

Matrix Weights, Maximal Operators, Calderón–Zygmund Operators, and Besov–Triebel–Lizorkin–Type Spaces — A Survey

Fan Bu, Dachun Yang*, Wen Yuan and Yuze Zhao

*Laboratory of Mathematics and Complex Systems (Ministry of Education of China),
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China*

Received 16 May 2025; Accepted (in revised version) 24 June 2025

Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

Abstract. The primary purpose of this survey is threefold. First, the authors recall some histories and present some recent developments of matrix weights, in which the authors not only improve some known results on the intrinsic properties of matrix weights, but also establish some new ones. Then the authors summarize matrix-weighted inequalities associated with various operators, such as Hardy–Littlewood-type maximal operators and Calderón–Zygmund operators. Finally, the authors overview matrix-weighted function spaces, including matrix-weighted Sobolev, BMO, and Besov–Triebel–Lizorkin-type spaces. Several open questions on these subjects are also presented.

Key Words: Sobolev space, BMO, Besov–Triebel–Lizorkin-type space, matrix weight, Hardy–Littlewood maximal operator, Calderón–Zygmund operator.

AMS Subject Classifications: 46E30, 46E35, 47A56, 42B20, 42B25, 42B35, 46E40, 47A56

1 Introduction

Throughout the whole article, we mainly work in \mathbb{R}^n and, unless necessary, we will not explicitly specify this underlying space.

The study of weighted norm inequalities in harmonic analysis began with the work of Muckenhoupt [113], who introduced the scalar A_p weights as a necessary and sufficient condition for the boundedness of the Hardy–Littlewood maximal operator on weighted Lebesgue spaces. This foundational result was soon extended by Hunt et al. [78] to the

*Corresponding author. *Email addresses:* fanbu@mail.bnu.edu.cn (F. Bu), dcyang@bnu.edu.cn (D. Yang), wenyuan@bnu.edu.cn (W. Yuan), yuzezhao@mail.bnu.edu.cn (Y. Zhao)

Hilbert transform. Coifman and Fefferman [40] later simplified Muckenhoupt's arguments and introduced the concept of A_∞ weights, along with reverse Hölder's inequality. Several equivalent characterizations of the A_∞ condition have since been established; see [57, 69, 96, 114, 135].

Over the past three decades, increasing attention has been devoted to quantitative weighted inequalities. Buckley [31] established the sharp bound for the Hardy–Littlewood maximal operator and provided a quantitative weighted inequality for Calderón–Zygmund operators. However, the sharp bound for the latter remained an open problem at that time. This problem, known as the A_2 conjecture, was completely solved by Hytönen [80], and one of its important applications can be found in [4]. A simpler proof of the A_2 conjecture was later given by Lerner [101]. Several other quantitative weighted inequalities have been studied; see [83, 98, 99, 104, 105].

To study the prediction theory of multivariate stochastic processes, Wiener and Masani [163, Section 4] introduced the matrix-weighted Lebesgue space $\mathcal{L}^2(W)$ over $[0, 2\pi]$. For any $p \in (0, \infty)$ and any matrix weight W on \mathbb{R}^n , the *matrix-weighted Lebesgue space* $\mathcal{L}^p(W)$ on \mathbb{R}^n is defined to be the set of all measurable vector-valued functions $\vec{f}: \mathbb{R}^n \rightarrow \mathbb{C}^m$ such that

$$\|\vec{f}\|_{\mathcal{L}^p(W)} := \left[\int_{\mathbb{R}^n} |W^{\frac{1}{p}}(x)\vec{f}(x)|^p dx \right]^{\frac{1}{p}} < \infty.$$

Motivated by problems arising from both the multivariate random stationary process and the Toeplitz operator, Treil and Volberg [149] introduced matrix \mathcal{A}_2 weights on \mathbb{R} and proved that

$$\text{the Hilbert transform is bounded on } \mathcal{L}^2(W) \text{ over } \mathbb{R} \text{ if and only if } W \in \mathcal{A}_2. \quad (1.1)$$

Subsequently, Nazarov and Treil [119] and Volberg [156] independently introduced matrix \mathcal{A}_p weights on \mathbb{R} and extended (1.1) to the full range $p \in (1, \infty)$ via different methods. The underlying space in both works is still \mathbb{R} . Since then, the study of $\mathcal{L}^p(W)$ on \mathbb{R}^n has attracted increasing attention. For further studies on the properties of matrix weights, see [18, 100, 156, 157].

The matrix analogue of the A_∞ condition has been relatively scarce, with notable references including [29, 52, 87, 88, 156]. A central difficulty in this area comes from the lack of a universally accepted definition that adequately characterizes this class in the matrix setting. A prominent advancement in this direction is the introduction of $\mathcal{A}_{p,\infty}$ -matrix weights on \mathbb{R} by Volberg [156], which can be interpreted as a refined matrix analogue of the scalar A_∞ weights. A comprehensive investigation of $\mathcal{A}_{p,\infty}$ -matrix weights on \mathbb{R}^n was carried out by Bu et al. [29], where they also introduced dimensions of weights and presented a detailed analysis of associated properties, such as equivalent characterizations, the reverse Hölder's inequality, and the self-improvement property.

The study of matrix-weighted inequalities is significantly more challenging than that in the scalar setting. The matrix-weighted maximal operator was introduced by