Commun. Comput. Phys. Vol. **40**, No. 1, pp. 1-26
doi: 10.4208/cicp.OA-2024-0201 July 2026

Score-fPINN: Fractional Score-Based Physics-Informed Neural Networks for High-Dimensional Fokker-Planck-Lévy Equations

Zheyuan Hu[1,*], Zhongqiang Zhang[2], George Em Karniadakis[3,4] and Kenji Kawaguchi[1]

[1] *Department of Computer Science, National University of Singapore, Singapore, 119077.*
[2] *Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609 USA.*
[3] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*
[4] *Advanced Computing, Mathematics and Data Division, Pacific Northwest National Laboratory, Richland, WA, United States.*

Received 20 August 2024; Accepted (in revised version) 8 January 2025

Abstract. We introduce an innovative approach for solving high-dimensional Fokker-Planck-Lévy (FPL) equations in modeling non-Brownian processes across disciplines such as physics, finance, and ecology. We utilize a fractional score function and Physical-informed neural networks (PINN) to lift the curse of dimensionality (CoD) and alleviate numerical overflow from exponentially decaying solutions with dimensions. The introduction of a fractional score function allows us to transform the FPL equation into a second-order partial differential equation without fractional Laplacian and thus can be readily solved with standard physics-informed neural networks (PINNs). We propose two methods to obtain a fractional score function: fractional score matching (FSM) and score-fPINN for fitting the fractional score function. While FSM is more cost-effective, it relies on known conditional distributions. On the other hand, score-fPINN is independent of specific stochastic differential equations (SDEs) but requires evaluating the PINN model's derivatives, which may be more costly. We conduct our experiments on various SDEs and demonstrate numerical stability and effectiveness of our method in dealing with high-dimensional problems, marking a significant advancement in addressing the CoD in FPL equations. Code is available at `https://github.com/zheyuanhu01/Score-fPINN`.

AMS subject classifications: 65M75, 65C30, 68T07

Key words: Physics-informed neural networks, high dimensional partial differential equations, curse of dimensionality, Fokker-Planck equations, Fokker-Planck-Lévy equations, score function, score matching.

*Corresponding author. *Email addresses:* `e0792494@u.nus.edu` (Z. Hu), `zzhang7@wpi.edu` (Z. Zhang), `george_karniadakis@brown.edu` (G. E. Karniadakis), `kenji@nus.edu.sg` (K. Kawaguchi)

1 Introduction

The Fokker-Planck-Lévy (FPL) equation, also known as the fractional Fokker-Planck equation, is a generalization of the traditional Fokker-Planck (FP) equation to incorporate Lévy processes, particularly those involving jumps or heavy-tailed distributions. The FPL equation is used in fields like physics for anomalous diffusion in complex systems, finance for pricing derivatives when the underlying asset exhibits jumps or heavy tails, and ecology for animal movement patterns that involve sudden, long-range moves. The classic Fokker-Planck equation describes the time evolution of the probability density function of the velocity of a particle under the influence of forces and Gaussian white noises. However, many physical and economic phenomena exhibit jumps and heavy tails, which are not adequately described by Gaussian processes. Lévy processes, which include a broader class of stochastic processes characterized by stable distributions and jumps, offer a more appropriate mathematical framework for such scenarios. The FPL equation extends the traditional Fokker-Planck equation by incorporating fractional derivatives, which can model these non-local, jump-like dynamics.

Despite its importance, obtaining numerical solutions to the FPL equation is still challenging due to the non-locality introduced by the fractional derivative, which requires special numerical schemes that can handle integral terms effectively and the need for handling both the small-scale behavior driven by the diffusion term as well as the large discrete changes introduced by the jump term.

The higher-dimensional FPL equations of interest in this paper pose more significant challenges owing to the curse of dimensionality (CoD), where traditional grid-based methods fail due to the exponential increase in computational requirements with the dimensionality of the PDE, rendering them unrealistic. Consider another branch of the traditional method, namely Monte Carlo simulation, which can tackle the CoD in certain PDEs. Though Monte Carlo methods can solve the FP equation with the Feynman–Kac formula and the corresponding stochastic differential equation (SDE), they solve the problem at one point and are also expensive when the solution at a large region is desired.

Physics-informed neural networks [40] have become popular in solving high-dimensional PDEs and tackling the CoD thanks to neural networks' strong universal approximation property [1], generalization capacities [21], robust optimization [28], and PINN's meshless plus grid-free training. Multiple methods [22, 23] recently have proposed to scale up and speed up PINNs to very high dimensions using random sampling. Although PINN offers the possibility of addressing the CoD in certain cases, in FPL equations, PINN accuracy is limited to moderately high dimensions (e.g., less than ten dimensions [9]) for computing probability density functions (PDFs) of interest in the FPL equations. They also exhibit significant numerical errors at higher dimensions, rendering them impractical. Specifically, FPL equations model PDFs and the most common Gaussian PDFs associated with Brownian motion exhibit exponential decay in numerical values as dimensionality increases. This phenomenon easily surpasses the numerical precision of computer simulations, leading to significant errors in PINN numerical solvers. The