Analysis in Theory and Applications
Anal. Theory Appl., Vol. **36**, No. 1 (2020), pp. 33-51

DOI: 10.4208/ata.OA-0019

Hölder Continuity of Spectral Measures for the Finitely Differentiable Quasi-Periodic Schrödinger Operators

Mei Sun and Xueyin Wang*

Institute of Applied System Analysis, Jiangsu University, Zhenjiang, Jiangsu 212013, China

Received 12 November 2019; Accepted (in revised version) 10 April 2020

Abstract. In the present paper, we prove the $\frac{1}{2}$-Hölder continuity of spectral measures for the C^k Schrödinger operators. This result is based on the quantitative almost reducibility and an estimate for the growth of the Schrödinger cocycles in [5].

Key Words: Schrödinger operator, quasi-periodic, almost reducibility, finitely differentiable.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

In this paper, we consider the Schrödinger operators defined on $\ell^2(\mathbb{Z})$

$$(H_{V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n,$$

where $V : \mathbb{T}^d \to \mathbb{R}$ is the potential, $\theta \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ is the phase, and $\alpha \in \mathbb{T}^d$ is the frequency.

These operators have been extensively and thoroughly studied for the deep connection with quasi-crystal and quantum Hall effects [11,18]. This paper concerns the regularity of the spectral measure of the quasi-periodic Schrödinger operators. For the analytic potential $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$, there is some significant progress [5, 21, 24]. However for the smooth potential $V \in C^k(\mathbb{T}^d, \mathbb{R})$, there is no similar result as far as we know, so we will give a supplementary answer to this situation.

Let us review some results on the Hölder continuity of the integrated density of states (IDS) and the individual spectral measures.

*Corresponding author. *Email addresses:* `sunm@ujs.edu.cn` (M. Sun), `2889190407@qq.com` (X. Wang)

1.1 Hölder continuity of IDS

Let $\Sigma_{V,\alpha,\theta}$ be the spectrum of $H_{V,\alpha,\theta}$, then $\Sigma_{V,\alpha,\theta} \subset \mathbb{R}$ since $H_{V,\alpha,\theta}$ is the bounded self-adjoint operator in $\ell^2(\mathbb{Z})$. The spectrum is independent of θ if $(\alpha, 1)$ is rational independent. For any $f \in \ell^2(\mathbb{Z})$, the spectral measure $\mu^f_{V,\alpha,\theta}$ of $H_{V,\alpha,\theta}$ can be defined as

$$\langle (H_{V,\alpha,\theta} - E)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{E' - E} d\mu^f_{V,\alpha,\theta}(E'), \quad \forall E \in \mathbb{C} \backslash \Sigma_{V,\alpha}. \tag{1.1}$$

Let $\mu_{V,\alpha,\theta} = \mu^{e_{-1}}_{V,\alpha,\theta} + \mu^{e_0}_{V,\alpha,\theta}$, where $\{e_i\}_{i\in\mathbb{Z}}$ is the cannonical basis of $\ell^2(\mathbb{Z})$. Let $N_{V,\alpha}$ be the IDS of $H_{V,\alpha,\theta}$, it is well known that IDS is the average of the spectral measure $\mu_{V,\alpha,\theta}$ with respect to θ, i.e.,

$$N_{V,\alpha}(E) = \int_{\mathbb{T}^d} \mu_{V,\alpha,\theta}(-\infty, E] d\theta.$$

Hence the regularity of IDS is closely related to that of the spectral measure.

Recall that $\alpha \in \mathbb{T}^d$ is Diophantine if there exist $\gamma > 0$ and $\tau > d-1$ such that $\alpha \in \mathrm{DC}_d(\gamma, \tau)$, where

$$\mathrm{DC}_d(\gamma, \tau) = \left\{ \alpha : \inf_{j\in\mathbb{Z}} |\langle n, \alpha \rangle - j| > \frac{\gamma}{|n|^\tau}, \ \forall n \in \mathbb{Z}^d \backslash \{0\} \right\}.$$

Let $\mathrm{DC}_d = \cup_{\gamma>0,\tau>d-1} \mathrm{DC}_d(\gamma, \tau)$. For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, let $\frac{p_n}{q_n}$ be the continued fraction approximants to α, then one can define

$$\beta(\alpha) = \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}.$$

Given the operator $H_{V,\alpha,\theta}$, one can define the Lyapunov exponent $L(\alpha, S^V_E)$ (see Section 2.1) of the corresponding Schrödinger cocycle $(\alpha, S^V_E(\theta))$, where $E \in \mathbb{R}$ and

$$S^V_E(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Hadj Amor [16] proved the $\frac{1}{2}$-Hölder continuity of IDS if $\alpha \in \mathrm{DC}_d$ and $V \in C^\omega(\mathbb{T}^d, \mathbb{R})$ is small, and her approach is based on the almost reducibility scheme developed by Eliasson [13]. Recall that the cocycle (α, A) is reducible if (α, A) can be conjugated to some constant cocycles and the cocycle (α, A) is almost reducible if the closure of its conjugates contains a constant. Avila and Jitomirskaya [4] proved the $\frac{1}{2}$-Hölder continuity of IDS for $\alpha \in \mathrm{DC}_1$ and the small analytic potential. Their result was non-perturbative, which means that the smallness is independent of α. After that, Avila [2,3] generalized the result for the small analytic potential with $\beta(\alpha) = 0$ if there is $\delta > 0$ such that $L(\alpha, S^V_{E+i\epsilon}) = 0$ for $|\epsilon| < \delta$. Note that Leguil-You-Zhao-Zhou [20] showed the same result as well by the global theory of the one-frequency Schrödinger operators [1].