

Singular Integral and Local Hardy Spaces in Dunkl Setting

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Received 22 October 2024; Accepted (in revised version) 19 July 2025

Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

Abstract. The objective of this paper is to establish the local Hardy space in the Dunkl setting, which pertains to the geometric framework defined by both the Euclidean metric and the Dunkl metric, the latter being influenced by finite reflection groups. This study leverages the weak local wavelet decomposition in L^2 space and the theory of nonhomogeneous singular integral operators as pivotal components.

Key Words: Nonhomogeneous Dunkl-Calderón-Zygmund singular integral, local Dunkl-Hardy space, weak local wavelet decomposition.

AMS Subject Classifications: 42B35, 43A85, 42B25, 42B30

1 Introduction

1.1 Preliminaries in the Dunkl setting

It is widely acknowledged that geometric structures play a pivotal role in harmonic analysis. In the context of Dunkl theory, this specific group structure guides the analytical process. Within the Dunkl framework, there exist corresponding Dunkl transforms, translation operators, convolution operators, and the Dunkl Laplacian. Additionally, two distinct metrics are at play: the Euclidean metric and the Dunkl metric, which is derived from finite reflection groups. Consequently, the Dunkl framework does not fit into the category of spaces of homogeneous type as defined by Coifman and Weiss [11]. We now provide background information on the Dunkl setting.

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The classical Fourier transform, initially defined on $L^1(\mathbb{R}^N)$, extends to an isometry of $L^2(\mathbb{R}^N)$ and exhibits certain properties with translation, dilation, and rotation groups. In [19], Dunkl introduced a similar transform, known as the Dunkl transform, which shares properties analogous to those of the classical Fourier transform. This transform is defined by

$$\hat{f}(x) = c_h \int_{\mathbb{R}^N} E(x, -iy) f(y) h_\kappa^2(y) dy,$$

where the conventional character $e^{-i\langle x,y \rangle}$ is supplanted by $E(x, -iy) = V_\kappa(e^{-i\langle \cdot, y \rangle})(x)$ for some positive linear operator V_κ and the weight functions h_κ remain invariant under a finite reflection group G on \mathbb{R}^N . Notably, the Dunkl transform adheres to the Plancherel identity, specifically,

$$\|\hat{f}\|_2 = \|f\|_2.$$

Should the parameter $\kappa = 0$, then $h_\kappa(x) = 1$ and $V_\kappa = id$, rendering the Dunkl transform equivalent to the classical Fourier transform.

Although the classical Fourier transform interacts seamlessly with the translation operator, the measure $h_\kappa^2(x) dx$ no longer remains invariant under standard translations. In [44], the translation operator associated with the Dunkl transform is consequently defined on the Dunkl transform side as

$$\widehat{\tau_y f}(x) = E(y, -ix) \hat{f}(x)$$

for all $x \in \mathbb{R}^N$.

When the function f belongs to the Schwartz class, the above equality holds pointwise. As an operator on $L^2(\mathbb{R}^N, h_\kappa^2)$, τ_y is bounded. However, it is not at all clear whether the translation operator can be defined for L^p functions with $p \neq 2$. Even the L^p boundedness of τ_y on the dense subspace of Schwartz functions for $p \neq 2$ remains an open question. So far, an explicit formula for τ_y is known only in specific cases: when f is a radial function or when $G = \mathbb{Z}_2^N$. For precision, it was proven in [38] that if f is a radial Schwartz function and $f(x) = f_0(\|x\|)$, then

$$\tau_y f(x) = V_\kappa \left[f_0 \left((\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\langle x', \cdot \rangle)^{\frac{1}{2}} \right) \right] (y'),$$

where $x' = \frac{x}{\|x\|}$ for non-zero $x \in \mathbb{R}^N$. See [20,21] for related works.

For $f, g \in L^2(\mathbb{R}^N, h_\kappa^2)$, their convolution can be defined in terms of the translation operator by

$$f *_\kappa g(x) = \int_{\mathbb{R}^N} f(y) \tau_x g^\vee(y) h_\kappa^2(y) dy,$$

where $g^\vee(y) = g(-y)$. See [44,45] for more details.

In [17], Dunkl also introduced so-called Dunkl operators, which are a family of first-order differential-difference operators playing a role similar to the usual partial differentiation for reflection groups. Specifically, denote the standard inner product in