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Convergence Analysis for a Finite Volume Evolution Galerkin Method for Multidimensional Hyperbolic Systems†

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Abstract. We study the convergence of a finite volume method based on the method of bicharacteristics for multidimensional hyperbolic conservation laws. In particular, we concentrate on the linear wave equation system and nonlinear Euler equations of gas dynamics. We show the stability and the consistency of the numerical approximations. By means of the generalized Lax equivalence principle we prove the convergence of numerical solutions to the strong solution on the lifespan.

AMS subject classifications: 65M08, 65M12, 35L65, 76N10

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1 Introduction

Hyperbolic conservation laws are fundamental to model conservation principles arising in fluid dynamics, physics or biology. In this paper we consider multidimensional hyperbolic conservation laws and concentrate on the linear wave equation system and the nonlinear Euler equations of gas dynamics.

These systems are approximated by a genuinely multidimensional finite volume evolution Galerkin method which is based on the method of bicharacteristics, that has been developed by Lukáčová, Morton and Warnecke [10]. Applying the generalized Lax

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equivalence principle, see [7], and the recently developed concept of generalized solutions, the dissipative solutions, we will analyze the convergence of the genuinely multidimensional finite volume evolution Galerkin method.

In our recent works [6,14], we have proved the convergence of the numerical solutions for the Euler equations obtained by the finite volume upwind method and the Godunov method, respectively. In general, we obtain only weak* convergence to a generalized, dissipative solution. If the limit is a weak entropy solution then the convergence is also strong. Moreover, if the Euler equations admit a strong solution then the numerical solutions converge strongly to the strong solution as long as the latter exists.

Our aim is to extend the previous convergence analysis to the genuinely multidimensional finite volume method based on the method of bicharacteristics. We point out that this is the first result available in the literature, where the convergence of the truly multidimensional scheme is studied for multidimensional hyperbolic systems. In this context we refer to the recent work of Lukáčová and Yuan [15], where the convergence of finite volume generalized Riemann problem method, see J. Li et al. [8,9], has been analysed for scalar hyperbolic equation.

The hyperbolic conservation law on a bounded domain $\Omega\subset\mathbb{R}^d\,(d=2,3)$ reads

$$\partial_t \boldsymbol{U}+\mathrm{div}_x \boldsymbol{F}(\boldsymbol{U})=0, \quad (t,x)\in(0,T)\times\Omega, \tag{1.1}$$

where $\boldsymbol{U}\in\mathbb{R}^N$ is the conservative variable and $\boldsymbol{F}\in\mathbb{R}^{N\times d}$ is the flux function. System (1.1) is accompanied with initial data $\boldsymbol{U}(0,\cdot)=\boldsymbol{U}_0$ on Ω and periodic boundary conditions. Taking the second law of thermodynamics into account we further require that the entropy inequality holds, i.e.

$$\partial_t S(\boldsymbol{U})+\mathrm{div}_x \boldsymbol{Q}(\boldsymbol{U})\geq 0, \quad (t,x)\in(0,T)\times\Omega. \tag{1.2}$$

We analyse specifically the linear wave equation system with

$$\boldsymbol{U}=(\phi,\boldsymbol{u})^T, \quad \boldsymbol{F}=(c\boldsymbol{u},\, c\phi\mathbb{I})^T, \quad c>0, \tag{1.3}$$

$$S=-\frac{1}{2}|\boldsymbol{U}|^2, \quad \boldsymbol{Q}=-c\phi\boldsymbol{U}, \tag{1.4}$$

and the nonlinear Euler equations with

$$\boldsymbol{U}=(\varrho,\boldsymbol{m},E)^T, \quad \boldsymbol{F}=\left(\boldsymbol{m},\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\varrho}+p\mathbb{I},\frac{\boldsymbol{m}(E+p)}{\varrho}\right)^T, \tag{1.5}$$

$$S=C_v\varrho s, \quad \boldsymbol{Q}=\frac{S\boldsymbol{m}}{\varrho} \quad \text{with } C_v=\frac{1}{\gamma-1} \text{ and } s=\ln\left(\frac{p}{\varrho^\gamma}\right), \quad \gamma>1. \tag{1.6}$$

In this paper we concentrate on two-dimensional problem, i.e. $d=2$. The generalization to $d=3$ is possible but technical and the details will be presented in a future work.

In (1.3), ϕ denotes the wave pressure, $\boldsymbol{u}=(u,v)$ is the velocity vector and $c=$constant is the wave speed. Further, in (1.5), ρ denotes the density, $\boldsymbol{m}=(m_1,m_2)=(\rho u,\rho v)$ is the momentum, E is the energy, $p=(\gamma-1)(E-\frac{1}{2}\frac{|\boldsymbol{m}|^2}{\rho})$ is the pressure and S is the entropy.