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Results About Parabolic-Like Mappings

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Abstract. In this paper we present the most important definitions and results of the theory of parabolic-like mappings, and we will give an example. The proofs of the results can be found in [2,4] and [3].

Key Words: Polynomials, rational maps, entire and meromorphic functions, renormalization, Holomorphic families of dynamical systems, the Mandelbrot set, bifurcations.

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1 Introduction

Complex Dynamics is concerned with the study of iteration of holomorphic maps on a Riemann surface. Let $z\in\widehat{\mathbb{C}}$, and let f be a holomorphic map on $\widehat{\mathbb{C}}$, the *orbit* of z under f is the sequence $\{z,f(z),f^2(z),\cdots,\}$ (where f^n means f composed to itself n-times). The main activity in Holomorphic dynamics is the study of the asymptotic behaviour of such orbits and the resulting classification of points in $\widehat{\mathbb{C}}$. The *Fatou set* is the set of points z such that the family (f^n) is equicontinuous near z; the dynamics is chaotic on the complementary *Julia set* (see [5]). An important special case is given by polynomial maps of $\widehat{\mathbb{C}}$. In the polynomial case the Julia set of a map f is the boundary of the basin of the superattracting fixed point at infinity. In this situation is useful to define the *filled Julia set* to be the complement of the basin of attraction of infinity. A classical theorem by Fatou in 1918 asserts that the filled Julia set is connected if and only if it contains all the finite critical points (those points where the derivative vanishes) of f (see [5]). This result motivates the consideration of the *connectedness locus* within a given family of maps, for example the family of polynomial of some degree d. The Mandelbrot set M is the connectedness locus of the quadratic family $z\mapsto z^2+c$.

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In 1985, Adrien Douady and John Hamal Hubbard published a groundbreaking paper entitled *On the dynamics of polynomial-like mappings* (see [1]). A polynomial-like mapping is a proper holomorphic map $f:U' \to U$, where $U',U \approx \mathbb{D}$, and $\overline{U'} \subset U$. The filled Julia set is defined in the polynomial-like case in the same fashion as for polynomials: the set of points which do not escape the domain. A polynomial-like map of degree d is determined up to holomorphic conjugacy by its internal and external classes, that is, the (conjugacy classes of) the restrictions to the filled Julia set and its complement. In particular an external map is a degree d real-analytic orientation preserving and strictly expanding self-covering of the unit circle. A central result of this theory gives a verifiable sufficient condition for the connectedness locus of an analytic family of quadratic polynomial-like maps to be homeomorphic to M. This theory provides a language and firm foundation for the formulation and resolution of numerous problems concerning renormalization, for example, the celebrated Branner-Hubbard description of the locus of cubic polynomials with one escaping critical point.

It has long been clear, from both heuristic considerations and numerical experimentations, that many of the parameter space consequences of the Douady-Hubbard theory should have appropriate analogues in parabolic settings. To understand and study such families we extend in [2] and [3] the polynomial-like theory to a class of parabolic mappings which we called parabolic-like mappings. A parabolic-like mapping is an object similar to a polynomial-like mapping, but with an external map with a parabolic fixed point (see [2]). In this paper we will state the most important definitions and results of the theory of parabolic-like mappings, and we will give an example.

The paper is organized as follows: in Section 2 we will remember some facts about polynomials dynamics on the Riemann Sphere and polynomial-like theory. In Section 3 we will give an idea of what kind of maps we are interested in. In Sections 4 and 5 we will give the main definitions and results of the parabolic-like mappings theory, and in Section 6 we will give an example.

2 Polynomials and polynomial-like mappings

Let P be a polynomial on the Riemann Sphere. Then P has a superattracting fixed point at infinity. Let $A_\infty(P):=\{z|P^n(z)\to\infty \text{ as } n\to\infty\}$ denotes the basin of attraction of infinity. Then $A_\infty(P)$ is a completely invariant Fatou component for the polynomial P. Define the filled Julia set as the complement of the basin of attraction of infinity: $K_P:=\widehat{\mathbb{C}}\setminus A_\infty(P)$. The Julia set for the polynomial P is the common boundary between the basin of attraction of infinity and the filled Julia set: $J_P=\partial K_P=\partial A_\infty(P)$.

2.1 Polynomials-like mappings

we said in the Introduction, the notion of polynomial-like mappings was introduced by Douady and Hubbard in the landmark paper *On the dynamics of Polynomial-like mappings*