Analysis in Theory and Applications
Anal. Theory Appl., Vol. **35**, No. 3 (2019), pp. 235-267

DOI: 10.4208/ata.OA-0013

KAM Theory for Partial Differential Equations

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Received 27 July 2018; Accepted (in revised version) 3 September 2018

Abstract. In the last years much progress has been achieved in KAM theory concerning bifurcation of quasi-periodic solutions of Hamiltonian or reversible partial differential equations. We provide an overview of the state of the art in this field.

Key Words: KAM for PDEs, quasi-periodic solutions, small divisors, infinite dimensional Hamiltonian and reversible systems, water waves, nonlinear wave and Schrödinger equations, KdV.

AMS Subject Classifications: 37K55, 37J40, 37C55, 76B15, 35S05

1 Introduction

Many partial differential equations (PDEs) arising in Physics are infinite dimensional dynamical systems

$$u_t = X(u), \quad u \in E, \tag{1.1}$$

defined on an infinite dimensional phase space E of functions $u:=u(x)$, whose vector field X (in general unbounded) is Hamiltonian or Reversible. A vector field X is Hamiltonian if

$$X(u) = J\nabla H(u),$$

where J is a non-degenerate antisymmetric linear operator, the function $H: E \to \mathbb{R}$ is the Hamiltonian and ∇ denotes the L^2-gradient. We refer to [106] for a general introduction to Hamiltonian PDEs. A vector field X is reversible if there exists an involution S of the phase space, i.e., a linear operator of E satisfying $S^2 = \mathrm{Id}$, see e.g., (1.21), such that

$$X \circ S = -S \circ X.$$

Such symmetries have important consequences on the dynamics of (1.1), as we describe below. Classical examples are

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1. **Nonlinear wave (NLW)/ Klein-Gordon.**

$$y_{tt}-\Delta y+V(x)y=f(x,y), \quad x\in\mathbb{T}^d:=(\mathbb{R}/2\pi\mathbb{Z})^d, \quad y\in\mathbb{R}, \tag{1.2}$$

with a real valued potential $V(x)\in\mathbb{R}$. If $V(x)=m$ is constant, (1.2) is also called a nonlinear Klein-Gordon equation. Eq. (1.2) can be written as the first order Hamiltonian system

$$\frac{d}{dt}\begin{pmatrix} y \\ p \end{pmatrix}=\begin{pmatrix} p \\ \Delta y-V(x)y+f(x,y) \end{pmatrix}=\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} \nabla_y H(y,p) \\ \nabla_p H(y,p) \end{pmatrix},$$

where $\nabla_y H$, $\nabla_p H$ denote the $L^2(\mathbb{T}^d_x)$-gradient of the Hamiltonian

$$H(y,p):=\int_{\mathbb{T}^d}\frac{p^2}{2}+\frac{1}{2}\big((\nabla_x y)^2+V(x)y^2\big)+F(x,y)dx \tag{1.3}$$

with potential $F(x,y):=-\int_0^y f(x,z)dz$ and $\nabla_x y:=(\partial_{x_1}y,\cdots,\partial_{x_d}y)$.

Considering in (1.3) an Hamiltonian density $F(x,y,\nabla_x y)$, which depends also on the first order derivatives $\nabla_x y$, the corresponding Hamiltonian PDE is a quasi-linear wave equation with a nonlinearity which depends (linearly) with respect to the second order derivatives $\partial^2_{x_i x_j}y$.

If the nonlinearity $f(x,y,\nabla_x y)$ in (1.2) depends on first order derivatives, the equation, called derivative nonlinear wave equation (DNLW), is no more Hamiltonian (at least with the usual symplectic structure) but it can admit a reversible symmetry, see e.g., [20].

2. **Nonlinear Schrödinger (NLS).**

$$\mathrm{i}u_t-\Delta u+V(x)u=\partial_{\bar u}F(x,u), \quad x\in\mathbb{T}^d, \quad u\in\mathbb{C}, \tag{1.4}$$

where $F(x,u)\in\mathbb{R}$, $\forall u\in\mathbb{C}$, and, for $u=a+\mathrm{i}b$, $a,b\in\mathbb{R}$, we define the operator $\partial_{\bar u}:=\frac{1}{2}(\partial_a+\mathrm{i}\partial_b)$. The NLS equation (1.4) can be written as the infinite dimensional complex Hamiltonian equation

$$u_t=\mathrm{i}\nabla_{\bar u}H(u), \quad H(u):=\int_{\mathbb{T}^d}|\nabla u|^2+V(x)|u|^2-F(x,u)dx.$$

A simpler pseudo-differential model equation which is often considered is (1.4) with the multiplicative potential replaced by a convolution potential $V*u$, defined as the Fourier multiplier

$$u(x)=\sum_{j\in\mathbb{Z}^d}u_j e^{\mathrm{i}j\cdot x}\mapsto V*u:=\sum_{j\in\mathbb{Z}^d}V_j u_j e^{\mathrm{i}j\cdot x}.$$

If the nonlinearity in the right hand side in (1.4) depends also on first and second order derivatives, we have, respectively, derivative NLS (DNLS) and fully-non-linear (or quasi-linear) Schrödinger type equations. According to the nonlinearity it may admit an Hamiltonian or reversible structure.