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Weak Boundedness for Commutators of n-Dimensional Hardy-Type Operators on Mixed Herz Spaces

Lei Ji[1,2], Mingquan Wei[3,*] and Dunyan Yan[2]

[1] *School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE, Beijing Normal University, Beijing 100875, China*
[2] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*
[3] *School of Mathematics and Statistics, Xinyang Normal University, Xinyang, Henan 464000, China*

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Dedicated to the memory of Prof. Donggao Deng on the occasion of his 90th birthday

Abstract. In this paper, we obtain the boundedness for the commutators of n-dimensional Hardy-type operators from mixed Herz spaces to weak mixed Herz spaces, where the symbol functions are in the intersection of the mixed central bounded mean oscillation space $\mathrm{CBMO}_{\vec{q}'}(\mathbb{R}^n)$ and the weak mixed central bounded mean oscillation space $\mathrm{W}_{\vec{q}}(\mathbb{R}^n)$.

Key Words: Weak mixed central bounded mean oscillation space, n-dimensional Hardy-type operators, commutator, weak mixed Herz space.

AMS Subject Classifications: 42B20, 47B47, 42B35

1 Introduction

Let f be a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hardy operator H was defined by Christ and Grafakos [4]:

$$Hf(x)=\frac{1}{|x|^n}\int_{|t|<|x|}f(t)dt,\quad x\in\mathbb{R}^n\backslash\{0\}.$$

The adjoint operator H^* of H, is given by

$$H^*f(x)=\int_{|t|\geq|x|}\frac{f(t)}{|t|^n}dt,\quad x\in\mathbb{R}^n\backslash\{0\}.$$

*Corresponding author. *Email addresses:* `jilei21@mails.ucas.ac.cn` (L. Ji), `weimingquan11@mails.ucas.ac.cn` (M. Wei), `ydunyan@ucas.ac.cn` (D. Yan)

By definition, H and H^* satisfy

$$\int_{\mathbb{R}^n} g(x)Hf(x)dx = \int_{\mathbb{R}^n} f(x)H^*g(x)dx$$

for a suitable function g. The boundedness and sharp estimates of H and H^* on $L^p(\mathbb{R}^n)$ were established in [4]. In recent years, the study of Hardy-type operators has attracted much attention. See, for instance, [1,7,8,20,24,27,29,33].

Let b be a locally integrable function on $\mathbb{R}^n$. The commutators of H and H^* with the function b are defined by

$$H_b f = b(Hf) - H(fb)$$

and

$$H_b^* f = b(H^*f) - H^*(fb).$$

Commonly, the commutators of classical singular integral operators and other important operators in harmonic analysis are used to characterize the bounded mean oscillation space. However, H_b and H_b^* are usually used to study central function spaces, such as the space $\mathrm{CBMO}_q(\mathbb{R}^n)$, since H and H^* are centrosymmetric. Here, $\mathrm{CBMO}_q(\mathbb{R}^n)$ represents the central bounded mean oscillation space introduced by Lu and Yang [21], which is the collection of all locally integrable functions f with the finite norm

$$\|f\|_{\mathrm{CBMO}_q} = \sup_{r>0} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx \right)^{\frac{1}{q}},$$

where $1 \le q < \infty$, and $B(0,r)$ denotes the ball centered at the origin with radius r and $f_{B(0,r)} = \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x)dx$. The space $\mathrm{CBMO}_q(\mathbb{R}^n)$ can be regarded as a local version of $\mathrm{BMO}(\mathbb{R}^n)$ at the origin. Here, $\mathrm{BMO}(\mathbb{R}^n)$ denotes the bounded mean oscillation space introduced by John and Nirenberg [14], which can be defined similarly to $\mathrm{CBMO}_q(\mathbb{R}^n)$, except that we take the supremum over all balls in $\mathbb{R}^n$ instead of balls centered at the origin. Different from $\mathrm{BMO}(\mathbb{R}^n)$, the space $\mathrm{CBMO}_q(\mathbb{R}^n)$ depends on q since the absence of the famous John–Nirenberg inequality.

During the past two decades, the boundedness of H_b and H_b^* has been intensively studied, see e.g., [15,18]. Fu et al. [6, Theorem 2.1] proved that H_b and H_b^* are bounded on $L^q(\mathbb{R}^n)$, if and only if $b \in \mathrm{CBMO}_{\max(q,q')}(\mathbb{R}^n)$, where $1 < q < \infty$ and $1/q + 1/q' = 1$. In [6, Theorem 2.2], the authors also obtained the boundedness of H_b and H_b^* on homogenous Herz spaces, which also gives a characterization of central bounded mean oscillation spaces. Subsequently, many researchers studied the boundedness of H_b and H_b^* on different function spaces, such as central Morrey spaces [5,28]. In addition, much attention has been paid to sufficient conditions for the boundedness of H_b and H_b^* when the symbol b belongs to more general function spaces, such as λ-central bounded mean oscillation spaces [35] and central Campanato spaces [9]. We also refer the readers to [23,34] for more studies on commutators of Hardy-type operators.