained in A. It is denoted by $S.\beta^* Iint(A)$, *i.e.* $S.\beta^* Iint(A) = \{\cup B : B \subset A, B \in S.\beta^* IO(X)\}$.

Definition 3.4. The strong $\beta^* - I-$ closure of a subset A of a space (X, τ, I) is defined to be an intersection of all $S.\beta^* - I-$ closed sets of X containing A. It is denoted by $S.\beta^* ICl(A)$, *i.e.* $S.\beta^* Icl(A) = \{\cap B : B \supset A, B \in S.\beta^* IC(X)\}$.

Lemma 3.1. *Let $A \subset (X, \tau, I)$. Then*

(i) $X - S.\beta^* Iint(A) = S.\beta^* Icl(X - A)$,

(ii) $X - S.\beta^* Icl(A) = S.\beta^* Iint(X - A)$.

Proof.

(i) Since $S.\beta^* Iint(A) = \{\cup B : B \subset A, B \in S.\beta^* IO(X)\}$, then

$$\begin{aligned} X - S.\beta^* Iint(A) &= X - \{\cup B : B \subset A, B \in S.\beta^* IO(X)\} \\ &= \{\cap (X - B) : X - B \supset X - A, X - B \in S.\beta^* IC(X)\} \\ &= \{\cap F : F \supset X - A, F \in S.\beta^* IC(X)\} \\ &= S.\beta^* Icl(X - A). \end{aligned}$$

(ii) Since $S.\beta^* Icl(A) = \{\cap B : B \supset A, B \in S.\beta^* IC(X)\}$, then

$$\begin{aligned} X - S.\beta^* Icl(A) &= X - \{\cap B : B \supset A, B \in S.\beta^* IC(X)\} \\ &= \{\cup (X - B) : X - B \subset X - A, X - B \in S.\beta^* IO(X)\} \\ &= \{\cup F : F \subset X - A, F \in S.\beta^* IO(X)\} \\ &= S.\beta^* Iint(X - A). \end{aligned}$$

□

4. Conclusion

In this paper, we introduce the notion of strong $\beta^* - I-$ open sets in ideal topological spaces. We demonstrate that the concept of strong $\beta^* - I-$ open sets is weaker that of open sets in ideal topological spaces. We discuss and prove several properties and relationships of strong $\beta^* - I-$ open sets and strong $\beta^* - I-$ closed sets. Furthermore, the concept of continuity can be studied in the light of the newly defined generalized open sets.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *$\beta-$ open sets and $\beta-$ continuous mappings*, Bull. Fac. Sci. Assiut. Univ., 1983, 12, 77–90.

[2] A.Açıkgöz, T. Noiri and S. Yüksel, *On $\delta - I -$ open sets and decomposition of $\alpha-$continuity*, Acta Math. hungar., 2004, 102, 349–357.

[3] A. Açıkgöz and S. Yüksel, *Some new sets and decomposition of $A_{I-R}-$ continuity, $\alpha - I-$continuity, continuity via idealization*, Acta. Math. Hungar., 2007, 144(1-2), 79–89.

[4] R. M. Aqeel, F. A. Ahmed and R. Gubran, *Between $\alpha - I-$ Open Sets and $S.P^* - I-$ Open Sets*, Acta Univ. Apulensis Math. Inform., 2023, 74(8), 117–136.

[5] R. M. Aqeel and A. A. Bin Kuddah, *On strong semi*-I-open sets in ideal topological spaces*, Univ.Aden J.Nat. Appl. Sc., 2019, 23(2).

[6] R. M. Aqeel and A. A. Bin Kuddah, *On strong pre*-I-open sets in ideal topological spaces*, Journal of New Theory, 2019, 28, 44–52.

[7] B. Chawalit, *Upper and lower $\beta(*)-$ continuity*, Heliyon, 2021, e05986, 7, 1–10.

[8] V. R. Devi, D. Sivaraj and T. T. Chelvam, *Properties of some $*-$dense in itself subsets*, Internat. J. Math. Sci., 2004, 72, 3989–3999.

[9] J. Dontchev, M. Ganster and D. Rose, *Ideal resolvability*, Topology Appl., 1999, 93(1), 1–16.

[10] E. Ekici, *On AC_I-sets, BC_I-sets, $\beta^*_I -$open sets and decompositions of continuity in ideal topological spaces*, Creat. Math. Inform., 2011, 20(1), 47–54.

[11] E. Ekici and T. Noiri, *$*$-extremally disconnected ideal topological spaces*, Acta Math. Hungar., 2009, 122, 81–90.

[12] E. Ekici and T.Noiri, *$*$-hyperconnected ideal topological spaces*, Analelestiin Ale Univ.A.I. Cuza Din Iasi-SerieNoua-Math., 2012, 122(1), 121–129.

[13] T. C. Glaisa, P. B. Jr Michael and N. P. Roberto *$\beta I-$ compactness, βI^*-heperconnectedness and $\beta I-$ Separatedness in ideal topological spaces* , DOI: 10.5772/intechopen.101524, 2022.

[14] A. C. Guler and G. Aslim, *b-I-open sets and decomposition of continuity via idealization*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb., 2005, 22, 27–32.

[15] E. Hatir and S. Jafari, *On weakly semi-I-open sets and another decomposition continuity via ideals*, Sarajevo J. Math., 2006, 2(14), 107–114.

[16] E. Hatir, A. Keskin and T. Noiri, *On a new decomposition of continuity via idealization*, JP J. Geometry Topology, 2003, 3(1), 53–64.

[17] E. Hatir, A. Keskin and T. Noiri, *A note on strong* $\beta - I -$ *open sets and strongly* $\beta - I -$ *continuous functions*, Acta Math. Hungar., 2005, 108(1-2), 87–94.

[18] E. Hatir and T. Noiri, *On decompositions of continuity via idealization*, Acta Math. Hungar., 2002, 96(4), 341–349.

[19] E. Hayashi, *Topologies defined by local properties*, Math. Ann., 1964, 156, 205–215.

[20] D. Janković and T.R.Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 1990, 97, 295–310.

[21] T.R.Hamlett and D. Janković, *Ideals in topological spaces and the set operator* Ψ, Boll. Un. Mat. Ital., 1990, 7, 863–874.

[22] D. Janković and T.R. Hamlett, *Compatible extensions of ideals* , Boll. U. M. I., 1991.

[23] K. Kuratowski, *Topologies I*, Warszawa, 1933.

[24] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 1963, 70(1), 36–41.

[25] A.S. Mashhour, M. E. Abd El-monsef and S. N. El-deeb, *On pre continuous and weak pre continuous mappings*, Proc. Math. Phys soc. Egypt, 1982, 53, 47–53.

[26] A. A. Nasef, *Ideals in general topology*, Ph. D. Faculty of Science, Tanta University, Egypt, 1992.

[27] P. L. Powar, T. Noiri and S. Bhadauria, *On* $\beta-$ *local function in ideal topological spaces*, European Journal of Pure and Applied Mathematics, 2020, 13(4), 758–765.

[28] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. A. M. S., 1937, 41(3), 375–481.

[29] J. Tong, *On decomposition of continuity in topological spaces*, Acta. Math. hungar., 1989, 54, 51–55.

[30] V. Vaidyanathaswamy, *The localisation theory in set topology* , Proc. Indian Acad. Sci. Math. Sci., 1945, 20, 51–61.

[31] S. Yüksel, A. Açıkgöz and E. Gursel, *On* $\beta - I -$ *Regular sets* , Far East J.Math. Sci.(FJMS), 2007, 25(2), 353–366.