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Role of Fluxes in High-Order Godunov Schemes[†]

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Abstract. This paper focuses on the evolution of some mathematical aspects related to high-resolution approximations to nonlinear hyperbolic balance laws. It addresses the crucial role of numerical fluxes in dealing with the three concepts of *consistency, stability and convergence.* The classical paper [15] by S. K. Godunov had a revolutionary effect on the field of numerical simulations of compressible fluid flows. The seminal paper of van Leer [30] has inaugurated the period of universal interest in high-resolution extensions of Godunov's scheme. The fundamental step consists of modifying the (locally) self-similar solution to the Riemann Problem (at discontinuities) by allowing piecewise polynomial (rather than piecewise constant) initial data. The GRP (Generalized Riemann Problem) analysis [1] provided analytical solutions (for piecewise linear data) that could be readily implemented in a high-resolution robust code. The treatment utilizes the framework of "balance laws", a common viewpoint in relevant physical conservation laws. The first significant observation is that under very mild conditions a weak solution is indeed a solution to the balance law (obtained by a formal application of the Gauss-Green formula), and the associated fluxes are Lipschitz continuous with respect to the spatial coordinates. Since high-resolution schemes require the computation of several quantities per mesh cell (e.g., slopes), the notion of "flux consistency" must be extended to this framework. A combination of consistency hypothesis with stability of the scheme leads to a suitable convergence theorem, generalizing the classical convergence theorem of Lax and Wendroff [17].

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1 Introduction

The seminal paper [15] by S. K. Godunov had a revolutionary effect on the field of numerical simulations of compressible fluid flows. However the initial evolution of this

[†]Dedicated to the memory of Professor Jiequan Li.

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effect was quite slow, when compared to other parallel advances related to the numerical simulations of fluid flows (e.g., vortex methods or finite elements). Eight years after Godunov's publication, the classical book of R.D. Richtmyer and K.W. Morton characterized his paper as follows [23, Section 12.15]:

"In 1959, Godunov described an ingenious method for one-dimensional problems with shocks." Yet, later on in the same section, they add the comment: "The method appears to have been extensively used in the Soviet Union."

The principles underlying the Godunov approach can be described as follows.

PRINCIPLES OF THE GODUNOV APPROACH

- Determining consistent numerical fluxes based on a local solution of the exact system.
- Invoking upwinding in computing the fluxes, i.e. selecting a unique local entropy solution.
- Updating the numerical solution via a discretized balance law, rather than a finite difference scheme.

The seminal paper of van Leer [30] has inaugurated the period of universal interest in high-resolution extensions of Godunov's scheme.

We concentrate here on a close inspection of the evaluation of numerical fluxes and their role in securing a high-resolution approximate solution. The first step to take is to modify the (locally) self-similar solution to the Riemann Problem (at discontinuities) by allowing piecewise polynomial (rather than piecewise-constant) initial data.

Our purpose is to present a general approach to the concept of *consistency* of numerical fluxes, in the framework of piecewise polynomial approximations. A fundamental fact is that the fluxes associated with any weak solution are Lipschitz continuous. They serve therefore as natural candidates for approximation. More explicitly, it is clear that a discontinuous function is less amenable to be reasonably approximated by regular functions. Thus, the flux associated with the exact solution is naturally approximated by an analytical evaluation of a flux associated with initial piecewise polynomial data. This is the basis of the GRP (Generalized Riemann Problem) method, as a direct analytical extension of the Godunov flux.

2 Fluxes – the heart of the matter

Hyperbolic conservation laws are often written in the divergence form of partial differential equations,

$$\mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \tag{2.1}$$

where t is the time of variable, $\nabla_{\mathbf{x}}\cdot$ is the divergence operator in terms of space variable $\mathbf{x}=(x_1,\cdots,x_m)$, $\mathbf{u}=(u_1,\cdots,u_D)^\top\in\mathbb{R}^D$ is the vector of conserved quantities and $\mathbf{f}(\mathbf{u})$ is the matrix of fluxes

$$\mathbf{f}(\mathbf{u})=(f_1(\mathbf{u}),\cdots,f_D(\mathbf{u}))\in\mathbb{R}^m\times\mathbb{R}^D, \tag{2.2}$$

and each $\mathbf{f}_i(\mathbf{u})$ is an m-vector. We only assume that the flux $\mathbf{f}(\mathbf{u})$ is locally bounded as the function of $\mathbf{u}$.

Since classical solutions of (2.1) in general break down and discontinuities appear in the solutions even when subject to very smooth initial data

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \tag{2.3}$$

we resort to the notion of a weak solution of (2.1) and (2.3), namely, the solution is defined in distributional sense:

Definition 2.1 (Weak solutions)**.** The function $\mathbf{u}\in L^\infty\cap L^1(\mathbb{R}^m;[0,T))\otimes\mathbb{R}^D$ is a weak solution of (2.1) and (2.3) if it satisfies

$$\int_0^T\int_{\mathbb{R}^m}[\mathbf{u}\phi_t+\mathbf{f}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\phi(\mathbf{x},t)]d\mathbf{x}dt+\int_{\mathbb{R}^m}\mathbf{u}_0(\mathbf{x})\phi(\mathbf{x},0)d\mathbf{x}=0, \tag{2.4}$$

for all smooth test functions $\phi\in C^\infty(\mathbb{R}^m\times[0,T))\otimes\mathbb{R}^D$. Note that ϕ is a D-vector, so that $\mathbf{u}\phi_t=\sum_{i=1}^{D}u_i\cdot\partial_t\phi_i$ and so on.

Alternatively, following the physical point-of-view, one has two approaches to *integral balance laws* in making sense for a solution of (2.1). We proceed to present these approaches.

Let $\Omega\subseteq\mathbb{R}^m$ be a bounded domain, $\Gamma=\partial\Omega$, and $0\le t_1<t<t_2<T$. Let ν be the outward unit normal. We formally apply the Gauss-Green theorem and carry out the integration of (2.1) in space to have:

Definition 2.2 (Instant Integral balance law)**.** Let $\mathbf{u}\in C((0,T);L^\infty(\mathbb{R}^m))\cap C([0,T);L^1(\mathbb{R}^m))\otimes\mathbb{R}^D$. Then

$$\frac{d}{dt}\int_\Omega\mathbf{u}(\mathbf{x},t)d\mathbf{x}=-\int_\Gamma\mathbf{f}(\mathbf{u})\cdot\nu dS_{\mathbf{x}} \tag{2.5}$$

is called the *instant balance law* of (2.1), where $dS_{\mathbf{x}}$ is surface Lebesgue measure, if the following two conditions are satisfied,

(i) For every $t\in[0,T)$, and every bounded domain $\Omega\subseteq\mathbb{R}^m$, the $D-$vector of integrated quantities (or "conserved quantities") $M(\Omega,t)=\int_\Omega\mathbf{u}(\mathbf{x},t)d\mathbf{x}$ is well defined and a continuously differentiable function of t.

(ii) The time dependent function

$$h(\Gamma;t):=\int_\Gamma \mathbf{f}(\mathbf{u})\cdot\nu dS_\mathbf{x} \tag{2.6}$$

is well defined and continuous in time t, and Eq. (2.5) is satisfied. Note that h is a D-vector.

Remark 2.1. In the context of theoretical continuum mechanics the quantity $\int_A \mathbf{f}(\mathbf{u})\cdot\nu dS_\mathbf{x}$ across a section $A\subseteq\Gamma$ is called *the Cauchy flux (across A)* and $\mathbf{f}(\mathbf{u})\cdot\nu$ is its density.

We can go further and integrate (2.5) (formally) over any time interval $[t_1,t_2]$, obtaining the following spacetime integral balance law.

Definition 2.3 (Spacetime integral balance laws)**.** Let the function $\mathbf{u}\in C((0,T);L^\infty(\mathbb{R}^m))\cap C([0,T);L^1(\mathbb{R}^m))\otimes\mathbb{R}^D$. Then

$$\int_\Omega \mathbf{u}(\mathbf{x},t_2)d\mathbf{x}-\int_\Omega \mathbf{u}(\mathbf{x},t_1)d\mathbf{x}=-\int_{t_1}^{t_2}\int_\Gamma \mathbf{f}(\mathbf{u})\cdot\nu dS_\mathbf{x}dt,\quad 0\le t_1<t_2<T, \tag{2.7}$$

is called the *spacetime balance law* of (2.1) if Eq. (2.7) holds, subject to the following conditions:

1. $M(\Omega,t)$ (see (i) in Definition 2.2) is continuous in t.
2. The *spacetime flux* across the boundary Γ over time interval $[t_1,t_2]$,

$$h(\Gamma;t_1,t_2):=\int_{t_1}^{t_2}\int_\Gamma \mathbf{f}(\mathbf{u})\cdot\nu dS_\mathbf{x}dt \tag{2.8}$$

is well defined and continuous with respect to suitable perturbations of the boundary Γ.

Remark 2.2. In analogy to the Cauchy flux, we call

$$H(A;t_1,t_2)=\int_{t_1}^{t_2}\int_A \mathbf{f}(\mathbf{u})\cdot\nu dS_\mathbf{x}dt,\quad A\subseteq\Gamma, \tag{2.9}$$

the *spacetime flux* through the boundary section A.

The validity of the conditions in Definitions 2.2 and 2.3, especially in what concerns the fluxes (2.6) and (2.8) is far from obvious. Note that the solution of (2.1) is typically discontinuous and $\mathbf{f}(\mathbf{u})$ is nonlinear so that the traces of fluxes need to be attended. Recall the following comment concerning this issue: "the drawback of this, functional analytic, demonstration is that it does not provide any clues on how the $q_\mathcal{D}$ may be computed from A" [10, Section 1.3], where $q_\mathcal{D}$ refers to the flux density, and A is the Cauchy flux. In the next section we justify the continuity of the spacetime flux with respect to space perturbation and establish the validity of the spacetime integral balance law (2.7) for a weak solution.

3 Regularity of fluxes admits integral balance laws

In this section we follow [6] in establishing (2.7) for weak solutions in the multidimensional case. In the case of one spatial coordinate this result was obtained in [5]. We start from $\Gamma_0=\Gamma$ and construct a tubular neighborhood [27] with the following properties. For some small $0<\delta<1$ there exists a family of expanding smooth bounded domains $\{\Omega_\mu\subseteq\mathbb{R}^m,\ \mu\in(-\delta,1-\delta)$ so that their respective boundaries $\{\Gamma_\mu,\ \mu\in(-\delta,1-\delta)\}$ form a foliation of a tubular neighborhood of Γ_0. The coordinate μ is normal to Γ_μ so that $\partial/\partial\mu=\nu_{\mathbf{x}}$ is the unit normal. Denote by dS_μ the Lebesgue surface measure on Γ_μ, $\mu\in(-\delta,1-\delta)$.

Theorem 3.1. *Let* $\mathbf{u}\in L^\infty_{loc}(\mathbb{R}^m\times(0,T))\cap L^1_{loc}(\mathbb{R}^m\times[0,T))\bigotimes\mathbb{R}^D$ *be a weak solution of* (2.1) *in the sense of Definition* 2.1. *Then we have*

(i) For every time interval $[t_1,t_2]$ *the function* $\mathbf{g}(\mathbf{x};t_1,t_2)=\int_{t_1}^{t_2}\mathbf{f}(\mathbf{u}(\mathbf{x},t))dt$ *satisfies*

$$\nabla_{\mathbf{x}}\cdot\mathbf{g}(\mathbf{x};t_1,t_2)\in L^\infty_{loc}(\mathbb{R}^m). \tag{3.1}$$

(ii) For every smooth domain and the geometric construction $\{\Omega_\mu\}$ *the trace function defined by*

$$h(\mu;t_1,t_2)=\int_{t_1}^{t_2}\left[\int_{\Gamma_\mu}\mathbf{f}(\mathbf{u})\cdot\nu_\mu dS_\mu\right]dt,\quad \mu\in(-\delta,1-\delta) \tag{3.2}$$

is Lipschitz continuous with respect to μ.

In this case Eq. (2.7) *holds for every* $0\le t_1<t_2\le T$.

Since a considerable part of the theoretical and numerical studies are still carried out in the one-dimensional case ($m=1$), it is useful to state the form of the theorem in this case.

Theorem 3.2. *Let* $\mathbf{u}(x,t)\in L^\infty_{loc}(\mathbb{R}\times(0,T))\cap L^1_{loc}(\mathbb{R}\times[0,T))\bigotimes\mathbb{R}^D$ *be a weak solution to one-dimensional conservation laws*

$$\mathbf{u}_t+\mathbf{f}(\mathbf{u})_x=0,\quad x\in\mathbb{R},\ t>0. \tag{3.3}$$

Then we have:

(i) For every interval $[t_1,t_2]$, *the spacetime flux* $\mathbf{g}(x)=\int_{t_1}^{t_2}\mathbf{f}(\mathbf{u}(x,t))dt$ *is locally Lipschitz continuous in* $x\in\mathbb{R}$.

(ii) the spacetime integral balance law holds over a spacetime domain $Q=[x_1,x_2]\times[t_1,t_2]$

$$\int_{x_1}^{x_2}\mathbf{u}(x,t_2)dx-\int_{x_1}^{x_2}\mathbf{u}(x,t_1)dx=\int_{t_1}^{t_2}\mathbf{f}(\mathbf{u}(x_1,t))dt-\int_{t_1}^{t_2}\mathbf{f}(\mathbf{u}(x_2,t))dt. \tag{3.4}$$

The proof can be found in [6] for Theorem 3.1 and in [5] for Theorem 3.2, relying on Sobolev estimates in $W^{1,p}$ to get Lipschitz continuity. This regularity property of spacetime fluxes is in sharp contrast to that of the Cauchy flux [7] since the discontinuous property of solution gives rise to the difficulty in defining the trace of $\mathbf{f}(\mathbf{u})$ on $A\subseteq\Gamma$.

4 Discretized fluxes and their consistency

In view of Theorems 3.1 and 3.2 the spacetime flux (2.8) is indeed continuous, while the instantaneous Cauchy flux (2.6), which is formally the time derivative of the spacetime flux is in general not well defined. We conclude that *the spacetime flux should be used for the approximation*, implying that the resulting finite volume scheme is fully discrete.

4.1 1D finite volume schemes

The integral balance law (2.7) is at the basis of the finite volume approximation to the conservation law (2.1). We first discuss the discretization in the one-dimensional setting, using a uniform grid. Let $\tau=\Delta t$ be a fixed time step. The spatial control volumes (intervals) are $I_j=(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}})$, $j\in\mathbb{N}$, $\Delta x=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$, and the spacetime control volumes are

$$Q_j^n=I_j\times(t_n,t_{n+1}),\quad t_{n+1}=t_n+\tau. \tag{4.1}$$

We denote by $\mathcal{U}$ the functional space of solutions of (3.3) and by $V^k\subseteq\mathcal{U}$ a finite dimensional subspace of order k when restricted to each I_j. In order to define the finite volume approximation, we need first to define approximate fluxes. We assume that there is a unique "entropy" solution, denoted by $\mathbf{u}(x,t;\xi)=S(t)\xi\in\mathcal{U}$, $0<t<\tau$, subject to the initial data $\xi\in V^k$. Due to the semigroup property of solutions to (3.3), we can focus our discussion on the interval $[0,\tau)$.

Definition 4.1 (1-D Approximate flux)**.** Let $\{\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t),\ 0\le t<\tau\}_{j=-\infty}^{\infty}$ be a family of D-dimensional functions of t. They are **approximate fluxes** (in the time interval $[0,\tau)$) corresponding to the initial function $\xi\in V^k$, if the following **finite propagation property** is satisfied for all $j\in\mathbb{N}$.

(i) $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)$, $0\le t<\tau$, depends only on the restriction of ξ to $I_j\cup I_{j+1}$.

(ii) If $\xi\equiv c=const.$ in $I_j\cup I_{j+1}$ then $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)\equiv\mathbf{f}(c)$.

Next we define the **consistency of the approximate fluxes**.

Definition 4.2 (Consistency in 1D)**.** The approximate flux $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)$ is consistent of order $q>0$ with the balance law (3.4) if there holds, for any $\xi\in V^k$,

$$\begin{aligned}&\left[\int_0^\tau \mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)dt-\int_0^\tau \mathbf{F}^{\xi}_{j-\frac{1}{2}}(t)dt\right]\\&\quad-\left[\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi)dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t;\xi))dt\right]=\mathcal{O}(\tau^{2+q}).\end{aligned} \tag{4.2}$$

Remark 4.1. Observe that the order of consistency strongly depends on the order of the approximating subspace V^k. This is clearly demonstrated in the case of the fundamental Godunov flux below.

The finite volume approximation to (3.4) is now presented in terms of the approximate fluxes.

Definition 4.3 (1D Finite Volume Approximation). (i) Let $\{\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t);\ 0<t<\tau\}$ be approximate fluxes consistent with (3.4) of order $q>0$ in the sense of Definition 4.2.

(ii) Let $\widetilde{S}(\tau): V^k \to \mathcal{U}$ be an approximate evolution operator associated with the approximate fluxes such that

$$\int_{I_j} \widetilde{S}(\tau)\xi dx - \int_{I_j} \xi dx + \int_0^{\tau} \mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)dt - \int_0^{\tau} \mathbf{F}^{\xi}_{j-\frac{1}{2}}(t)dt = 0, \tag{4.3}$$

for $j \in \mathbb{N}$.

(iii) There exists a projection map $P^k: \mathcal{U} \to V^k$ such that the average is preserved

$$\int_{I_j} P^k \xi(x)dx = \int_{I_j} \xi(x)dx, \quad j \in \mathbb{N}. \tag{4.4}$$

Then a family of maps $\{\Phi^k : V^k \to V^k\}$ is a finite volume scheme (FVS) for the conservation law (3.3) if

$$\Phi^k = P^k \widetilde{S}(\tau). \tag{4.5}$$

Thus given initial data $\mathbf{u}(x,0) = \mathbf{u}_0(x) \in \mathcal{U}$ to (3.3), we construct the sequence of finite volume approximate solutions by taking first $\mathbf{u}^0(x) = P^k \mathbf{u}_0(x)$ and then proceed for $n = 0,1,2\cdots$, by

$$\mathbf{u}^{n+1}(x) = \Phi^k \mathbf{u}^n(x). \tag{4.6}$$

4.2 Consistency of fluxes: Godunov and beyond

It is clear that the error of a finite volume approximation comes from two parts: the flux approximation and the projection. The literature concerning the projection error (i.e. slope-limiters) is quite extensive. Here we concentrate on the flux approximation, which strongly depends on the space of approximation. In this section, we suppress the dependence of notation on ξ if no confusion can occur.

Godunov flux. We first assume that the initial data $\xi(x) \in V^0$ is piecewise constant

$$\mathbf{u}_0(x) = \xi(x) = \mathbf{u}^0_j, \quad x \in I_j. \tag{4.7}$$

Then (assuming a CFL condition) the solution $\mathbf{u}(x_{j+\frac{1}{2}},t;\xi)$ is constant for $0<t<\tau$ and can be obtained by solving the local Riemann problem. The value is denoted by $\mathbf{u}_{j+\frac{1}{2}} := R(0;\mathbf{u}_j^0,\mathbf{u}_{j+1}^0)$. The Godunov flux is defined as [15]

$$\mathbf{F}_{j+\frac{1}{2}}(t)=\mathbf{f}(\mathbf{u}_{j+\frac{1}{2}}). \tag{4.8}$$

Hence, if $\xi\in V^0$, the Godunov flux fully agrees with the exact flux for the piecewise constant initial data (4.7) and no error exists. Formally, it means that the order of consistency of the Godunov flux is $q=\infty$!

In general, as $\xi(x)\in V^k$, the Godunov flux uses the leading term for the approximation. To be more precise, the solution $\mathbf{u}(x_{j+\frac{1}{2}},t;\xi)$ is no longer constant for $0<t<\tau$ and the solution $\mathbf{u}(x,t;\xi)$ along $x=x_{j+\frac{1}{2}}$ can be expanded as

$$\mathbf{u}(x_{j+\frac{1}{2}},t;\xi)=\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi)+t\cdot\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}},0+;\xi)+\mathcal{O}(t^2), \tag{4.9}$$

and

$$\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))=\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))+\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}},0+;\xi)t+\mathcal{O}(t^2). \tag{4.10}$$

The Godunov flux uses the leading term of the expansion (4.10),

$$\mathbf{F}^G_{j+\frac{1}{2}}(t)=\mathbf{f}(\mathbf{u}_{j+\frac{1}{2}}),\quad \mathbf{u}_{j+\frac{1}{2}}=\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi). \tag{4.11}$$

Then we have

$$\int_0^\tau \mathbf{F}^G_{j+\frac{1}{2}}(t)dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt=\frac{\tau^2}{2}\mathbf{f}'(\mathbf{u}_{j+\frac{1}{2}})\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}},0+;\xi)+\mathcal{O}(\tau^3). \tag{4.12}$$

Taking the difference of the two boundary values

$$\begin{aligned}&\int_0^\tau \mathbf{F}^G_{j+\frac{1}{2}}(t)dt-\int_0^\tau \mathbf{F}^G_{j-\frac{1}{2}}(t)dt-\left[\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t;\xi))dt\right]\\ &=\left[\mathbf{f}'(\mathbf{u}_{j+\frac{1}{2}})\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}},0+;\xi)-\mathbf{f}'(\mathbf{u}_{j-\frac{1}{2}})\frac{\partial \mathbf{u}}{\partial t}(x_{j-\frac{1}{2}},0+;\xi)\right]\frac{\tau^2}{2}+\mathcal{O}(\tau^3).\end{aligned} \tag{4.13}$$

If the solution $\mathbf{u}(x,t;\xi)$ is smooth the difference in the right-hand side of (4.13) contributes (via the CFL condition) another factor of τ. Otherwise, the error is $\mathcal{O}(\tau^2)$. We therefore arrive at the following conclusion.

Proposition 4.1 (Godunov Flux). Assume that $\xi(x)\in V^k, k\geq 1$. Then the Godunov scheme has first order accuracy for smooth solutions but just zero order if the solution contains discontinuities.

First order flux approximation. In practice, for the given initial data $\xi(x)\in V^k$, there is an alternative way of defining first order flux approximations [14],

$$\mathbf{F}_{j+\frac{1}{2}}(t)=\frac{1}{2}(\mathbf{f}(\mathbf{u}_-)+\mathbf{f}(\mathbf{u}_+))-\frac{\alpha}{2\lambda}(\mathbf{u}_+-\mathbf{u}_-),\quad \mathbf{u}_\pm:=\xi(x_{j+\frac{1}{2}}\pm), \tag{4.14}$$

for some $\alpha>0$, $\lambda=\tau/\Delta x$. If we try to obtain its order of consistency as in Definition 4.2 we get

$$\begin{aligned}&\Big[\int_0^\tau \mathbf{F}_{j+\frac{1}{2}}(t)dt-\int_0^\tau \mathbf{F}_{j-\frac{1}{2}}(t)dt\Big]\\ &\quad -\Big[\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t;\xi))dt\Big]=\mathcal{O}(|\mathbf{u}_+-\mathbf{u}_-|)\tau.\end{aligned} \tag{4.15}$$

Thus the error is estimated in terms of the total variation $TV(\mathbf{u})$. This is true even for $\xi(x)\in V^0$. In general it cannot be converted to estimates in terms of τ due to discontinuities. Furthermore, while for scalar conservation laws the total variation is not increasing [10], this is not true for hyperbolic systems, where solutions involve very complex wave interactions. We conclude that for such approximate fluxes the order of consistency (even the notion of consistency) cannot be addressed in our framework.

High order flux approximations. As discussed above, in order to achieve high order accuracy, we have to adopt high order flux approximation $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)$. **It is precisely here that we can use the Lipschitz continuity of fluxes** as expressed in Theorem 3.2. Indeed, the theorem guarantees that the difference

$$\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t;\xi))dt$$

provides (using the CFL condition) a factor $\mathcal{O}(\tau)$. In view of Definition 4.2 this means that we can focus on one endpoint and attempt to obtain a high value of α in the estimate of

$$\int_0^\tau \mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt=\mathcal{O}(\tau^{1+\alpha}), \tag{4.16}$$

for some $\alpha>0$. The error is measured in terms of the temporal increment τ or equivalently the spatial grid size Δx. Thanks to (4.9) and (4.10), we have

$$\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)=\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial t}(x_{j+\frac{1}{2}},0+;\xi)t+\cdots+\mathcal{O}(t^\alpha). \tag{4.17}$$

This is equivalent to the Taylor method for ordinary differential equations and requires the knowledge of the instantaneous values $\frac{\partial \mathbf{u}}{\partial t},\cdots,\frac{\partial^\alpha \mathbf{u}}{\partial t^\alpha}$. In numerical approximations this approach is replaced by multi-stage methods [19], in order to avoid high order temporal derivatives.

In Section 5 we will see how the Taylor method can be implemented by introducing the generalized Riemann problem (GRP) methodology. In a suitable sense, it can be considered as a Lax-Wendroff approach (normally associated with analytic setting) in a discontinuous nonlinear framework. The evaluation of temporal derivatives is carried out by using spatial slopes on the two sides of a discontinuity [2] and careful inspection of the propagation of the solution along characteristics (including shock formation).

4.3 Multi-D extensions

We briefly discuss how the $1-D$ methodology can be adopted in order to establish a finite volume approximation to multidimensional conservation laws (2.7). Let Ω be a computational domain covered by a set of closed control volume Ω_j, $\Omega = \cup_{j\in J}\Omega_j$, $\Omega_j \cap \Omega_\ell = \Gamma_{j\ell}$. They are assumed to be pairwise disjoint except for common boundaries. Then the spacetime integral balance law (2.7), when applied to Ω_j, becomes,

$$\int_{\Omega_j} \mathbf{u}(\mathbf{x},\tau;\xi)d\mathbf{x} - \int_{\Omega_j} \mathbf{u}(\mathbf{x},0;\xi)d\mathbf{x} + \sum_\ell \int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u})\cdot \nu_{j\ell} dS_{\mathbf{x}} dt = 0, \tag{4.18}$$

where $\mathbf{u}(\mathbf{x},0;\xi) = \xi$ is the initial data.

The general approach, as in the $1-D$ case, emphasizes the role of approximate fluxes.

Definition 4.4 (Multi-D approximate flux)**.** The functions of the family $\{\mathbf{F}^\xi_{j\ell}(t), 0 \le t < \tau\}_{j=-\infty}^{\infty}$ are **approximate fluxes** (in the time interval $[0,\tau)$) corresponding to the initial function $\xi \in V^k$, if the following **finite propagation property** is satisfied.

(i) $\mathbf{F}^\xi_{j\ell}(t)$, $0 \le t < \tau$, depends only on the restriction of ξ to $\Omega_j \cup \Omega_\ell$, where the index ℓ is taken such that $\Gamma_{j\ell} \neq \emptyset$.

(ii) If $\xi \equiv c = const.$ in $\Omega_j \cup \Omega_\ell$ then $\mathbf{F}^\xi_{j\ell}(t) \equiv \mathbf{f}(c)\|\Gamma_{j\ell}\|$.

The multi-dimensional case is complicated since the exact flux depends on $x \in \Gamma_{j\ell}$, and needs to be approximated at every boundary point. Remark that $\mathbf{F}^\xi_{j\ell}(t)$ implicitly contains the approximate integration along the common boundary $\Gamma_{j\ell}$.

Numerically, the boundary integrals are handled by using suitably high order integration formulas, such as Gaussian quadrature:

$$\int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u})\cdot \nu_{j\ell} dS_{\mathbf{x}} dt \approx \sum_{\ell,r} \int_0^\tau \omega_r \mathbf{f}(\mathbf{u})\cdot \nu_{j\ell}(\mathbf{x}_r) dt, \tag{4.19}$$

within desired order of accuracy, where ω_r is the weight at the Gaussian point $(\mathbf{x}_r,t)$ on $\Gamma_{j\ell}$. Then we can construct the approximate flux at each point $\mathbf{x}_r$.

Definition 4.5 (Consistency in Multi-D)**.** The approximate flux $\mathbf{F}^\xi_{j\ell}(t)$ is consistent of order $q>0$ with the balance law (4.18) if there holds

$$\sum_\ell \int_0^\tau \mathbf{F}^\xi_{j\ell}(t)dt - \sum_\ell \int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u}(\mathbf{x},t;\xi))\cdot \nu_{j\ell} dS_{\mathbf{x}} dt = \mathcal{O}(\tau^{2+q}). \tag{4.20}$$

Observe that the boundary integral $\int_0^\tau\int_{\Gamma_{j\ell}}\mathbf{f}(\mathbf{u}(\mathbf{x},t;\xi))\cdot\nu_{j\ell}dS_{\mathbf{x}}$ is well defined precisely due to Theorem 3.1. We refer to [6] for more details.

5 GRP fluxes – a necessary step for high resolution

The Lax-Wendroff approach was proposed in a finite difference version for hyperbolic conservation laws [17], assuming very regular solutions. Essentially it can be viewed as the numerical realization of Cauchy-Kovalevskaya theorem for partial differential equations [11, Chapter 4]. In this section we show how an appropriate modification of it can be applied in our setting of discontinuous solutions. Specifically, this modification is used in the construction of approximate fluxes with high order of consistency. We shall do it only in the $1-D$ setting. We therefore consider the $1-D$ version of Eq. (2.1):

$$\mathbf{u}_t+\mathbf{f}(\mathbf{u})_x=0,\quad x\in\mathbb{R}. \tag{5.1}$$

In general, solutions $\mathbf{u}(x,t;\xi)$ are known to develop discontinuities even for very smooth initial data ξ. In particular, the same is true for the fluxes $\mathbf{f}(\mathbf{u}(x,t;\xi))$. Nevertheless, in light of Theorem 3.2 the integral $\int_0^\tau\mathbf{f}(\mathbf{u}(x,t;\xi))dt$ is a Lipschitz function of x, hence it is legitimate to consider its point value at every fixed point, in particular the point $x=x_{j+\frac{1}{2}}$, that is a point of discontinuity of the initial data ξ. Then we are led to study the behavior of $\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))$ as a function of $t\in(0,\tau)$.

It boils down to solving the Generalized Riemann Problem (GRP) [2, 4]) which we proceed to discuss.

Let $\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi)$ be the instantaneous value of the solution (obtained by solving a Riemann problem) and let

$$F^{\xi}_{j+\frac{1}{2}}(0+)=\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi)), \tag{5.2}$$

be the corresponding instantaneous flux.

Let $\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)$ be the instantaneous value of the time-derivative of the solution. From

$$\begin{aligned}&\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))\\ =&\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))+\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)t+\mathcal{O}(t^2),\end{aligned} \tag{5.3}$$

it follows that

$$\begin{aligned}&\int_0^\tau F^{\xi}_{j+\frac{1}{2}}(0+)dt-\int_0^\tau\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt\\ =&\frac{1}{2}\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)\tau^2+\mathcal{O}(\tau^3).\end{aligned}$$

Hence

$$\int_0^\tau \left[F^\xi_{j+\frac{1}{2}}(0+)-F^\xi_{j-\frac{1}{2}}(0+)\right]dt-\int_0^\tau \left[\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))-\mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t;\xi))\right]dt$$
$$=\frac{1}{2}[\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)-\mathbf{f}'(\mathbf{u}(x_{j-\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j-\frac{1}{2}},0+;\xi)]\tau^2+\mathcal{O}(\tau^3). \quad (5.4)$$

If no regularity of the solution $\mathbf{u}(x,t;\xi)$ is assumed (in particular, if it is discontinuous) then the approximate flux $F^\xi_{j+\frac{1}{2}}(0+)$ is only consistent of order zero ($q=0$ in (4.2)). However, in regions where the solution is smooth the difference

$$\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)-\mathbf{f}'(\mathbf{u}(x_{j-\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j-\frac{1}{2}},0+;\xi)=\mathcal{O}(\tau), \quad (5.5)$$

thus raising the order of consistency to $q=1$.

The remedy here is to upgrade the approximate flux (5.2) by adding the GRP solution, thus introducing the GRP fluxes.

Definition 5.1 (GRP Approximate Flux). The GRP approximate flux is given by

$$F^\xi_{j+\frac{1}{2}}(t)=\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))+\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)t. \quad (5.6)$$

Now

$$\int_0^\tau F^\xi_{j+\frac{1}{2}}(t)dt-\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\xi))dt=\mathcal{O}(\tau^3),$$

so that the order of consistency is $q=1$ in all cases. For smooth solutions we obtain second-order consistency ($q=2$), since in analogy with (5.5)

$$\mathbf{f}''(\mathbf{u}(x_{j+\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\xi)-\mathbf{f}''(\mathbf{u}(x_{j-\frac{1}{2}},0+;\xi))\mathbf{u}_t(x_{j-\frac{1}{2}},0+;\xi)=\mathcal{O}(\tau). \quad (5.7)$$

Thus, when reduced to the smooth setting, the common statement about the second order consistency of this approximate flux (as well as the MUSCL flux) is recovered.

Remark 5.1 (GRP fluxes in multi-D). In Definition 4.5 we have introduced the notion of consistency (of arbitrary order) of a flux in the multi-dimensional setting. However, it is a challenge to extend the one-dimensional GRP approximate flux (Definition 5.1) to a suitable multi-D GRP flux. Observe that this is different from the implementation of a "Strang spatial splitting" [2, Chapter 7], but rather a genuine multi-dimensional singularity analysis. Over the last thirty years there have several attempts in this direction, but we cannot give here a more detailed account.

In the final paragraphs of this section we outline this methodology as applied to Euler's system for compressible, nonisentropic flows, sort of a "flagship" representing nonlinear systems of conservation laws. We also point out a few related systems to which the GRP scheme has been successfully implemented.

5.1 Euler equations of compressible inviscid flow

As the prototype of hyperbolic conservation laws, the system of compressible Euler equations (in one space dimension)

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mathbf{v}) = 0, \\ (\rho\mathbf{v})_t + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} + p) = 0, \\ (\rho E)_t + \nabla\cdot(\mathbf{v}(\rho E + p)) = 0, \end{cases} \tag{5.8}$$

plays an important role in the development of theory, numerics and applications, where ρ, $\mathbf{v}$, p, $E = |\mathbf{v}|^2/2 + e$ are the density, velocity, pressure and total energy, e is the internal energy.

As $\xi(x) \in V^1$, i,e., $\xi(x)$ is piecewise linear, the generalized Riemann problem (GRP) method was developed in [1,2] and then improved in the direct Eulerian version [3]. As $\xi(x) \in V^k$, $k \geq 2$, the GRP method was extended in [22] to achieve high order approximate fluxes in the sense of (4.16).

Some remarks are in order.

(i) There are acoustic versions of GRP methods provided that waves involved are weak so that the equations (5.8) could be linearized. The popular ADER solvers were developed along this line [28]. Hence the GRP method could be regarded, as already mentioned, as a nonlinear version of discontinuous Lax-Wendroff method.

(ii) It is amazing to find that the GRP method effectively reflects the thermodynamics of compressible flows [21].

(iii) There are extensions to various systems, e.g., the relativistic fluid dynamics [32] and the blood model [24].

6 Lax-Wendroff type convergence

The notion of high order consistency of the approximate fluxes (Definition 4.2) is crucial in the study of the convergence of the approximate solutions to a solution of the balance law. We discuss it in the $1-D$ case.

Applying the finite volume approximation (4.6), we construct the discrete sequence

$$\widetilde{\theta^{n+1}}(x) = \Phi^k(\widetilde{\theta^n}) \in V^k, \quad n = 0,1,2,\cdots,N-1. \tag{6.1}$$

The initial data is given by taking the projection of the initial function $u_0 \in \mathcal{U}$ on the subspace V^k. Recall that $\mathcal{U}$ is the functional space of solutions of (3.3) and $V^k \subseteq \mathcal{U}$ a finite dimensional subspace of order k when restricted to each mesh interval I_j

$$\theta^0 = \widetilde{\theta^0} = P^k u_0 \in V^k. \tag{6.2}$$

Observe that at each step $\widetilde{\theta^n}\in V^k$ is discontinuous at cell boundaries $x=x_{j+\frac{1}{2}}$ since the element of V^k should preserve the average over I_j.

We shall further assume that these fluxes are consistent of order $q>0$ (Definition 4.2). It follows from Definition 4.3 (see Eq. (4.3)) that for all grid intervals I_j,

$$\begin{aligned}&\int_{I_j}[\widetilde{\theta^{n+1}}(x)-\widetilde{\theta^n}(x)]dx\\ =&-\int_{t_n}^{t_{n+1}}[F^{\widetilde{\theta^n}}_{j+\frac{1}{2}}(t-t_n)-F^{\widetilde{\theta^n}}_{j-\frac{1}{2}}(t-t_n)]dt,\quad -\infty<j<\infty.\end{aligned}\tag{6.3}$$

We now construct an interpolation function (in spacetime) $\widetilde{Y^k}(x,t)$ as follows.

$$\widetilde{Y^\tau}(x,t)=\frac{1}{\tau}[(t_{n+1}-t)\widetilde{\theta^n}(x)+(t-t_n)\widetilde{\theta^{n+1}}(x)],\quad t\in[t_n,t_{n+1}],\quad n=0,1,\cdots,N-1.\tag{6.4}$$

Observe that $t_n=n\tau$ depends on τ.

Instead of the classical Lax-Wendroff theorem [14, Section 3.1] we get here the following convergence theorem for the FVS (finite volume scheme, see Definition 4.3). We refer to [5] for the proof. It should be pointed out that the hypotheses imposed in our theorem are quite analogous to those imposed in the original Lax-Wendroff theorem. Admittedly, from the viewpoint of a numerical practitioner they are not easy to validate.

Theorem 6.1. *Assume that the FVS* (6.1) *is consistent of order* $q>0$. *Let* $\{\tau_m\downarrow 0\}$ *be a decreasing sequence of time steps. Let* $u_0\in\mathcal{U}$ *and let* $\{\widetilde{Y^{\tau_m}}(x,t)\}_{m=1}^\infty$ *be the corresponding functions defined in* (6.4).

Suppose that

(i) The sequence $\{\widetilde{Y^{\tau_m}}(x,t)\}_{m=1}^\infty$ *is uniformly bounded in* $L^\infty([0,T],L^\infty(\mathbb{R}))$.

(ii) The sequence $\{\widetilde{Y^{\tau_m}}(x,t)\}_{m=1}^\infty$ *converges in* $C([0,T],L^1_{loc}(\mathbb{R}))$ *to a function* $v(x,t)$ *(in particular it is uniformly bounded in this space).*

Then $v(x,t)$ *is a solution of the balance law* (3.4) *in the sense of Theorem* 3.2.

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