Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2024-0262

Vol. **39**, No. 5, pp. 1421-1453
May 2026

A Lagrangian GRP Algorithm for Axisymmetric Problems of Compressible Fluids

Zijin Zhu[1], Min Xiao[2], Jiwei Zhang[1] and Guoxi Ni[3,*]

[1] *School of Mathematics and Statistics, Wuhan University, Wuhan, China.*
[2] *College of Science, China University of Petroleum, Beijing, China.*
[3] *Institute of Applied Physics and Computational Mathematics, Beijing, China.*

Communicated by Kun Xu

Received 29 October 2024; Accepted (in revised version) 23 March 2025

Abstract. We devise a Lagrangian Generalized Riemann Problem (GRP) algorithm for axisymmetric hydrodynamics, which pays attention to high-resolution boundary treatment at the symmetric axis. The numerical boundary condition here is first formulated the same as the scheme on interior cells. Then we uniformly obtain the requisite interface values in constructing numerical fluxes and sources through the newly-tailored GRP solver and its one-sided variant. There also exist some innovations in the other two critical procedures: the derivation of vertex velocities and the compliance with the geometry conservation law (GCL). Several challenging numerical examples are utilized to demonstrate the performance of our algorithm in resolving discontinuities, maintaining symmetry, alleviating overheating phenomena, and dealing with complex fluids.

AMS subject classifications: 76M12, 35L65, 65M08, 76N15

Key words: Axisymmetry, cell-centered Lagrangian algorithm, high-resolution numerical boundary condition, GRP solver, excess heating.

1 Introduction

The numerical simulations of axisymmetric multi-material flows play a vital role in numerous engineering applications, such as inertial confinement fusion, astrophysics, and weaponry equipment [8, 12, 36]. The dimensionality reduction induced by axisymmetry offers a signal computational advantage. And the Lagrangian algorithms are famous for

*Corresponding author. *Email addresses:* `zhuzijin@whu.edu.cn` (Z. Zhu), `xiaomin@cup.edu.cn` (M. Xiao), `jiweizhang@whu.edu.cn` (J. Zhang), `gxni@iapcm.ac.cn` (G. Ni)

the effective capture of material interfaces. However, except for the resolution of discontinuities, there still persist three tasks to confront in their collocation: the accordance with conservation law [34], the maintenance of spherical symmetry [10], and the appropriate boundary treatment at the symmetric axis $r=0$ [24]. Overcoming these intertwined issues unavoidably involves multiple trade-offs.

The axisymmetric schemes can fall into two main categories concerning the discrete manners: area-weighted and volume-weighted [22]. Each has its own pros and cons. The former only conserves mass. In contrast, the latter, primarily combined with the cell-centered style, can strictly conserve mass, momentum, and total energy [11]. As for the preservation of spherical symmetry on the polar mesh, the latter is much inferior [30].

The singularity of geometrical source at the inner boundary $r=0$ is another inescapable difficulty. We cannot totally predict its function in the absence of mathematical theories, especially when a reflected shock is formed. Well, that is exactly the most essential and anticipated part. Previous efforts on this issue include the extrapolation of neighboring solution [37], the import of heat viscosity [24], and so on. The straightforward use of reflective boundary condition is most common among them [9, 10, 20, 32]. Whereas, the research in [19] suggests its actual ignorance of source effect. Excess heating, numerical instabilities, and even non-physical solution can thus be produced.

Upon these investigations, we embark on the choice of a discrete manner, since the numerical solution at $r=0$ relies on it likewise [31]. A volume-weighted scheme on the flexible quadrilateral mesh is precisely the one we tend to adopt. Conceivably, the relevant algorithm will have strengths in ensuring the conservation and weaknesses in preserving the spherical symmetry. We hope that the latter can be improved under the applicable numerical boundary.

Our scheme is established through a geometrically compatible analysis under the finite volume framework, which reminds us that the numerical boundary condition at $r=0$ can take the same form. Its consistency with the conservation law is thus guaranteed. Then we proceed to the numerical approximations of the fluxes and sources at the interior interfaces and $r=0$. A feasible option comes from the Generalized Riemann Problem (GRP) solver and its one-sided version.

The studies on the GRP series could date back to [3] and have been widely exploited since then [4, 6, 14, 27, 36]. This type of solver can serve as a high-order extension of the associated Riemann problem solver and brings us two major benefits. Firstly, we can skip the operator splitting and naturally exhibit the source effect via the Lax-Wendroff approach [5, 18]. Secondly, we notice that the radial velocity vanishes at $r=0$ due to the symmetry argument. The one-sided GRP solver can utilize this to precisely describe the fluids here. With the experience on the Eulerian algorithms [38,39], we believe that it can also make a difference in the Lagrangian ones, particularly in preserving the spherical symmetry and reducing the heating error.

Turning to the Lagrangian velocities, the linearized GRP solver has once been directly applied at vertices in [23], since the fluid structure with initial four regions is too complicated to be distinguished. In contrast, we opt to propose an entire version at interfaces