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Role of Fluxes in High-Order Godunov Schemes†

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Abstract. This paper focuses on the evolution of some mathematical aspects related to high-resolution approximations to nonlinear hyperbolic balance laws. It addresses the crucial role of numerical fluxes in dealing with the three concepts of *consistency, stability and convergence.* The classical paper [15] by S. K. Godunov had a revolutionary effect on the field of numerical simulations of compressible fluid flows. The seminal paper of van Leer [30] has inaugurated the period of universal interest in high-resolution extensions of Godunov's scheme. The fundamental step consists of modifying the (locally) self-similar solution to the Riemann Problem (at discontinuities) by allowing piecewise polynomial (rather than piecewise constant) initial data. The GRP (Generalized Riemann Problem) analysis [1] provided analytical solutions (for piecewise linear data) that could be readily implemented in a high-resolution robust code. The treatment utilizes the framework of "balance laws", a common viewpoint in relevant physical conservation laws. The first significant observation is that under very mild conditions a weak solution is indeed a solution to the balance law (obtained by a formal application of the Gauss-Green formula), and the associated fluxes are Lipschitz continuous with respect to the spatial coordinates. Since high-resolution schemes require the computation of several quantities per mesh cell (e.g., slopes), the notion of "flux consistency" must be extended to this framework. A combination of consistency hypothesis with stability of the scheme leads to a suitable convergence theorem, generalizing the classical convergence theorem of Lax and Wendroff [17].

AMS subject classifications: 65M06, 35L67, 35L65, 76N15

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1 Introduction

The seminal paper [15] by S. K. Godunov had a revolutionary effect on the field of numerical simulations of compressible fluid flows. However the initial evolution of this

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effect was quite slow, when compared to other parallel advances related to the numerical simulations of fluid flows (e.g., vortex methods or finite elements). Eight years after Godunov's publication, the classical book of R.D. Richtmyer and K.W. Morton characterized his paper as follows [23, Section 12.15]:

"In 1959, Godunov described an ingenious method for one-dimensional problems with shocks." Yet, later on in the same section, they add the comment: "The method appears to have been extensively used in the Soviet Union."

The principles underlying the Godunov approach can be described as follows.

PRINCIPLES OF THE GODUNOV APPROACH

- Determining consistent numerical fluxes based on a local solution of the exact system.
- Invoking upwinding in computing the fluxes, i.e. selecting a unique local entropy solution.
- Updating the numerical solution via a discretized balance law, rather than a finite difference scheme.

The seminal paper of van Leer [30] has inaugurated the period of universal interest in high-resolution extensions of Godunov's scheme.

We concentrate here on a close inspection of the evaluation of numerical fluxes and their role in securing a high-resolution approximate solution. The first step to take is to modify the (locally) self-similar solution to the Riemann Problem (at discontinuities) by allowing piecewise polynomial (rather than piecewise-constant) initial data.

Our purpose is to present a general approach to the concept of *consistency* of numerical fluxes, in the framework of piecewise polynomial approximations. A fundamental fact is that the fluxes associated with any weak solution are Lipschitz continuous. They serve therefore as natural candidates for approximation. More explicitly, it is clear that a discontinuous function is less amenable to be reasonably approximated by regular functions. Thus, the flux associated with the exact solution is naturally approximated by an analytical evaluation of a flux associated with initial piecewise polynomial data. This is the basis of the GRP (Generalized Riemann Problem) method, as a direct analytical extension of the Godunov flux.

2 Fluxes – the heart of the matter

Hyperbolic conservation laws are often written in the divergence form of partial differential equations,

$$\mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \tag{2.1}$$