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Rearrangement Free Method for Hardy-Littlewood-Sobolev Inequalities on $\mathbb{S}^n$

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Abstract. For conformal Hardy-Littlewood-Sobolev(HLS) inequalities [22] and reversed conformal HLS inequalities [8] on $\mathbb{S}^n$, a new proof is given for the attainability of their sharp constants. Classical methods used in [22] and [8] depends on rearrangement inequalities. Here, we use the subcritical approach to construct the extremal sequence and circumvent the blow-up phenomenon by renormalization method. The merit of the method is that it does not rely on rearrangement inequalities.

Key Words: Hardy-Littlewood-Sobolev inequality, reversed Hardy-Littlewood-Sobolev inequality, rearrangement free method.

AMS Subject Classifications: 39B62, 26A33, 26D10

1 Introduction

The conformal Hardy-Littlewood-Sobolev(HLS) inequality on $\mathbb{R}^n$ is

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy\right|\le N_{n,\alpha}\|f\|_{p_\alpha}\|g\|_{p_\alpha},\quad 0<\alpha<n,\quad p_\alpha=\frac{2n}{n+\alpha},\tag{1.1}$$

where

$$N_{n,\alpha}=\pi^{(n-\alpha)/2}\frac{\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)}\left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{\alpha/n}$$

is the best constant. Lieb [22] proved that the extremal functions of (1.1) are radial symmetric by rearrangement inequalities, and obtained the sharp constant by the conformal symmetries of (1.1). Different discussions can be found in [3,23]. Recently, the classification of the solutions for the Euler-Lagrange equation of (1.1) was given in [4] and [21] by the method of moving planes and the method of moving spheres, respectively.

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For $\alpha > n$, Dou and Zhu [8] (also see [2,24]) established a class of reversed conformal HLS inequalities

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\alpha}}dxdy \geq \tilde{N}_{n,\alpha}\|f\|_{p_\alpha}\|g\|_{p_\alpha}, \tag{1.2}$$

where $\tilde{N}_{n,\alpha} = N_{n,\alpha}$ is the best constant. Employing the rearrangement inequalities and the method of moving spheres, they obtained the sharp constant and classified the solutions of the corresponding Euler-Lagrange equation.

As stated above, it can be found that rearrangement inequalities, the method of moving planes and the method of moving spheres are basic and important tools for studying the HLS inequalities. More applications of these techniques can be found in the study of HLS type inequalities and reversed HLS type inequalities on the upper half space (see [6,9,18,25] and the references therein).

Heisenberg group is one of the simplest noncommutative geometries and is the model space of CR manifolds. It is natural that we want to generalize these traditional methods on Heisenberg group. But, because of the non-commutativity, rearrangement inequalities, the method of moving planes and the method of moving spheres don't work efficiently on Heisenberg group. In this paper, we will try a class of rearrangement free method and give a new proof for the existence of the extremal functions of (1.1) and (1.2). Recently, we successfully generalize the method to study the reversed HLS inequalities on the Heisenberg group (see [15]).

From [22] and [8], we know that the extremal functions of (1.1) and (1.2) have the form

$$f_\epsilon(x) = c_1 g_\epsilon(x) = c\left(\frac{\epsilon}{\epsilon^2+|x-x_0|^2}\right)^{(n+\alpha)/2},$$

where c_1, c and ϵ are constants, x_0 is some point in $\mathbb{R}^n$. Note that f_ϵ and g_ϵ will blow up as $\epsilon \to 0^+$, and vanish as $\epsilon \to +\infty$. The phenomenon makes it difficult to study the extremal problems. To overcome the difficulty, we often renormalize the extremal sequence. For example, Lieb [22] renormalized the extremal sequence $\{f_j(x)\}$ so that it satisfies $f_j(x) > \beta > 0$ if $|x| = 1$. The technique can also be found in [8].

Recently, Dou, Guo and Zhu [6] adopted a subcritical approach to study sharp HLS type inequalities on the upper half space. By Young inequality, they first established two classes of HLS type inequalities with subcritical power on a ball. Then, using the conformal transformation between ball and upper half space and the method of moving planes, they proved that the extremal functions of HLS type inequalities with subcritical power are constant functions. Passing to the limit from subcritical power to critical power, they obtained two classes of sharp HLS type inequalities on the upper half space.

In their approach, we note three advantages. First, extremal functions of HLS type inequalities with subcritical power satisfy the corresponding Euler-Lagrange equation, by which we can study the regularity of theses functions. Second, as power approach to critical, the corresponding extremal functions form a extremal sequence to the extremal problem of HLS type inequalities with critical power. Third, since these extremal functions of

HLS type inequalities with subcritical power are constant functions, we can choose every extremal function to be $f \equiv 1$ and avoid efficiently the blow-up phenomenon.

Since the method of moving planes and the method of moving spheres don't work efficiently on the Heisenberg group, it is not easy to prove the extremal functions of HLS inequalities with subcritical power on the CR sphere to be constant function. So, we will combine the subcritical approach and renormalization method and give a new proof of the existence of the extremal functions of (1.1) and (1.2). Moreover, our method doesn't depend on rearrangement inequalities, the method of moving planes and the method of moving spheres, and can circumvent the blow-up phenomenon.

By stereographic projection $\mathcal{S}: x \in \mathbb{R}^n \to \xi \in \mathbb{S}^n \backslash \mathfrak{S}$ defined by

$$\xi^j := \frac{2x^j}{1+|x|^2} \quad \text{for} \quad j = 1, 2, \cdots, n, \quad \xi^{n+1} := \frac{1-|x|^2}{1+|x|^2},$$

with $\mathfrak{S} = (0, \cdots, 0, -1)$ being the south pole, (1.1) is equivalent to the HLS inequality on $\mathbb{S}^n$ stated as, for $0 < \alpha < n$ and any $f, g \in L^{p_\alpha}(\mathbb{S}^n)$,

$$\left| \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(\xi) |\xi - \eta|^{\alpha - n} g(\eta) d\xi d\eta \right| \leq N_{n,\alpha} \|f\|_{L^{p_\alpha}(\mathbb{S}^n)} \|g\|_{L^{p_\alpha}(\mathbb{S}^n)}, \tag{1.3}$$

and (1.2) is equivalent to the reversed HLS inequality on $\mathbb{S}^n$ stated as, for $\alpha > n$ and any nonnegative function $f, g \in L^{p_\alpha}(\mathbb{S}^n)$,

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(\xi) |\xi - \eta|^{\alpha - n} g(\eta) d\xi d\eta \geq \tilde{N}_{n,\alpha} \|f\|_{L^{p_\alpha}(\mathbb{S}^n)} \|g\|_{L^{p_\alpha}(\mathbb{S}^n)}, \tag{1.4}$$

where $d\xi$ and $d\eta$ denote the surface measure of $\mathbb{S}^n$.

When $0 < \alpha < n$, we introduce conventionally the following duality form of HLS inequality

$$\left(\int_{\mathbb{S}^n} |I_\alpha f|^{q_\alpha} d\xi \right)^{1/q_\alpha} \leq C(n, \alpha) \left(\int_{\mathbb{S}^n} |f|^{p_\alpha} d\xi \right)^{1/p_\alpha}, \quad \forall f \in L^{p_\alpha}(\mathbb{S}^n), \tag{1.5}$$

where

$$I_\alpha f(\xi) = \int_{\mathbb{S}^n} |\xi - \eta|^{\alpha - n} f(\eta) d\eta,$$

$|\xi - \eta|$ is the chord distance on $\mathbb{S}^n$ and $q_\alpha = \frac{2n}{n-\alpha}$ is the conjugate number of p_α. Define the corresponding extremal problem as

$$\begin{aligned} N_{n,\alpha} &:= \sup\{ \|I_\alpha f\|_{L^{q_\alpha}(\mathbb{S}^n)} : \|f\|_{L^{p_\alpha}(\mathbb{S}^n)} = 1 \} \\ &= \sup\Big\{ \frac{\|I_\alpha f\|_{L^{q_\alpha}(\mathbb{S}^n)}}{\|f\|_{L^{p_\alpha}(\mathbb{S}^n)}} : f \in L^{p_\alpha}(\mathbb{S}^n) \backslash \{0\} \Big\}. \end{aligned} \tag{1.6}$$

Then, by our method, we can prove the attainability of $N_{n,\alpha}$.

Theorem 1.1. *For $\alpha \in (0,n)$, sharp constant $N_{n,\alpha}$ is attained.*

Since $q_\alpha < 0$ when $\alpha > n$, we think that the bilinear form (1.4) is easier to study than its duality form. As convention, the extremal problem of reversed HLS inequality is defined as

$$\begin{aligned}\tilde{N}_{n,\alpha} :=& \inf\Big\{\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{f(\xi)g(\eta)}{|\xi-\eta|^{\alpha-n}}d\xi d\eta:\ f\geq 0,\ g\geq 0,\ \|f\|_{L^{p_\alpha}(\mathbb{S}^n)}=\|g\|_{L^{p_\alpha}(\mathbb{S}^n)}=1\Big\}\\ =&\inf\Big\{\frac{\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}f(\xi)|\xi-\eta|^{\alpha-n}g(\eta)d\xi d\eta}{\|f\|_{L^{p_\alpha}(\mathbb{S}^n)}\|g\|_{L^{p_\alpha}(\mathbb{S}^n)}}:f\geq 0,\ g\geq 0,\ f,g\in L^{p_\alpha}(\mathbb{S}^n)\setminus\{0\}\Big\}. \quad (1.7)\end{aligned}$$

Then, we can prove the attainability of $\tilde{N}_{n,\alpha}$.

Theorem 1.2. *For $\alpha > n$, sharp constant $\tilde{N}_{n,\alpha}$ is attained.*

Since $p_\alpha \in (0,1)$ and nonlinear terms with negative power appears in the Euler-Lagrange equations of (1.7) (see Section 3), the variational problem (1.7) is analytically different from the case $\alpha \in (0,n)$. Moreover, we need not only a upper bound to control the blow up of the sequence, but also a lower bound to avoid the blow up of terms with negative power. So, different techniques are needed for the extremal problem (1.7). More details can be seen in Section 3.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3, we consider the case $\alpha > n$ and prove Theorem 1.2.

2 Case of $0 < \alpha < n$

2.1 Subcritical HLS inequalities

Let $p \in (p_\alpha, \min\{2, \frac{n}{\alpha}\})$.

Proposition 2.1. *There exists positive constant $N_{n,\alpha,p}$ such that for any $f \in L^p(\mathbb{S}^n)$, it holds*

$$\Big(\int_{\mathbb{S}^n}|I_\alpha f|^{q_\alpha}d\xi\Big)^{1/q_\alpha}\leq N_{n,\alpha,p}\Big(\int_{\mathbb{S}^n}|f|^p d\xi\Big)^{1/p}, \tag{2.1}$$

where $N_{n,\alpha,p}$ is sharp and can be attained by some positive function $f_p \in L^p(\mathbb{S}^n)$ satisfying $\|f_p\|_{L^p(\mathbb{S}^n)} = 1$. Moreover, f_p satisfies the following Euler-Lagrange equation

$$N^{q_\alpha}_{n,\alpha,p}f_p^{p-1}(\xi)=\int_{\mathbb{S}^n}|\xi-\eta|^{\alpha-n}(I_\alpha f_p)^{q_\alpha-1}(\eta)d\eta \tag{2.2}$$

and $f_p \in C^\gamma(\mathbb{S}^n)$ with

$$\gamma=\begin{cases}\alpha, & \text{if } 0<\alpha<1,\\ \frac{1}{2}, & \text{if } \alpha=1,\\ 1, & \text{if } \alpha>1.\end{cases}$$

Proof. By Young's inequality, we know that there exists some positive constant C such that

$$\|I_\alpha f\|_{L^{q_\alpha}(\mathbb{S}^n)} \le C\|f\|_{L^p(\mathbb{S}^n)}$$

holds for any $f \in L^p(\mathbb{S}^n)$. Through a similar argument as Proposition 2.3 of [27], we prove that $I_\alpha : L^p(\mathbb{S}^n) \to L^{q_\alpha}(\mathbb{S}^n)$ is compact. Combining the theory of reflexive space, we get the existence of nonnegative extremal function f_p of (2.1). So, f_p satisfies (2.2) and f_p is positive.

It is routine that $f_p \in L^\infty(\mathbb{S}^n)$ and $I_\alpha f_p \in L^\infty(\mathbb{S}^n)$ are got by iteration (details can be found in [7,11,13]).

If $\alpha > 1$, we find that

$$\int_{\mathbb{S}^n} \left|\frac{\partial}{\partial \xi_i}|\xi-\eta|^{\alpha-n}(I_\alpha f_p)^{q_\alpha-1}(\eta)\right| d\eta \le C\int_{\mathbb{S}^n}|\xi-\eta|^{\alpha-n-1}, \quad i=1,2,\cdots,n+1,$$

converges uniformly for $\xi \in \mathbb{S}^n$. So,

$$\begin{aligned}
&N_{n,\alpha,p}^{q_\alpha}\frac{\partial f_p^{p-1}}{\partial \xi_i} = \int_{\mathbb{S}^n}\frac{\partial}{\partial \xi_i}|\xi-\eta|^{\alpha-n}(I_\alpha f_p)^{q_\alpha-1}(\eta)d\eta,\\
&\frac{\partial f_p}{\partial \xi_i} = (q-1)(f_p^{p-1})^{q-2}\frac{\partial f_p^{p-1}}{\partial \xi_i},
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$. Namely, $f_p \in C^1(\mathbb{S}^n)$ for $\alpha > 1$.

If $\alpha \in (0,1)$, we can prove $f \in C^\alpha(\mathbb{S}^n)$ by a similar argument of Lemma 4.3 of [13]. In fact, for any $\xi_1, \xi_2 \in \mathbb{S}^n$ and denoting

$$g(\xi) = N_{n,\alpha,p}^{q_\alpha} f_p^{p-1}(\xi),$$

we have

$$\begin{aligned}
|g(\xi_1)-g(\xi_2)| \le & C\int_{\mathbb{S}^n}\Big||\xi_1-\eta|^{\alpha-n}-|\xi_2-\eta|^{\alpha-n}\Big|d\eta\\
=& C\int_{|\xi_2-\eta|\ge 2|\xi_1-\xi_2|}\Big||\xi_1-\eta|^{\alpha-n}-|\xi_2-\eta|^{\alpha-n}\Big|d\eta\\
&+C\int_{|\xi_2-\eta|< 2|\xi_1-\xi_2|}\Big||\xi_1-\eta|^{\alpha-n}-|\xi_2-\eta|^{\alpha-n}\Big|d\eta\\
=:& I+II.
\end{aligned} \tag{2.3}$$

If $|\xi_2-\eta| \ge 2|\xi_1-\xi_2|$, then

$$\begin{aligned}
&|\xi_1-\eta| \ge |\xi_2-\eta|-|\xi_1-\xi_2| \ge \frac{1}{2}|\xi_2-\eta|,\\
&\Big||\xi_1-\eta|^{\alpha-n}-|\xi_2-\eta|^{\alpha-n}\Big| \le C|\xi_2-\eta|^{\alpha-n-1}|\xi_1-\xi_2|.
\end{aligned}$$

So,

$$I \le C|\xi_1-\xi_2| \int_{|\xi_2-\eta|\ge 2|\xi_1-\xi_2|} |\xi_2-\eta|^{\alpha-n-1} d\eta \le C|\xi_1-\xi_2|^{\alpha}. \tag{2.4}$$

On the other hand,

$$\begin{aligned} II \le & C\int_{|\xi_1-\eta|\le 3|\xi_1-\xi_2|} |\xi_1-\eta|^{\alpha-n} d\eta + C\int_{|\xi_2-\eta|<2|\xi_1-\xi_2|} |\xi_2-\eta|^{\alpha-n} d\eta \\ \le & C|\xi_1-\xi_2|^{\alpha}. \end{aligned} \tag{2.5}$$

Combining (2.3), (2.4) and (2.5), we have $g \in C^{\alpha}(\mathbb{S}^n)$ and then $f \in C^{\alpha}(\mathbb{S}^n)$.

If $\alpha = 1$, then

$$\begin{aligned} & |g(\xi_1)-g(\xi_2)| \\ \le & \int_{\mathbb{S}^n} |\xi_1-\eta|^{\frac{1}{2}} \Big| |\xi_1-\eta|^{\frac{1}{2}-n} - |\xi_2-\eta|^{\frac{1}{2}-n} \Big| d\eta \\ & + \int_{\mathbb{S}^n} \Big| |\xi_1-\eta|^{\frac{1}{2}} - |\xi_2-\eta|^{\frac{1}{2}} \Big| |\xi_2-\eta|^{\frac{1}{2}-n} d\eta \\ \le & 2^{\frac{1}{2}} \int_{\mathbb{S}^n} \Big| |\xi_1-\eta|^{\frac{1}{2}-n} - |\xi_2-\eta|^{\frac{1}{2}-n} \Big| d\eta + |\xi_1-\xi_2|^{\frac{1}{2}} \int_{\mathbb{S}^n} |\xi_2-\eta|^{\frac{1}{2}-n} d\eta \\ \le & C|\xi_1-\xi_2|^{\frac{1}{2}}, \end{aligned}$$

namely, $g \in C^{1/2}(\mathbb{S}^n)$. So, $f \in C^{1/2}(\mathbb{S}^n)$. □

2.2 Maximizing sequence

In this subsection, we will prove that sequence $\{f_p\}$ is a maximizing sequence of (1.6) as $p \to p_\alpha^+$.

Lemma 2.1. *$N_{n,\alpha,p} \to N_{n,\alpha}$ as $p \to p_\alpha^+$.*

Proof. Take $f_p \in L^p(\mathbb{S}^n)$ be the maximizer given by Proposition 2.1. Namely, f_p satisfies

$$N_{n,\alpha,p} = \|I_\alpha f_p\|_{L^{q_\alpha}(\mathbb{S}^n)} \quad \text{with} \quad \|f_p\|_{L^p(\mathbb{S}^n)} = 1.$$

Let

$$\tilde{f}_p = \frac{f_p}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^n)}}$$

and then

$$\begin{aligned} N_{n,\alpha,p} = & \|f_p\|_{L^{p_\alpha}} \|I_\alpha \tilde{f}_p\|_{L^{q_\alpha}(\mathbb{S}^n)} \le |\mathbb{S}^n|^{\frac{1}{p_\alpha}-\frac{1}{p}} \|I_\alpha \tilde{f}_p\|_{L^{q_\alpha}(\mathbb{S}^n)} \\ \le & |\mathbb{S}^n|^{\frac{1}{p_\alpha}-\frac{1}{p}} N_{n,\alpha} \to N_{n,\alpha} \quad \text{as} \ \ p \to p_\alpha^+. \end{aligned}$$

So,

$$\limsup_{p\to p_\alpha^+} N_{n,\alpha,p} \le N_{n,\alpha}. \tag{2.6}$$

Take smooth function sequence $\{f_k\} \subset L^{p_\alpha}(\mathbb{S}^n)$ being a maximizing sequence of $N_{n,\alpha}$. Namely,

$$N_{n,\alpha} = \lim_{k\to+\infty} \frac{\|I_\alpha f_k\|_{L^{q_\alpha}(\mathbb{S}^n)}}{\|f_k\|_{L^{p_\alpha}(\mathbb{S}^n)}}.$$

For any $p \geq p_\alpha$, let $\tilde{f}_k = \frac{f_k}{\|f_k\|_{L^p(\mathbb{S}^n)}}$ and then

$$N_{n,\alpha,p} \geq \|I_\alpha \tilde{f}_k\|_{L^{q_\alpha}(\mathbb{S}^n)} = \frac{\|I_\alpha f_k\|_{L^{q_\alpha}(\mathbb{S}^n)}}{\|f_k\|_{L^p(\mathbb{S}^n)}}. \tag{2.7}$$

We firstly send p to p_α^+ in (2.7) and get

$$\liminf_{p\to p_\alpha^+} N_{n,\alpha,p} \geq \frac{\|I_\alpha f_k\|_{L^{q_\alpha}(\mathbb{S}^n)}}{\|f_k\|_{L^{p_\alpha}(\mathbb{S}^n)}}.$$

Then let $k \to +\infty$ and deduce that

$$\liminf_{p\to p_\alpha^+} N_{n,\alpha,p} \geq N_{n,\alpha}. \tag{2.8}$$

Combining (2.6) and (2.8), we complete the proof. □

Lemma 2.2. *Function sequence $\{f_p\}$ given by Proposition* 2.1 *is a maximizing sequence of $N_{n,\alpha}$, namely,*

$$N_{n,\alpha} = \lim_{p\to p_\alpha^+} \frac{\|I_\alpha f_p\|_{L^{q_\alpha}(\mathbb{S}^n)}}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^n)}}. \tag{2.9}$$

Proof. By the definition of $N_{n,\alpha}$,

$$N_{n,\alpha} \geq \frac{\|I_\alpha f_p\|_{L^{q_\alpha}(\mathbb{S}^n)}}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^n)}} \geq \frac{\|I_\alpha f_p\|_{L^{q_\alpha}(\mathbb{S}^n)}}{|\mathbb{S}^n|^{\frac{1}{p_\alpha}-\frac{1}{p}}}.$$

Letting $p \to p_\alpha^+$ and using Lemma 2.1, we get (2.9). □

Remark 2.1. By Lemma 2.1 and Lemma 2.2, we have

$$\lim_{p\to p_\alpha^+} \|f_p\|_{L^{p_\alpha}(\mathbb{S}^n)} = 1.$$

2.3 Proof of Theorem 1.1

Proof. In the following, we will construct a maximizer for the extremal problem (1.6) from the maximizing sequence $\{f_p\}$. Since the rotational invariance of (2.1) and (2.2), we assume without loss of generality that

$$f_p(\mathfrak{N}) = \max_{\xi\in\mathbb{S}^n} f_p(\xi) \quad \text{with} \quad \mathfrak{N} = (0,\cdots,0,1)$$

being the north pole.

Case 1: For some subsequence $p_j \to p_\alpha^+$, $f_{p_j}(\mathfrak{N})$ is uniformly bounded. Then, by Proposition 2.1, we know that sequences $\{f_{p_j}\}$ and $\{I_\alpha(f_{p_j})\}$ are uniformly bounded and equicontinuous on $\mathbb{S}^n$. So, by Arzelà-Ascoli theorem, there exists a subsequence of $\{f_{p_j}\}$ (still denoted by $\{f_{p_j}\}$) and some nonnegative function $f \in C(\mathbb{S}^n)$ such that

$$\begin{aligned} &f_{p_j} \to f \quad \text{uniformly on} \quad \mathbb{S}^n, \\ &I_\alpha(f_{p_j}) \to I_\alpha f \quad \text{uniformly on} \quad \mathbb{S}^n. \end{aligned}$$

Then,

$$\int_{\mathbb{S}^n} f^{p_\alpha}(\xi) d\xi = \lim_{p_j \to p_\alpha} \int_{\mathbb{S}^n} f_{p_j}^{p_j}(\xi) d\xi = 1$$

and by (2.2) and Lemma 2.1,

$$N_{n,\alpha}^{q_\alpha} f^{p_\alpha - 1}(\xi) = \int_{\mathbb{S}^n} |\xi - \eta|^{\alpha - n} (I_\alpha f)^{q_\alpha - 1}(\eta) d\eta.$$

So, $f(\xi)$ satisfies

$$N_{n,\alpha}^{q_\alpha} = \int_{\mathbb{S}^n} (I_\alpha f)^{q_\alpha}(\eta) d\eta, \quad \int_{\mathbb{S}^n} f^{p_\alpha}(\xi) d\xi = 1,$$

namely, f is a maximizer.

Case 2: $f_p(\mathfrak{N}) \to +\infty$ as $p \to p_\alpha^+$. Let $u_p = f_p^{p-1}$ and $v_p = I_\alpha f_p$. Then, u_p and v_p satisfy

$$\int_{\mathbb{S}^n} u_p^q d\xi = 1, \quad \lim_{p \to p_\alpha^+} \int_{\mathbb{S}^n} v_p^{q_\alpha} d\xi = N_{n,\alpha}^{q_\alpha}, \tag{2.10}$$

and by (2.2),

$$\begin{cases} N_{n,\alpha,p}^{q_\alpha} u_p(\xi) = \displaystyle\int_{\mathbb{S}^n} |\xi - \eta|^{\alpha - n} v_p^{q_\alpha - 1}(\eta) d\eta, \\ v_p(\xi) = \displaystyle\int_{\mathbb{S}^n} |\xi - \eta|^{\alpha - n} u_p^{q-1}(\eta) d\eta, \end{cases} \tag{2.11}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Applying stereographic projection and dilations on $\mathbb{R}^n$, we get from (2.11) that

$$\begin{cases} N_{n,\alpha,p}^{q_\alpha} \left(\dfrac{2\lambda}{1 + |\lambda x|^2} \right)^{\frac{n-\alpha}{2}} u_p(\mathcal{S}(\lambda x)) = \displaystyle\int_{\mathbb{R}^n} \frac{\left(\frac{2\lambda}{1+|\lambda y|^2} \right)^{\frac{n+\alpha}{2}} v_p(\mathcal{S}(\lambda y))^{q_\alpha - 1}}{|x - y|^{n-\alpha}} dy, \\ \left(\dfrac{2\lambda}{1 + |\lambda x|^2} \right)^{\frac{n-\alpha}{2}} v_p(\mathcal{S}(\lambda x)) = \displaystyle\int_{\mathbb{R}^n} \frac{\left(\frac{2\lambda}{1+|\lambda y|^2} \right)^{\frac{n+\alpha}{2}} u_p(\mathcal{S}(\lambda y))^{q-1}}{|x - y|^{n-\alpha}} dy. \end{cases} \tag{2.12}$$

Take $\lambda = \lambda_p$ satisfying

$$(2\lambda_p)^{(n-\alpha)/2} u_p(\mathcal{S}(0)) = 1$$

and denote

$$\begin{cases} U_p(x) = \left(\dfrac{2\lambda_p}{1+|\lambda_p x|^2}\right)^{\frac{n-\alpha}{2}} u_p(\mathcal{S}(\lambda_p x)), \\ V_p(x) = \left(\dfrac{2\lambda_p}{1+|\lambda_p x|^2}\right)^{\frac{n-\alpha}{2}} v_p(\mathcal{S}(\lambda_p x)). \end{cases}$$

Then, $U_p(x) \le U_p(0) = 1$ and U_p, V_p satisfy the following renormalized equations

$$\begin{cases} N_{n,\alpha,p}^{q_\alpha} U_p(x) = \displaystyle\int_{\mathbb{R}^n} |x-y|^{\alpha-n} V_p^{q_\alpha-1}(y)dy, \\ V_p(x) = \displaystyle\int_{\mathbb{R}^n} |x-y|^{\alpha-n} \left(\dfrac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q)} U_p^{q-1}(y)dy. \end{cases} \tag{2.13}$$

Next, we will prove that sequences $\{U_p, V_p\}$ satisfy some compactness property. Then, a maximizer can be got through limitation.

For any $x \in \mathbb{R}^n$,

$$\begin{aligned} V_p(x) &= \int_{\mathbb{R}^n} |y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^{\frac{(n-\alpha)(q_\alpha-q)}{2}} U_p^{q-1}(x-y)dy \\ &= \int_{|y|\le 2} |y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^{\frac{(n-\alpha)(q_\alpha-q)}{2}} U_p^{q-1}(x-y)dy \\ &\quad + \int_{|y|>2} |y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^{\frac{(n-\alpha)(q_\alpha-q)}{2}} U_p^{q-1}(x-y)dy \\ &:= V_p^1(x) + V_p^2(x), \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} V_p^1(x) &= \int_{|y|\le 2} |y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^{\frac{(n-\alpha)(q_\alpha-q)}{2}} U_p^{q-1}(x-y)dy, \\ V_p^2(x) &= \int_{|y|>2} |y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^{\frac{(n-\alpha)(q_\alpha-q)}{2}} U_p^{q-1}(x-y)dy. \end{aligned}$$

As $p \to p_\alpha^+$,

$$\begin{aligned} V_p^2(x) &\le \int_{|y|>2} |y|^{\alpha-n} \left(\left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^{\frac{n}{q}} u_p(x-y)\right)^{q-1} dy \\ &\le \left(\int_{|y|>2} |y|^{(\alpha-n)q} dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left(\frac{2\lambda_p}{1+|\lambda_p(x-y)|^2}\right)^n u_p^q(x-y)dy\right)^{\frac{q-1}{q}} \end{aligned}$$

$$\le \left(\int_{|y|>2} |y|^{(\alpha-n)q} dy \right)^{\frac{1}{q}}$$
$$= |\mathbb{S}^{n-1}|^{\frac{1}{q}} 2^{\alpha-\frac{n}{p}} \le |\mathbb{S}^{n-1}|^{\frac{1}{q}}$$

and

$$V_p^1(x) \le \int_{|y|\le 2} |y|^{\alpha-n} dy = |\mathbb{S}^{n-1}| 2^{\alpha}.$$

So, $\{V_p(x)\}$ are uniformly bounded. If $\alpha > 1$, noting that

$$\Big| \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} |x-y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p y|^2} \right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^{q-1}(y) dy \Big|$$
$$\le \int_{\mathbb{R}^n} (n-\alpha)|x-y|^{\alpha-n-1} \left(\frac{2\lambda_p}{1+|\lambda_p y|^2} \right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^{q-1}(y) dy,$$

converges uniformly on $\mathbb{R}^n$ by a similar argument as (2.14), we know that

$$\frac{\partial V_p(x)}{\partial x_i} = (\alpha - n) \int_{\mathbb{R}^n} \frac{x_i - y_i}{|x-y|^{2+n-\alpha}} \left(\frac{2\lambda_p}{1+|\lambda_p y|^2} \right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^{q-1}(y) dy, \tag{2.15a}$$
$$\Big| \frac{\partial V_p(x)}{\partial x_i} \Big| \le |\mathbb{S}^{n-1}|^{\frac{1}{q}} 2^{\alpha-1-\frac{n}{p}} + |\mathbb{S}^{n-1}| 2^{\alpha-1}. \tag{2.15b}$$

So, as $p \to p_\alpha^+$, $\{V_p(x)\}$ are equicontinuous if $\alpha > 1$.

For any given constant $R_0 > 0$ and any $x, y \in B(0, R_0)$, as $p \to p_\alpha^+$, we have

$$\begin{aligned}
&|V_p(x) - V_p(y)| \\
=& \Big| \int_{\mathbb{R}^n} \left(|x-y-z|^{\alpha-n} - |z|^{\alpha-n} \right) \left(\frac{2\lambda_p}{1+|\lambda_p (y+z)|^2} \right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^{q-1}(y+z) dz \Big| \\
\le& \int_{\mathbb{R}^n} \left| |x-y-z|^{\alpha-n} - |z|^{\alpha-n} \right| dz \\
=& \int_{2|x-y|<|z|} \left| |x-y-z|^{\alpha-n} - |z|^{\alpha-n} \right| dz + \int_{|z|\le 2|x-y|} \left| |x-y-z|^{\alpha-n} - |z|^{\alpha-n} \right| dz \\
=:& I + II.
\end{aligned} \tag{2.16}$$

If $|z| > 2|x-y|$, then

$$\left| |x-y-z|^{\alpha-n} - |z|^{\alpha-n} \right| \le C|z|^{\alpha-n-1}|x-y|.$$

So, $I \le C|x-y|^\alpha$ if $0 < \alpha < 1$. On the other hand, if $|z| \le 2|x-y|$, then

$$II \le 2 \int_{|z|\le 3|x-y|} |z|^{\alpha-n} dz \le C|x-y|^\alpha.$$

By now, we deduce that $\{V_p(x)\}$ are also equicontinuous on $B(0,R_0)$ as $p\to p_\alpha^+$ and $0<\alpha<1$.

If $\alpha=1$, as the case $0<\alpha<1$, we can prove that for any $x, x+y, x-y\in B_{R_0}(0)$,

$$|V_p(x+y)+V_p(x-y)-2V_p(x)|\le C|y|.$$

Namely, $V_p(x)\in\Lambda_1(B_{R_0}(0))$, where $\Lambda_1(B_{R_0}(0))$ is the space of Lipschitz continuous functions introduced in Chapter V, Section 4 of [26]. By the classical theory about Lipschitz continuous functions in [26], we know that $V_p(x)\in C^\tau(B_{R_0}(0))$ for any $\tau\in(0,1)$. So, $\{V_p(x)\}$ are also equicontinuous on $B(0,R_0)$ as $p\to p_\alpha^+$ and $\alpha=1$.

By Arzelà-Ascoli theorem, we know that there exists a subsequence of $\{V_p(x)\}$ (still denoted as $\{V_p(x)\}$) and $V(x)\in C(\mathbb{R}^n)$ such that, as $p\to p_\alpha^+$,

$$V_p(x)\to V(x)\quad \text{uniformly on}\quad B(0,R_0). \tag{2.17}$$

Similarly, we can prove that there exists a subsequence of $\{U_p(x)\}$ (still denoted by $\{U_p(x)\}$) and $U(x)\in C(\mathbb{R}^n)$ such that

$$U_p(x)\to U(x)\quad \text{uniformly on}\quad B(0,R_0). \tag{2.18}$$

If it holds that

$$\lim_{p\to p_\alpha^+}(2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha-q)}=1, \tag{2.19}$$

whose proof is postponed to the end for readability, then as $p\to p_\alpha^+$,

$$\left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q)}U_p^{q-1}(y)\to U^{q_\alpha-1}(y)\quad \text{uniformly on}\quad B(0,R)$$

and

$$\begin{aligned}1&=\int_{\mathbb{R}^n}\left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q)}U_p^{q}(y)dy\\&>\int_{|y|\le R}\left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q)}U_p^{q}(y)dy\to\int_{|y|\le R}U^{q_\alpha}(y)dy.\end{aligned} \tag{2.20}$$

So,

$$\int_{\mathbb{R}^n}U^{q_\alpha}(y)dy\le 1.$$

Moreover, take the limit in (2.13) as $p\to p_\alpha^+$ and obtain

$$\begin{cases}V(x)=\displaystyle\int_{\mathbb{R}^n}|x-y|^{\alpha-n}U^{q_\alpha-1}(y)dy=I_\alpha(U^{q_\alpha-1})(x) & \text{in }\mathbb{R}^n,\\ N_{n,\alpha}^{q_\alpha}U(x)=\displaystyle\int_{\mathbb{R}^n}|x-y|^{\alpha-n}V^{q_\alpha-1}(y)dy & \text{in }\mathbb{R}^n,\end{cases} \tag{2.21}$$

which imply by (2.20) and $q_\alpha > p_\alpha$ that

$$\begin{aligned} N_{n,\alpha} =& \frac{\left(\int_{\mathbb{R}^n} I_\alpha (U^{q_\alpha-1})^{q_\alpha}(y)dy\right)^{1/q_\alpha}}{\left(\int_{\mathbb{R}^n} U^{(q_\alpha-1)p_\alpha}dx\right)^{1/q_\alpha}} \\ \le& \frac{\left(\int_{\mathbb{R}^n} I_\alpha (U^{q_\alpha-1})^{q_\alpha}(y)dy\right)^{1/q_\alpha}}{\left(\int_{\mathbb{R}^n} U^{(q_\alpha-1)p_\alpha}dx\right)^{1/p_\alpha}} \le N_{n,\alpha}. \end{aligned}$$

Namely, $U^{q_\alpha-1}(x)$ satisfies

$$N_{n,\alpha} = \left(\int_{\mathbb{R}^n} I_\alpha (U^{q_\alpha-1})^{q_\alpha}(y)dy\right)^{1/q_\alpha}, \qquad \int_{\mathbb{R}^n} U^{(q_\alpha-1)p_\alpha}dx = 1.$$

Then, using stereographic projection, we construct a maximizer $f(\xi)$ on $\mathbb{S}^n$ as

$$f(\xi) = (1+\xi_{n+1})^{-\frac{n+\alpha}{2}} U^{q_\alpha-1}(\mathcal{S}^{-1}(\xi)).$$

Now, we give the proof of (2.19). In fact, noting that

$$0 \le (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha-q)} \le 1,$$

we need only to prove

$$\liminf_{p\to {}_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha-q)} = 1.$$

If

$$\liminf_{p\to p_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha-q)} = 0,$$

then there exists a subsequence of $\{\lambda_p\}$ (still denoted as $\{\lambda_p\}$) such that

$$\lim_{p\to p_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha-q)} = 0.$$

So

$$\left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q)} U_p^{q-1}(y) \to 0 \quad \text{uniformly on} \quad B(0,R),$$

and for any $x\in\mathbb{R}^n$,

$$V(x) = \lim_{p\to p_\alpha^+} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} \left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q)} U_p^{q-1}(y)dy = 0. \tag{2.22}$$

By (2.13) and (2.22), we get $U(0)=0$, which is contradict to $U(0)=1$. Therefore,

$$\liminf_{p\to p_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha-q)} > 0.$$

If

$$\liminf_{p_m \to_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha - q)} = c \in (0,1),$$

then there exists a subsequence of $\{\lambda_p\}$ (still denoted as $\{\lambda_p\}$) such that

$$\begin{aligned} &\lim_{p_m \to_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha - q)} = c, \\ &\left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^{q-1}(y) \to cU^{q_\alpha - 1}(y) \quad \text{uniformly on} \quad B(0,R). \end{aligned}$$

It deduces that

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} \left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^q(y)dy \\ &> \int_{|y|\le R} \left(\frac{2\lambda_p}{1+|\lambda_p y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q)} U_p^q(y)dy \to c\int_{|y|\le R} U^{q_\alpha}(y)dy, \end{aligned}$$

which implies

$$c\int_{\mathbb{R}^n} U^{q_\alpha}(y)dy \le 1.$$

Then, take the limitation on (2.13) as $p \to p_\alpha^+$ and get

$$\begin{cases} V(x) = c\int_{\mathbb{R}^n} |x-y|^{\alpha-n} U^{q_\alpha - 1}(y)dy = cI_\alpha(U^{q_\alpha-1})(x), \\ N_{n,\alpha}^{q_\alpha} U(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} V^{q_\alpha-1}(y)dy. \end{cases} \tag{2.23}$$

Combining the facts $p_\alpha < q_\alpha$ and $0 < c < 1$, we have

$$\begin{aligned} N_{n,\alpha} &= \frac{c\left(\int_{\mathbb{R}^n} I_\alpha(U^{q_\alpha-1})^{q_\alpha}(y)dy\right)^{1/q_\alpha}}{\left(c\int_{\mathbb{R}^n} U^{(q_\alpha-1)p_\alpha}dx\right)^{1/q_\alpha}} \\ &\le \frac{c\left(\int_{\mathbb{R}^n} I_\alpha(U^{q_\alpha-1})^{q_\alpha}(y)dy\right)^{1/q_\alpha}}{\left(c\int_{\mathbb{R}^n} U^{(q_\alpha-1)p_\alpha}dx\right)^{1/p_\alpha}} \le c^{\frac{1}{q_\alpha}} N_{n,\alpha}, \end{aligned}$$

which is a contradiction. So,

$$\liminf_{p \to p_\alpha^+} (2\lambda_p)^{\frac{n-\alpha}{2}(q_\alpha - q)} = 1.$$

Thus, we complete the proof. □

3 Case of $\alpha > n$

3.1 Subcritical HLS inequalities

Let $p \in (0, p_\alpha)$.

Lemma 3.1. *There exists some positive constant $C = C(n,\alpha,p)$ such that*

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f(\xi)|\xi-\eta|^{\alpha-n}g(\eta)d\xi d\eta \geq C\|f\|_{L^p(\mathbb{S}^n)}\|g\|_{L^p(\mathbb{S}^n)} \tag{3.1}$$

holds for any nonnegative $f,g \in L^p(\mathbb{S}^n)$.

Proof. It is easy to prove that (3.1) holds for any nonnegative $f,g \in L^p(\mathbb{S}^n) \cap L^{p_\alpha}(\mathbb{S}^n)$ by (1.4) and Hölder inequality. Then we can complete the proof by a density argument. □

Define the extremal problem of (3.1) as

$$\begin{aligned}\tilde{N}_{n,\alpha,p} =& \inf\Big\{\int_{\mathbb{S}^n}\int_{\mathbb{S}^n}\frac{f(\xi)g(\eta)}{|\xi-\eta|^{n-\alpha}}d\xi d\eta :\ f\geq 0,\ g\geq 0,\ \|f\|_{L^p(\mathbb{S}^n)} = \|g\|_{L^p(\mathbb{S}^n)} = 1\Big\}\\ =& \inf\Big\{\frac{\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f(\xi)|\xi-\eta|^{\alpha-n}g(\eta)d\xi d\eta}{\|f\|_{L^p(\mathbb{S}^n)}\|g\|_{L^p(\mathbb{S}^n)}} :\ f\geq 0,\ g\geq 0,\ f,g\in L^p(\mathbb{S}^n)\setminus\{0\}\Big\}.\end{aligned} \tag{3.2}$$

Obviously, we know that $\tilde{N}_{n,\alpha,p} \geq C > 0$ by (3.1). Moreover, along the idea of Lemma 3.2 of [5] and Proposition 2.5 of [6], we can prove the attainability of $\tilde{N}_{n,\alpha,p}$.

Proposition 3.1. *There exist a pair of nonnegative functions $f,g \in L^1(\mathbb{S}^n)$ such that*

$$\begin{aligned}&\|f\|_{L^p(\mathbb{S}^n)} = \|g\|_{L^p(\mathbb{S}^n)} = 1,\\ &\tilde{N}_{n,\alpha,p} = \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f(\xi)|\xi-\eta|^{\alpha-n}g(\eta)d\xi d\eta.\end{aligned}$$

Then, they satisfy the following Euler-Lagrange equations

$$\begin{cases}\tilde{N}_{n,\alpha,p}f^{p-1}(\xi) = \int_{\mathbb{S}^n}|\xi-\eta|^{\alpha-n}g(\eta)d\eta,\\ \tilde{N}_{n,\alpha,p}g^{p-1}(\xi) = \int_{\mathbb{S}^n}|\xi-\eta|^{\alpha-n}f(\eta)d\eta.\end{cases} \tag{3.3}$$

Moreover, $f,g \in C^1(\mathbb{S}^n)$ and there exists some positive constant $C = C(n,\alpha,p)$ such that

$$\begin{aligned}&0 < \frac{1}{C} < f,g < C < +\infty,\\ &\|f\|_{C^1(\mathbb{S}^n)}, \|f\|_{C^1(\mathbb{S}^n)} \leq C.\end{aligned}$$

Proof. Without loss of generality, as in the Proposition 2.5 of [6], we can choose a nonnegative minimizing sequence $\{f_j, g_j\}_{j=1}^{+\infty} \subset L^p(\mathbb{S}^n) \times L^p(\mathbb{S}^n)$ such that

$$\|f_j\|_{L^{p_\alpha}(\mathbb{S}^n)} = \|g_j\|_{L^{p_\alpha}(\mathbb{S}^n)} = 1, \quad j = 1, 2, \cdots,$$
$$\tilde{N}_{n,\alpha,p} = \lim_{j\to+\infty} \frac{\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f_j(\xi)|\xi-\eta|^{\alpha-n} g_j(\eta)d\xi d\eta}{\|f_j\|_{L^p(\mathbb{S}^n)}\|g_j\|_{L^p(\mathbb{S}^n)}}.$$

So sequences $\{f_j^p\}_{j=1}^{+\infty}$ and $\{g_j^p\}_{j=1}^{+\infty}$ are bounded in $L^{p_\alpha/p}(\mathbb{S}^n)$. By the theory of reflexive space, we know that there exist two subsequences of $\{f_j^p\}$ and $\{g_j^p\}$ (still denoted by $\{f_j^p\}$ and $\{g_j^p\}$) and two nonnegative functions $f^p, g^p \in L^{p_\alpha/p}(\mathbb{S}^n)$ such that

$$f_j^p \rightharpoonup f^p \quad \text{and} \quad g_j^p \rightharpoonup g^p \quad \text{weakly in } L^{p_\alpha/p}(\mathbb{S}^n).$$

Since $1 \in L^{p_\alpha/(p_\alpha-p)}(\mathbb{S}^n)$, then

$$\int_{\mathbb{S}^n} f_j^p d\xi \to \int_{\mathbb{S}^n} f^p d\xi, \quad \int_{\mathbb{S}^n} g_j^p d\xi \to \int_{\mathbb{S}^n} g^p d\xi \quad \text{as } j\to+\infty. \tag{3.4}$$

Moreover, if we can prove that

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f(\xi)|\xi-\eta|^{\alpha-n} g(\eta)d\xi d\eta \le \liminf_{j\to+\infty} \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f_j(\xi)|\xi-\eta|^{\alpha-n} g_j(\eta)d\xi d\eta, \tag{3.5}$$

then we can prove the attainability of $\tilde{N}_{n,\alpha,p}$.

To prove (3.5), we need the claim

$$\|f_i\|_{L^1(\mathbb{S}^n)} \le C, \quad \|f_i\|_{L^1(\mathbb{S}^n)} \le C \quad \text{uniformly}. \tag{3.6}$$

For readability, assume firstly that the claim holds and relegate the proof to the end.

By (3.6), we have up to a subsequence that

$$f_j^p \rightharpoonup f^p \quad \text{and} \quad g_j^p \rightharpoonup g^p \quad \text{weakly in } L^{1/p}(\mathbb{S}^n), \tag{3.7a}$$

and

$$\int_{\mathbb{S}^n} f_j^p d\xi \ge C > 0, \quad \int_{\mathbb{S}^n} g_j^p d\xi \ge C > 0 \tag{3.7b}$$

via an interpolation inequality. So,

$$\|f\|_{L^1(\mathbb{S}^n)} \le C \quad \text{and} \quad \|g\|_{L^1(\mathbb{S}^n)} \le C.$$

As in Lemma 3.2 of [5], we have that, as $j\to+\infty$,

$$\int_{\mathbb{S}^n} |\xi-\eta|^{\alpha-n} g_j^p(\eta) g^{1-p}(\eta)d\eta \to I_\alpha g(\xi) \tag{3.8}$$

uniformly for all $\xi \in \mathbb{S}^n$. Then, for any $\epsilon > 0$, there exists some $N > 0$ such that for any $j > N$,

$$\Big|\int_{\mathbb{S}^n} |\xi-\eta|^{\alpha-n} g_j^p(\eta) g^{1-p}(\eta) d\eta - I_\alpha g(\xi)\Big| \le \epsilon$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{S}^n} f_j^p(\xi) f^{1-p}(\xi) \int_{\mathbb{S}^n} \frac{g_j^p(\eta) g^{1-p}(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi - \int_{\mathbb{S}^n} f_j^p(\xi) f^{1-p}(\xi) \int_{\mathbb{S}^n} \frac{g(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi\Big| \\ \le&\epsilon \int_{\mathbb{S}^n} f_j^p(\xi) f^{1-p}(\xi) d\xi \le C\epsilon. \end{aligned} \tag{3.9}$$

On the other hand, noting $f^{1-p}(\xi) \in L^{1/(1-p)}(\mathbb{S}^n)$ and

$$\int_{\mathbb{S}^n} |\xi-\eta|^{\alpha-n} g(\eta) d\eta \le C \int_{\mathbb{S}^n} g(\eta) d\eta \le C,$$

we have by the weak convergence that, as $j \to +\infty$,

$$\int_{\mathbb{S}^n} f_j^p(\xi) f^{1-p}(\xi) \int_{\mathbb{S}^n} \frac{g(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi \ \ \to \ \ \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(\xi) |\xi-\eta|^{\alpha-n} g(\eta) d\eta d\xi. \tag{3.10}$$

Combining (3.9) and (3.10), it holds that

$$\lim_{j\to+\infty} \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{f_j^p(\xi) f^{1-p}(\xi) g_j^p(\eta) g^{1-p}(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi = \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{f(\xi) g(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi. \tag{3.11}$$

Then, for any $\epsilon > 0$, there exists some $N_1 > 0$ such that for any $j > N_1$,

$$\begin{aligned}&\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{f(\xi) g(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi - \epsilon \\ \le& \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{f_j^p(\xi) f^{1-p}(\xi) g_j^p(\eta) g^{1-p}(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi \\ \le& \Big(\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{f_j(\xi) g_j(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi\Big)^p \Big(\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{f(\xi) g(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta d\xi\Big)^{1-p}. \end{aligned}$$

Because of the arbitrariness of ϵ, we obtain (3.5) and then prove that the function pair $(f,g) \in L^1(\mathbb{S}^n) \times L^1(\mathbb{S}^n)$ is a minimizer.

Now, we give the proof of the claim (3.6). From

$$\|f_j\|_{L^{p_\alpha}(\mathbb{S}^n)} = \|g_j\|_{L^{p_\alpha}(\mathbb{S}^n)} = 1,$$

we get

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f_j(\xi) |\xi-\eta|^{\alpha-n} g_j(\eta) d\xi d\eta \ge C_1 > 0$$

by (1.4) and

$$\|f_j\|_{L^p(\mathbb{S}^n)} \leq C_2, \quad \|g_j\|_{L^p(\mathbb{S}^n)} \leq C_2,$$

by Hölder's inequality. Then, by the definition of $N_{n,\alpha,p}$, we have

$$\|f_j\|_{L^p(\mathbb{S}^n)} \geq C_3 > 0, \quad \|g_j\|_{L^p(\mathbb{S}^n)} \geq C_3 > 0,$$
$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f_j(\xi)|\xi-\eta|^{\alpha-n} g_j(\eta)d\xi d\eta \leq C_4.$$

It follows, via the reversed Hölder's inequality, that

$$\|I_\alpha f_j\|_{L^{p'}(\mathbb{S}^n)} \leq C_5 < \infty \quad \text{and} \quad \|I_\alpha g_j\|_{L^{p'}(\mathbb{S}^n)} \leq C_5 < \infty,$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Then, by a similar argument as the proof of Lemma 3.2 of [5], we have (3.6).

By renormalization, we assume that the minimizer $f,g \in L^1(\mathbb{S}^n)$ satisfy

$$\|f\|_{L^p(\mathbb{S}^n)} = \|g\|_{L^p(\mathbb{S}^n)} = 1.$$

Then, f, g satisfy the Euler-Lagrange equations (3.3).

Since $f,g \in L^1(\mathbb{S}^n)$ and $0<p<p_\alpha<1$, it is easy to prove from (3.3) that $f \geq C_6 > 0$ and $g \geq C_6 > 0$. Then, by (3.3), we have $f < C_7$ and $g < C_7$. Moreover, since $\alpha > n \geq 1$, we have $f,g \in C^1(\mathbb{S}^n)$ and

$$\|f\|_{C^1(\mathbb{S}^n)}, \|g\|_{C^1(\mathbb{S}^n)} \leq C_8 < +\infty.$$

Thus, we complete the proof. □

3.2 Minimizer of critical HLS inequalities, namely Theorem 1.2

As Lemma 2.1 and Lemma 2.2, we have

Lemma 3.2. *$\tilde{N}_{n,\alpha,p} \to \tilde{N}_{n,\alpha}$ as $p \to p_\alpha^-$ and the corresponding minimizer pairs $\{f_p, g_p\} \in C^1(\mathbb{S}^n) \times C^1(\mathbb{S}^n)$ satisfying*

$$\|f_p\|_{L^p(\mathbb{S}^n)} = \|g_p\|_{L^p(\mathbb{S}^n)} = 1$$

form a minimizing sequence for sharp constant $\tilde{N}_{n,\alpha}$, namely,

$$\tilde{N}_{n,\alpha} = \lim_{p \to p_\alpha^-} \frac{\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f_p(\xi)|\xi-\eta|^{\alpha-n} g_p(\eta)d\xi d\eta}{\|f_p\|_{L^{p_\alpha}(\mathbb{S}^n)}\|g_p\|_{L^{p_\alpha}(\mathbb{S}^n)}}.$$

Proof. Since the proof is similar to Lemma 2.1 and Lemma 2.2, then we omit the details for conciseness. □

Proof of Theorem 1.2. As in Lemma 3.2, we take the minimizer $\{f_p, g_p\} \in C^1(\mathbb{S}^n) \times C^1(\mathbb{S}^n)$ as a minimizing sequence for $\tilde{N}_{n,\alpha}$. Then, $\{f_p, g_p\}$ satisfy (3.3). By the rotational invariance, we assume without loss the generality that

$$f_p(\mathfrak{N}) = \max_{\xi \in \mathbb{S}^n} f_p(\xi).$$

Case 1: For some subsequence $p_j \to p_\alpha^-$, $\max\{\max_{\xi\in\mathbb{S}^n} f_{p_j}, \max_{\xi\in\mathbb{S}^n} g_{p_j}\}$ is uniformly bounded. Then, sequences $\{f_{p_j}\}$ and $\{g_{p_j}\}$ are uniformly bounded and equicontinuous on $\mathbb{S}^n$. Moreover, by (3.3), there exists some positive constant C independent of p_j such that $f_{p_j}, g_{p_j} \geq C > 0$. So, by Arzelà-Ascoli theorem, there exist two subsequences of $\{f_{p_j}\}$ and $\{g_{p_j}\}$ (still denoted by $\{f_{p_j}\}$ and $\{g_{p_j}\}$) and two nonnegative functions $f, g \in C^1(\mathbb{S}^n)$ such that

$$f_{p_j} \to f \quad \text{and} \quad g_{p_j} \to g \quad \text{uniformly on} \quad \mathbb{S}^n.$$

Then,

$$\begin{aligned} \int_{\mathbb{S}^n} f^{p_\alpha}(\xi) d\xi &= \lim_{p_j \to p_\alpha} \int_{\mathbb{S}^n} f_{p_j}^{p_j}(\xi) d\xi = 1, \\ \int_{\mathbb{S}^n} g^{p_\alpha}(\xi) d\xi &= \lim_{p_j \to p_\alpha} \int_{\mathbb{S}^n} g_{p_j}^{p_j}(\xi) d\xi = 1, \end{aligned}$$

and by (3.3) and Lemma 3.2, as $j \to +\infty$

$$\begin{cases} \tilde{N}_{n,\alpha} f^{p-1}(\xi) = \displaystyle\int_{\mathbb{S}^n} |\xi - \eta|^{\alpha-n} g(\eta) d\eta, \\ \tilde{N}_{n,\alpha} g^{p-1}(\xi) = \displaystyle\int_{\mathbb{S}^n} |\xi - \eta|^{\alpha-n} f(\eta) d\eta, \end{cases}$$

namely, $\{f, g\}$ are minimizers.

Case 2: For any subsequence $p_j \to p_\alpha^-$, $f_{p_j}(\mathfrak{N}) \to +\infty$ or $\max_{\xi\in\mathbb{S}^n} g_{p_j} \to +\infty$. Without loss of generality, we assume $f_{p_j}(\mathfrak{N}) \to +\infty$.

Case 2a:

$$\limsup_{j\to+\infty} \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^n} g_{p_j}} = +\infty.$$

Then, there exists a subsequence of $\{p_j\}$ (still denoted by $\{p_j\}$) such that

$$f_{p_j}(\mathfrak{N}) \to +\infty \quad \text{and} \quad \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^n} g_{p_j}} \to +\infty.$$

Let $u_j = f_{p_j}^{p_j-1}$ and $v_j = g_{p_j}^{p_j-1}$. Then, u_j and v_j satisfy

$$\int_{\mathbb{S}^n} u_j^{q_j} d\xi = \int_{\mathbb{S}^n} v_j^{q_j} d\xi = 1 \tag{3.12}$$

and by (3.3),

$$\begin{cases} \tilde{N}_{n,\alpha,p_j} u_j(\xi) = \int_{\mathbb{S}^n} |\xi-\eta|^{\alpha-n} v_j^{q_j-1}(\eta) d\eta, \\ \tilde{N}_{n,\alpha,p_j} v_j(\xi) = \int_{\mathbb{S}^n} |\xi-\eta|^{\alpha-n} u_j^{q_j-1}(\eta) d\eta, \end{cases} \tag{3.13}$$

where $\frac{1}{p_j}+\frac{1}{q_j}=1$. Applying stereographic projection and dilations on $\mathbb{R}^n$, we get from (3.13) that

$$\begin{cases} \tilde{N}_{n,\alpha,p_j} \left(\frac{2}{1+|\lambda x|^2}\right)^{\frac{n-\alpha}{2}} u_j(\mathcal{S}(\lambda x)) = \lambda^\alpha c \int_{\mathbb{R}^n} \frac{\left(\frac{2}{1+|\lambda y|^2}\right)^{\frac{n+\alpha}{2}} v_j(\mathcal{S}(\lambda y))^{q_j-1}}{|x-y|^{n-\alpha}} dy, \\ \tilde{N}_{n,\alpha,p_j} \left(\frac{2}{1+|\lambda x|^2}\right)^{\frac{n-\alpha}{2}} v_j(\mathcal{S}(\lambda x)) = \lambda^\alpha \int_{\mathbb{R}^n} \frac{\left(\frac{2}{1+|\lambda y|^2}\right)^{\frac{n+\alpha}{2}} u_j(\mathcal{S}(\lambda y))^{q_j-1}}{|x-y|^{n-\alpha}} dy. \end{cases} \tag{3.14}$$

Take $\lambda=\lambda_j$ satisfying

$$2^{(n-\alpha)/2} \lambda_j^{\alpha/(q_j-2)} u_j(\mathcal{S}(0)) = 1$$

and denote

$$\begin{cases} U_j(x) = \lambda_j^{\frac{\alpha}{q_j-2}-\frac{\alpha}{q_\alpha-2}} \left(\frac{2\lambda_j}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}} u_j(\mathcal{S}(\lambda_j x)), \\ V_j(x) = \lambda_j^{\frac{\alpha}{q_j-2}-\frac{\alpha}{q_\alpha-2}} \left(\frac{2\lambda_j}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}} v_j(\mathcal{S}(\lambda_j x)). \end{cases} \tag{3.15}$$

Then, U_j, V_j satisfy the following renormalized equations

$$\begin{cases} \tilde{N}_{n,\alpha,p_j} U_j(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y) dy, \\ \tilde{N}_{n,\alpha,p_j} V_j(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} U_j^{q_j-1}(y) dy, \end{cases} \tag{3.16}$$

$U_j(x) \geq U_j(0) = 1$ and

$$V_j(x) \geq \lambda_j^{\alpha/(q_j-2)} 2^{(n-\alpha)/2} \min_{\xi\in\mathbb{S}^n} v_j = \frac{\min_{\xi\in\mathbb{S}^n} v_j}{u_j(\mathcal{S}(0))} \to +\infty \tag{3.17}$$

uniformly for any x as $j\to+\infty$.

Claim: There exist $C_1, C_2 > 0$ such that, for any x, when $j\to\infty$

$$0 < C_1(1+|x|^{\alpha-n}) \leq U_j(x) \leq C_2(1+|x|^{\alpha-n}) \quad \text{uniformly}. \tag{3.18}$$

Once the claim holds,

$$\begin{aligned}\tilde{N}_{n,\alpha,p_j}V_j(0)=&\int_{\mathbb{R}^n}|y|^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}U_j^{q_j-1}(y)dy\\ \le& C\int_{\mathbb{R}^n}|y|^{\alpha-n}(1+|y|^{\alpha-n})^{q_j-1}dy\le C,\end{aligned}$$

which is contradict to (3.17). So, Case 2a does not appear.

Now, we give the proof of the claim (3.18). Noting that

$$\begin{aligned}\tilde{N}_{n,\alpha,p_j}=&\int_{\mathbb{R}^n}|y|^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\\ \le& C<+\infty \end{aligned}\tag{3.19}$$

uniformly as $j\to\infty$, we obtain from (3.17) that as $j\to\infty$ and $|x|\ge 1$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\\ \le&\int_{|y|\le 1}C\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}dy\\ &+\int_{|y|>1}|y|^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\\ \le& C<+\infty\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{|x-y|^{\alpha-n}}{|x|^{\alpha-n}}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\\ \le& C\int_{\mathbb{R}^n}(1+|y|^{\alpha-n})\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\\ \le& C<+\infty\end{aligned}\tag{3.21}$$

uniformly. Then, by dominated convergence theorem,

$$\lim_{|x|\to+\infty}\frac{U_j(x)}{|x|^{\alpha-n}}=\frac{1}{\tilde{N}_{n,\alpha,p_j}}\int_{\mathbb{R}^n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\le C.\tag{3.22}$$

On the other hand, if we can prove

$$\int_{\mathbb{R}^n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}V_j^{q_j-1}(y)dy\ge c_1>0\quad\text{as }\ j\to\infty,\tag{3.23}$$

then we have the claim (3.18). By contradiction, we assume that (3.23) does not hold. Then, there exists a subsequence of $\{V_j\}$ (still denoted as $\{V_j\}$) such that

$$\int_{\mathbb{R}^n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y)dy\to 0 \quad \text{as } j\to+\infty. \tag{3.24}$$

For any $x\in B(0,R_0)$, where R_0 is given arbitrarily, and taking $R>>R_0$ large enough,

$$\begin{aligned}1\le &U_j(x)=\frac{1}{\tilde{N}_{n,\alpha,p_j}}\int_{\mathbb{R}^n}|x-y|^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y)dy\\ \le&\frac{1}{\tilde{N}_{n,\alpha,p_j}}\Big((R+R_0)^{\alpha-n}\int_{|y|\le R}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y)dy\\ &+\int_{|y|>R}\Big(1+\frac{R_0}{R}\Big)^{\alpha-n}|y|^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y)dy\Big)\\ \le&\frac{(R+R_0)^{\alpha-n}}{\tilde{N}_{n,\alpha,p_j}}\int_{|y|\le R}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y)dy+\Big(1+\frac{R_0}{R}\Big)^{\alpha-n}.\end{aligned}$$

Then, for any $\epsilon>0$, we firstly take R large enough and then let j large enough such that

$$1\le U_j(x)\le 1+\epsilon,$$

which imply that $U_j(x)\to 1$, $x\in B(0,R_0)$ uniformly as $j\to\infty$. Then, for $|x|\ge 2$ and by (3.16), we have

$$\begin{aligned}V_j(x)\ge&\frac{1}{\tilde{N}_{n,\alpha,p_j}}\int_{|y|\le 1}|x-y|^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} U_j^{q_j-1}(y)dy\\ \ge&\frac{1}{\tilde{N}_{n,\alpha,p_j}}\int_{|y|\le 1}\Big(\frac{|x|}{2}\Big)^{\alpha-n}\left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} U_j^{q_j-1}(y)dy\\ \ge&C|x|^{\alpha-n},\end{aligned}\tag{3.25}$$

where, in the last inequality, we have used facts: as $j\to\infty$,

$$U_j(y)\to 1 \quad\text{and}\quad \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}\to 1 \quad \text{uniformly on } B(0,1).$$

Then, by (3.25) and letting p_j close to p_α, we take $R >> 2$ and obtain

$$\begin{aligned}\tilde{N}_{n,\alpha,p_j} =& \int_{\mathbb{R}^n} |y|^{\alpha-n} \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q_j)} V_j^{q_j-1}(y)dy \\ \leq & R^{\alpha-n} \int_{|y|\leq R} \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q_j)} V_j^{q_j-1}(y)dy \\ &+ \int_{|y|>R} |y|^{\alpha-n} \cdot C|y|^{(\alpha-n)(q_j-1)} dy \\ \leq & R^{\alpha-n} \int_{|y|\leq R} \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q_j)} V_j^{q_j-1}(y)dy + CR^{(\alpha-n)q_j+n}.\end{aligned}$$

Taking firstly R large enoug and then letting $j \to +\infty$, we have $\tilde{N}_{n,\alpha,p_j} \to 0$, which is contradict to $\tilde{N}_{n,\alpha,p_j} \to N_{n,\alpha}$. So, (3.23) holds.

Case 2b:

$$\limsup_{j\to+\infty} \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^n} g_{p_j}} = 0.$$

Then, there exists a subsequence of $\{p_j\}$ (still denoted as $\{p_j\}$) such that

$$f_{p_j}(\mathfrak{N}) \to +\infty \quad \text{and} \quad \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^n} g_{p_j}} \to 0,$$

which implies that $\max_{\xi\in\mathbb{S}^n} g_{p_j} \to +\infty$. Similar to Case 2a, we can prove that Case 2b does not appear.

Case 2c:

$$\limsup_{j\to+\infty} \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^n} g_{p_j}} = c_0 \in (0,+\infty).$$

Then, there exists a subsequence of $\{p_j\}$ (still denoted as $\{p_j\}$) such that

$$f_{p_j}(\mathfrak{N}) \to +\infty, \quad \max g_{p_j} \to +\infty \quad \text{and} \quad \frac{f_{p_j}(\mathfrak{N})}{\max_{\xi\in\mathbb{S}^n} g_{p_j}} \to c_0 \in (0,+\infty).$$

As Case 2a, we introduce function pairs $\{U_j, V_j\}$ defined as (3.15) and then they satisfy (3.16), $U_j(x) \geq U_j(0) = 1$ and

$$V_j(x) \geq \lambda_j^{\alpha/(q_j-2)} 2^{(n-\alpha)/2} \min_{\xi\in\mathbb{S}^n} v_j = \frac{\min_{\xi\in\mathbb{S}^n} v_j}{u_j(\mathcal{S}(0))} \to c_0^{1-p_\alpha} \in (0,+\infty) \tag{3.26}$$

uniformly for any x as $j \to +\infty$. So, $\{V_j(x)\}$ have uniformly lower bound $C > 0$.

Repeating the proof of (3.18), there exist two positive constant C_1 and C_2 such that, as $j \to +\infty$,

$$0 < C_1(1+|x|^{\alpha-n}) \le U_j(x) \le C_2(1+|x|^{\alpha-n}), \tag{3.27a}$$
$$0 < C_1(1+|x|^{\alpha-n}) \le V_j(x) \le C_2(1+|x|^{\alpha-n}), \tag{3.27b}$$

uniformly for any x.

For any given constant $R_0 > 0$ and any $x \in B(0,R_0)$, as $j \to +\infty$, we have by (3.27b) that

$$\begin{aligned}
\tilde{N}_{n,\alpha,p_j} U_j(x) =& \int_{\mathbb{R}^n} |y|^{\alpha-n} \left(\frac{2}{1+|\lambda_j(x-y)|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha - q_j)} V_j^{q_j-1}(x-y)dy \\
\le& \int_{|y|\le 2R_0} |y|^{\alpha-n} 2^{\frac{n-\alpha}{2}(q_\alpha-q_j)} C_1^{q_j-1} dy \\
&+ \int_{|y|>2R_0} |y|^{\alpha-n} 2^{\frac{n-\alpha}{2}(q_\alpha-q_j)} C_1^{q_j-1} |x-y|^{(\alpha-n)(q_j-1)} dy \\
\le& C(2R_0)^\alpha + C\int_{|y|>2R_0} |y|^{\alpha-n}\left(\frac{|y|}{2}\right)^{(\alpha-n)(q_j-1)} dy \le C,
\end{aligned} \tag{3.28}$$

namely, $U_j(x)$ is uniformly bounded on $B(0,R_0)$. Similarly, $V_j(x)$ is also uniformly bounded on $B(0,R_0)$.

By a similar computation of (3.28) and noting $\alpha > n \ge 1$, we have that, as $j \to +\infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x-y|^{\alpha-n-1} \left(\frac{2}{1+|\lambda_j y|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)} V_j^{q_j-1}(y)dy \\
\le& C(2R_0)^{\alpha-1} + C(2R_0)^{(\alpha-n)q_j-1+n} \le C
\end{aligned}$$

uniformly for any $x \in B(0,R_0)$. Since the arbitrariness of R_0, we know that $U_j(x) \in C^1(\mathbb{R}^n)$ and $\|U_j\|_{C^1(B(0,R_0))}$ is uniformly bounded. Similarly, we can prove that $V_j(x) \in C^1(\mathbb{R}^n)$ and $\|V_j\|_{C^1(B(0,R_0))}$ is uniformly bounded.

By Arzelà-Ascoli theorem, there exist two subsequences of $\{U_j\}$ and $\{V_j\}$ (still denoted as $\{U_j\}$ and $\{V_j\}$) and two functions $U, V \in C^1(\mathbb{R}^n)$ with lower bound $C > 0$ such that

$$U_j \to U \quad \text{and} \quad V_j \to V \quad \text{uniformly on} \quad B(0,R_0). \tag{3.29}$$

Combining the arbitrariness of R_0, we can prove that $U(x)$ and $V(x)$ satisfy

$$\begin{cases}
\tilde{N}_{n,\alpha} U(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} V^{q_\alpha-1}(y)dy & \text{in } \mathbb{R}^n, \\
\tilde{N}_{n,\alpha} V(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} U^{q_\alpha-1}(y)dy & \text{in } \mathbb{R}^n.
\end{cases} \tag{3.30}$$

Since

$$1=\int_{\mathbb{S}^n} u_j^{q_j}(\xi)d\xi=\int_{\mathbb{R}^n} U_j^{q_j}(x)\lambda_j^{n-\frac{\alpha q_j}{q_j-2}}\left(\frac{2}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}dx$$
$$\leq\int_{\mathbb{R}^n} U_j^{q_j}(x)\left(\frac{2}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}dx$$

and

$$U_j^{q_j}(x)\left(\frac{2}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}\to U^{q_\alpha}(x)$$

uniformly on any compact domain, then it holds by (3.27a) that

$$\int_{\mathbb{R}^n} U^{q_\alpha}dx=\lim_{j\to+\infty}\int_{\mathbb{R}^n} U_j^{q_j}(x)\left(\frac{2}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}dx\geq 1.$$

Similarly, it also holds by (3.27b) that

$$\int_{\mathbb{R}^n} V^{q_\alpha}dx=\lim_{j\to+\infty}\int_{\mathbb{R}^n} V_j^{q_j}(x)\left(\frac{2}{1+|\lambda_j x|^2}\right)^{\frac{n-\alpha}{2}(q_\alpha-q_j)}dx\geq 1.$$

Let

$$F(x)=U^{q_\alpha-1}(x)\quad\text{and}\quad G(x)=V^{q_\alpha-1}(x).$$

Then

$$\int_{\mathbb{R}^n} F^{p_\alpha}dx\geq 1,\quad \int_{\mathbb{R}^n} G^{p_\alpha}dx\geq 1,$$

and F, G satisfy

$$\begin{cases}\tilde{N}_{n,\alpha}F^{p_\alpha-1}(x)=\int_{\mathbb{R}^n}|x-y|^{\alpha-n}G(y)dy & \text{in } \mathbb{R}^n,\\ \tilde{N}_{n,\alpha}G^{p_\alpha-1}(x)=\int_{\mathbb{R}^n}|x-y|^{\alpha-n}F(y)dy & \text{in } \mathbb{R}^n.\end{cases}$$

Combining $2>p_\alpha$, it holds that

$$N_{n,\alpha}=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}F(x)|x-y|^{\alpha-n}G(y)dxdy\quad\text{and}\quad \|F\|_{L^{p_\alpha}(\mathbb{R}^n)}=\|G\|_{L^{p_\alpha}(\mathbb{R}^n)}=1.$$

Applying stereographic projection, we can construct a pair of minimizer $\{f(\xi),g(\xi)\}$ on $\mathbb{S}^n$ as

$$f(\xi)=(1+\xi_{n+1})^{-\frac{n+\alpha}{2}}F(\mathcal{S}^{-1}(\xi)),$$
$$g(\xi)=(1+\xi_{n+1})^{-\frac{n+\alpha}{2}}G(\mathcal{S}^{-1}(\xi)).$$

Thus, we complete the proof. □

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