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Optimal Recovery of Functions on the Sphere on a Sobolev Spaces with a Gaussian Measure in the Average Case Setting

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Abstract. In this paper, we study optimal recovery (reconstruction) of functions on the sphere in the average case setting. We obtain the asymptotic orders of average sampling numbers of a Sobolev space on the sphere with a Gaussian measure in the $L_q(\mathbb{S}^{d-1})$ metric for $1 \le q \le \infty$, and show that some worst-case asymptotically optimal algorithms are also asymptotically optimal in the average case setting in the $L_q(\mathbb{S}^{d-1})$ metric for $1 \le q \le \infty$.

Key Words: Optimal recovery on the sphere, average sampling numbers, optimal algorithm, Gaussian measure.

AMS Subject Classifications: 41A25, 41A35

1 Introduction

This paper is devoted to studying the optimal recovery (reconstruction) of functions on the sphere on a Sobolev space with a Gaussian measure in the average case setting. Let F be a Banach space of functions defined on D, G be a normed linear spaces with norm $\|\cdot\|_G$, and let γ be a centered Gaussian probability measure on F. We want to reconstruct functions f from F using finitely many arbitrary function values $f(x)$ for some $x \in D$. It is well known that, in the average case setting with the average being respect to a centered Gaussian measure, adaptive choice of the above function values as well as nonlinear algorithms do not essentially help, see [20, 24]. Hence, we can restrict our attention to linear algorithms, i.e., algorithms of the form

$$A_N(f) := \sum_{j=1}^{N} f(x_j) h_j, \quad h_j \in G, \quad x_j \in D. \tag{1.1}$$

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For $0<p<\infty$, the p-average error of an algorithm A_N in G with respect to the measure γ is defined by

$$e^{\rm avg}(A_N,\gamma,G)_p:=\Big(\int_F \|f-A_N(f)\|_G^p\gamma(df)\Big)^{\frac{1}{p}}.$$

We define the p-average sampling numbers of F in G by

$$g_N^{(a)}(F,\gamma,G)_p:=\inf_{x_j\in D,\ h_j\in G,\ j=1,\cdots,N} e^{\rm avg}(A_N,\gamma,G)_p.$$

We stress that for a centered Gaussian measure, the averaging parameter p is irrelevant up to a constant (see [11, Theorem 1.2] or [27, Corollary 1]).

There are a few papers devoted to studying average case approximation, see for example, [4, 5, 9, 10, 12–17, 20, 22–29]. However, much less attention has been devoted to average sampling numbers; for exceptions see, e.g., [13, 14, 23]. In [23] and [14], among others, the authors obtained upper bounds for average sampling numbers on the Wiener space in the uniform norm and on the weighted Korobov spaces in the L_2 metric, respectively. In [13] the authors investigate average sampling numbers of a periodic Sobolev space with a Gaussian measure in the L_q metric for $1\le q\le\infty$, and obtain their asymptotic orders. More information about average case setting results can be found in [20] and [24].

In the paper, we shall investigate average sampling numbers in the L_q metric for $1\le q\le\infty$ on a Sobolev space on the sphere with a Gaussian measure, and obtain the asymptotic orders. We show that some worst-case asymptotically optimal algorithms are also asymptotically optimal in the average case setting.

2 Main results

Let $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d : |x|=1\}$ be the unit sphere of $\mathbb{R}^d$ endowed with the usual rotation variant measure $d\sigma(x)$, and let $d(x,y)=\arccos(x\cdot y)$ be the geodesic distance between two points $x,y\in\mathbb{S}^{d-1}$, where $x\cdot y$ is the usual inner product and $|x|=(x\cdot x)^{1/2}$ is the Euclidean norm. For $1\le q\le\infty$, denote by $L_q\equiv L_q(\mathbb{S}^{d-1})$ the collection of real measurable functions f on $\mathbb{S}^{d-1}$ with finite norm

$$\|f\|_q=\Big(\int_{\mathbb{S}^{d-1}}|f(x)|^q d\sigma(x)\Big)^{\frac{1}{q}},\quad 1\le q<\infty,$$

and for $q=\infty$, the essential supremum is understood instead of the integral. We denote by Π_n^d the space of all spherical polynomials of degree at most n, and by $\mathcal{H}_l^d$ the space of all spherical harmonics of degree l on $\mathbb{S}^{d-1}$. It is well known that the spaces $\mathcal{H}_l^d$, $l=0,1,\cdots$, are just the eigenspaces corresponding to the eigenvalues $-l(l+d-2)$ of the Laplace-Beltrami operator Δ on the sphere and are mutually orthogonal with respect to the inner product

$$\langle f,g\rangle:=\int_{\mathbb{S}^{d-1}}f(x)g(x)d\sigma(x),$$